Symmetry and Symmetry Breaking in Dynamical Systems

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1 Introduction

The same symmetries may underly diverse contexts such as phase transitions of crystals (Landau theory), fluid dynamics, and problems in biology and chemical engineering. Hence, seemingly unrelated systems may exhibit similar phenomena in regard to symmetries of patterns and transitions between patterns (spontaneous symmetry breaking). It is natural to focus attention on aspects of pattern formation that are *universal* or *model-independent* — aspects depending on underlying symmetries rather than model-specific details.

The general framework is that the underlying system is governed by an evolution equation

$$\dot{x} = f(x), \tag{1.1}$$

with symmetry group Γ. To avoid technicalities, we assume that (1.1) is an ordinary differential equation (ODE), the vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is as smooth as desired, and Γ is a compact Lie group acting linearly on $\mathbb{R}^n$. An inner product may be chosen so that Γ acts orthogonally. The vector field in (1.1) is Γ-*equivariant* if

$$f(\gamma x) = \gamma f(x) \text{ for all } x \in \mathbb{R}^n,\ \gamma \in \Gamma. \tag{1.2}$$

Equivalently, if $x(t)$ is a solution and $\gamma \in \Gamma$, then $\gamma x(t)$ is a solution.

In this article, we are interested in the dynamics to be expected for equivariant vector fields, and transitions that arise as parameters are varied. The symmetry group Γ is taken as given, whereas f is a general Γ-equivariant vector field. (Other features such as energy-conservation or time-reversibility must be built into the general setup, but are excluded in this article.)

2 Isotropy subgroups and commuting linear maps

Let Γ be a compact Lie group acting linearly on $\mathbb{R}^n$. The *isotropy subgroup* of $x \in \mathbb{R}^n$ is defined to be

$$\Sigma_x = \{\gamma \in \Gamma : \gamma x = x\}.$$

Note that $\Sigma_{\gamma x} = \gamma \Sigma_x \gamma^{-1}$ for all $x \in \mathbb{R}^n$, $\gamma \in \Gamma$.

Given an isotropy subgroup $\Sigma \subset \Gamma$, define the *fixed-point subspace*

$$\mathrm{Fix}\,\Sigma = \{y \in \mathbb{R}^n : \sigma y = y \text{ for all } \sigma \in \Sigma\}.$$

If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Γ-equivariant vector field, then $f(\mathrm{Fix}\,\Sigma) \subset \mathrm{Fix}\,\Sigma$ for each isotropy subgroup Σ. Hence $\mathrm{Fix}\,\Sigma$ is flow-invariant.

The normalizer $N(\Sigma) = \{\gamma \in \Gamma : \gamma \Sigma \gamma^{-1} = \Sigma\}$ is the largest subgroup of Γ that acts on $\mathrm{Fix}\,\Sigma$, and $f_\Sigma = f|_{\mathrm{Fix}\,\Sigma}$ is $(N(\Sigma)/\Sigma)$-equivariant.

An isotropy subgroup Σ is *axial* if $\dim \mathrm{Fix}\,\Sigma = 1$, and then $N(\Sigma)/\Sigma \cong \mathbb{Z}_2$ or $\mathbf{1}$. More generally, Σ is *maximal* if there are no isotropy subgroups T with $\Sigma \subset T \subset \Gamma$ other than $T = \Sigma$ and $T = \Gamma$. Then $N(\Sigma)/\Sigma$ acts fixed-point freely on $\mathrm{Fix}\,\Sigma$ and the connected component of the identity $(N(\Sigma/\Sigma)^0 \cong \mathbf{1}$, $\mathbf{SO}(2)$ or $\mathbf{SU}(2)$. Correspondingly Σ is called *real*, *complex* or *quaternionic*. In the complex case $\dim \mathrm{Fix}\,\Sigma$ is even; in the quaternionic case $\dim \mathrm{Fix}\,\Sigma \equiv 0 \bmod 4$.

The dihedral group $\Gamma = \mathbb{D}_m$ of order m is the symmetry group of the regular m-gon, $m \geq 3$. Its standard action on $\mathbb{R}^2$ is generated by

$$\rho = \begin{pmatrix} \cos 2\pi/m & -\sin 2\pi/m \\ \sin 2\pi/m & \cos 2\pi/m \end{pmatrix}, \qquad \kappa = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For m even, the isotropy subgroups up to conjugacy are

$$\mathbb{D}_m, \quad \mathbb{Z}_2(\kappa), \quad \mathbb{Z}_2(\rho\kappa), \quad \mathbf{1},$$

where $\mathbb{Z}_j(g)$ denotes the cyclic group of order j generated by g. The maximal isotropy subgroups $\Sigma = \mathbb{Z}_2(\kappa)$, $\mathbb{Z}_2(\kappa\rho)$ are axial with $N(\Sigma)/\Sigma \cong \mathbb{Z}_2$. For m odd, $\mathbb{Z}_2(\rho\kappa)$ is conjugate to $\mathbb{Z}_2(\kappa)$ leaving three conjugacy classes of isotropy subgroups, and $\Sigma = \mathbb{Z}_2(\kappa)$ is axial with $N(\Sigma)/\Sigma = \mathbf{1}$.

The space of commuting linear maps

$$\mathrm{Hom}_\Gamma(\mathbb{R}^n) = \{L : \mathbb{R}^n \to \mathbb{R}^n \text{ linear} : L\gamma = \gamma L \text{ for all } \gamma \in \Gamma\}$$

is completely described representation-theoretically. Recall that Γ acts *irreducibly* on $\mathbb{R}^n$ if the only Γ-invariant subspaces of $\mathbb{R}^n$ are $\mathbb{R}^n$ and $\{0\}$. Then $\mathrm{Hom}_\Gamma(\mathbb{R}^n)$ is a real division ring (skew field) $\mathcal{D} \cong \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. The representation is called *absolutely irreducible* when $\mathcal{D} = \mathbb{R}$ and *nonabsolutely irreducible* when $\mathcal{D} = \mathbb{C}$ or $\mathbb{H}$.

If the action of Γ is not irreducible, write $\mathbb{R}^n = V_1 \oplus \cdots \oplus V_k$ (nonuniquely) as a sum of irreducible subspaces. Summing together irreducible subspaces that are isomorphic to form *isotypic components* W, gives the (unique) *isotypic decomposition* $\mathbb{R}^n = W_1 \oplus \cdots \oplus W_\ell$. If $L \in \mathrm{Hom}_\Gamma(\mathbb{R}^n)$, then $L(W_j) \subset W_j$ for each j, hence $\mathrm{Hom}_\Gamma(\mathbb{R}^n) = \mathrm{Hom}_\Gamma(W_1) \oplus \cdots \oplus \mathrm{Hom}_\Gamma(W_\ell)$. Each W_j consists of k_j isomorphic copies of an irreducible representation with division ring $\mathcal{D}_j$. Let $M_k(\mathcal{D})$ denote the space of $k \times k$ matrices with entries in $\mathcal{D}$. Then

$$\mathrm{Hom}_\Gamma(\mathbb{R}^n) \cong M_{k_1}(\mathcal{D}_1) \oplus \cdots \oplus M_{k_\ell}(\mathcal{D}_\ell). \tag{2.1}$$

Spectral properties of commuting linear maps can be recovered from the decomposition (2.1), paying due attention to multiplicity and complex conjugates of eigenvalues.

3 Equivariant dynamics

The dynamics of equivariant systems includes (relative) equilibria and periodic solutions, robust heteroclinic cycles/networks, and symmetric chaotic attractors.

3.1 Equilibria

Consider the ODE (1.1) with Γ-equivariant vector field f satisfying (1.2). If $x(t) \equiv x_0$ is an equilibrium, $f(x_0) = 0$, then there is a group orbit Γx_0 of equilibria.

Let $\Sigma = \Sigma_{x_0}$ be the isotropy subgroup of x_0. If $\dim \Sigma = \dim \Gamma$, then generically (for an open dense set of Γ-equivariant vector fields), the eigenvalues of $(df)_{x_0}$ have nonzero real part, hence x_0 is hyperbolic. If the eigenvalues all have negative real part, then x_0 is asymptotically stable. If at least one eigenvalue has positive real part, then x_0 is unstable. Hyperbolic equilibria are isolated and persist under perturbations of f; the perturbed equilibria continue to have isotropy Σ. Since $(df)_{x_0} \in \mathrm{Hom}_\Sigma(\mathbb{R}^n)$, decomposition (2.1) for the action of Σ on $\mathbb{R}^n$ facilitates stability computations for x_0.

If $\dim \Sigma < \dim \Gamma$, then Γx_0 is a continuous group orbit of equilibria. Generically $\dim \ker(df)_{x_0} = \dim \Gamma - \dim \Sigma$ and $\ker(df)_0 = \{\xi x_0 : \xi \in L\Gamma\}$ where $L\Gamma$ is the Lie algebra of Γ. The remaining $k = n - \dim \Gamma + \dim \Sigma$ eigenvalues generically have nonzero real part so Γx_0 is normally hyperbolic. If all k eigenvalues have nonzero real part, then Γx_0 is asymptotically stable. If at least one has positive real part, then Γx_0 is unstable. When $N(\Sigma)/\Sigma$ is finite, generically x_0 is an isolated equilibrium in $\mathrm{Fix}(\Sigma)$ and persists as an equilibrium with isotropy Σ under perturbation.

3.2 Relative equilibria and skew products

A point $x_0 \in \mathbb{R}^n$ (or the corresponding group orbit Γx_0) is a *relative equilibrium* if $f(x_0) \in T_{x_0}\Gamma x_0 = L\Gamma x_0$. If x_0 has isotropy Σ, then x_0 is a relative equilibrium if $f(x_0) \in LD_\Sigma\, x_0$, where $D_\Sigma = \big(N(\Sigma)/\Sigma\big)^0$.

Write $f(x_0) = \xi x_0$ where $\xi \in LD_\Sigma$. The closure of the one-parameter subgroup $\exp(t\xi)$ is a maximal torus in D_Σ for almost every ξ. All maximal tori are conjugate with common dimension $d = \operatorname{rank} D_\Sigma$. The solution $x(t) = \exp(t\xi)x_0$ is typically a d-dimensional quasi-periodic motion. "Typically" holds in both the topological and probabilistic sense and there is no phase-locking. When $d = 1$, $x(t)$ is periodic, often called a *rotating wave*.

Choose a Σ-invariant local cross-section X to the group orbit Γx_0 at x_0. There is a Γ-invariant neighborhood of Γx_0 that is Γ-equivariantly diffeomorphic to $(\Gamma \times X)/\Sigma$, where Σ acts freely on $\Gamma \times X$ by

$$\sigma \cdot (\gamma, x) = (\gamma\sigma^{-1}, \sigma x),$$

and Γ acts by left multiplication on the first factor. The Γ-equivariant ODE on $(\Gamma \times X)/\Sigma$ lifts to a $(\Gamma \times \Sigma)$-equivariant skew product on $\Gamma \times X$

$$\dot{\gamma} = \gamma\xi(x), \quad \dot{x} = h(x), \tag{3.1}$$

where $\xi : X \to L\Gamma$, $h : X \to X$ satisfy the Σ-equivariance conditions

$$\xi(\sigma x) = \mathrm{Ad}_\sigma \xi(x) = \sigma\xi(x)\sigma^{-1}, \quad h(\sigma x) = \sigma h(x),$$

and $h(x_0) = 0$.

Thus, dynamics near the relative equilibrium $\Gamma x_0 \subset \mathbb{R}^n$ reduces to dynamics near the ordinary equilibrium $x_0 \in X$ for the Σ-equivariant vector $h : X \to X$, coupled with Γ drifts. In particular, the stability of Γx_0 is determined by $(dh)_{x_0}$.

3.3 Periodic solutions

A nonequilibrium solution $x(t)$ is periodic if $x(t+T) = x(t)$ for some $T > 0$. The least such T is the *(absolute) period*. The *spatial* symmetry group Δ is the isotropy subgroup of $x(t)$ for some, and hence all, $t \in \mathbb{R}$. The periodic solution $P = \{x(t) : 0 \le t < T\}$ lies inside $\operatorname{Fix}\Delta$. Define the *spatiotemporal* symmetry group $\Sigma = \{\gamma \in \Gamma : \gamma P = P\}$. Note that Δ is a normal subgroup of Σ and either $\Sigma/\Delta \cong S^1$ (P is a rotating wave) or $\Sigma/\Delta \cong \mathbb{Z}_q$ and P is called a *standing wave* or a *discrete rotating wave*. For each $\sigma \in \Sigma$, there exists $T_\sigma \in [0, T)$ such that $\sigma x(t) = x(t + T_\sigma)$. The *relative period* of $x(t)$ is the least $T > 0$ such that $x(T) \in \Sigma x_0$.

If $\dim\Sigma = \dim\Gamma$, then generically P is hyperbolic, hence isolated, the stability of P is determined by its Floquet exponents, and P persists under perturbation as a periodic solution with spatial symmetry Δ and spatiotemporal symmetry Σ. For Γ infinite and $N(\Delta)/\Delta$ finite, generically P is isolated in $\mathrm{Fix}\,\Delta$ and the neutral Floquet exponent has multiplicity $\dim\Gamma - \dim\Sigma + 1$.

3.4 Relative periodic solutions

A solution $x(t)$ is a *relative periodic solution* if it is not a relative equilibrium and $x(T) \in \Gamma x(0)$ for some $T > 0$. The least such T is the *relative period*. The spatial symmetry group $\Delta = \Sigma_{x(t)}$ for some, hence all, t. The spatiotemporal symmetry group Σ is the closed subgroup of Γ generated by Δ and σ, where $x(T) = \sigma x(0)$, and generically $\Sigma/\Delta \cong \mathbb{T}^d \times \mathbb{Z}_q$ is a maximal topologically cyclic (Cartan) subgroup of $N(\Delta)/\Delta$ containing $\sigma\Delta$. Then $x(t)$ is a $(d+1)$-dimensional quasiperiodic motion.

The dynamics near the relative periodic solution is again governed by a skew product. There exists $n \geq 1$ such that $\sigma^n = \exp(n\xi)$ where $\xi \in LZ(\Sigma)$ and $Z(\Sigma) \subset \Gamma$ is the centralizer of Σ. Define $\alpha = \exp(-\xi)\sigma$. Form a semidirect product $\Delta \rtimes \mathbb{Z}_{2n}$ by adjoining to Δ an element Q of order $2n$ such that $Q\delta Q^{-1} = \sigma\delta\sigma^{-1}$ for $\delta \in \Delta$.

In a comoving frame with velocity ξ, a neighborhood of the relative periodic orbit is Γ-equivariantly diffeomorphic to $(\Gamma \times X \times S^1)/\Delta \rtimes \mathbb{Z}_{2n}$ where X is a $\Delta \rtimes \mathbb{Z}_{2n}$-invariant cross-section, $S^1 = \mathbb{R}/2n\mathbb{Z}$ and $\Delta \rtimes \mathbb{Z}_{2n}$ acts on $\Gamma \times X \times S^1$ as

$$\delta \cdot (\gamma, x, \theta) = (\gamma\delta^{-1}, \delta x, \theta), \quad Q \cdot (\gamma, x, \theta) = (\gamma\alpha^{-1}, Qx, \theta + 1).$$

The Γ-equivariant ODE on $(\Gamma \times X \times S^1)/\Delta \rtimes \mathbb{Z}_{2n}$ lifts to a $\Gamma \times (\Delta \rtimes \mathbb{Z}_{2n})$-equivariant skew product

$$\dot\gamma = \gamma\xi(x,\theta), \quad \dot x = h(x,\theta), \quad \dot\theta = 1, \tag{3.2}$$

where $\xi : X \times S^1 \to L\Gamma$, $h : X \times S^1 \to X$ satisfy appropriate $\Delta \rtimes \mathbb{Z}_{2n}$-equivariance conditions.

3.5 Robust heteroclinic cycles

Heteroclinic cycles, degenerate in systems without symmetry, arise robustly in equivariant systems. Let $x_1, \ldots, x_m \in \mathbb{R}^n$ be saddles with $W^u(x_i) - \{x_i\} \subset \Gamma W^s(x_{i+1})$ (where $m+1=1$). If $\Sigma_1, \ldots, \Sigma_m \subset \Gamma$ are isotropy subgroups, $W^u(x_i) \subset \mathrm{Fix}\,\Sigma_i$, and x_{i+1} is a sink in $\mathrm{Fix}\,\Sigma_i$, then saddle-sink connections from x_i to x_{i+1} persist for nearby Γ-equivariant flows. The union $\bigcup_{i=1}^m \Gamma W^u(x_i)$ forms a robust heteroclinic cycle. (See Subsection 4.6 for an example.) Such cycles, when asymptotically stable, are a mechanism for intermittency or bursting, notably in rotating Rayleigh-Bénard convection (where rolls disappear and reorient themselves at approximately 60°), and provide a possible intrinsic explanation for irregular reversals of the Earth's magnetic field.

Asymmetric perturbations (deterministic or noisy) destroy the cycles, but the perturbed attractors inherit the bursting behavior.

Establishing the existence of heteroclinic connections is often straightforward when $\dim \mathrm{Fix}\,\Sigma_i = 2$ and nontrivial with $\dim \mathrm{Fix}\,\Sigma_i \geq 3$. Criteria for asymptotic stability of heteroclinic cycles are given in terms of real parts of eigenvalues of $(df)_{x_i}$, and depend on the geometry of the representation of Γ.

Robust cycles exist also between more complicated dynamical states such as periodic solutions or chaotic sets (*cycling chaos*). When $W^u(x_i)$ connects to two or more distinct states, the collection of unstable manifolds forms a *heteroclinic network* leading to competition between various subnetworks.

3.6 Symmetric attractors

Suppose that Γ is a finite group acting linearly on $\mathbb{R}^n$. A closed subset $A \subset \mathbb{R}^n$ has symmetry groups $\Delta = \{\gamma \in \Gamma : \gamma x = x \text{ for all } x \in A\}$, $\Sigma = \{\gamma \in \Gamma : \gamma A = A\}$. Here, Δ is an isotropy subgroup and $\Delta \subset \Sigma \subset N(\Delta)$. In applications, Δ corresponds to *instantaneous symmetry* and Σ to *symmetry on average.*

If A is an attractor (a Lyapunov stable ω-limit set) for a Γ-equivariant vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, then Σ fixes a connected component of $\mathrm{Fix}\,\Delta - L$ where L is the union of proper fixed-point spaces in $\mathrm{Fix}\,\Delta$.

Provided $\dim \mathrm{Fix}\,\Delta \geq 3$, all pairs Δ, Σ satisfying the above restrictions arise as symmetry groups of a nonperiodic attractor A. If $\dim \mathrm{Fix}\,\Delta \geq 5$, then A is realized by a uniformly hyperbolic (Axiom A) attractor.

If $\dim \mathrm{Fix}\,\Delta \geq 3$ and Σ fixes a connected component of $\mathrm{Fix}\,\Delta - L$, then A is realized by a periodic sink provided Σ/Δ is cyclic. If $\dim \mathrm{Fix}\,\Delta = 2$, then in addition either $\Sigma = \Delta$ or $\Sigma = N(\Delta)$.

Suppose A is an attractor and $\gamma \in \Gamma - \Sigma$. Then $\gamma A \cap A = \emptyset$. Varying a parameter, A may undergo a *symmetry-increasing bifurcation*: A grows until it collides with γA producing a larger attractor with symmetry on average generated by Σ and γ.

Determining symmetries of an attractor by inspection is often infeasible. A *detective* is a Γ-equivariant polynomial $\phi : \mathbb{R}^n \to V$ where every subgroup of Γ is an isotropy subgroup for the action on V, and each component of ϕ is nonzero. Suppose that $A \subset \mathbb{R}^n$ is an attractor with physical (SRB) measure μ. By ergodicity, the time average $\psi_A = \lim_{T\to\infty} \frac{1}{T}\int_0^T \phi(x(t))dt \in V$ is well-defined for almost every trajectory $x(t)$ in $\mathrm{supp}\,\mu$. Generically, $\Sigma_{\psi_A} = \Sigma_A$ so computing the symmetry of A reduces to computing the symmetry of a point.

If Γ is an infinite compact Lie group, and A is an ω-limit set containing points of trivial isotropy, then A cannot be uniformly hyperbolic. Hence *partially hyperbolic flows* arise naturally in systems with continuous symmetry. Consider the skew product (3.1) where $\Sigma = \mathbf{1}$ and $h : X \to X$ possesses a hyperbolic basic set $\Lambda \subset X$ with equilibrium measure μ (for a Hölder potential). Let ν denote Haar measure on Γ. Then $\Lambda \times \Gamma$ is partially hyperbolic, and $\mu \times \nu$ is ergodic (even Bernoulli) for an open dense set of equivariant flows. Such *stably ergodic* flows possess strong statistical properties (rapid decay of correlations, central limit theorem); a possible explanation for *hypermeander* (Brownian-like motion) of spiral waves in planar excitable media.

3.7 Forced symmetry breaking

In applications, symmetry is not perfect and account should be taken of Γ'-equivariant perturbations of (1.1) for Γ' a subgroup of Γ (including $\Gamma' = \mathbf{1}$). This topic is not discussed in this article, except in Subsections 3.5 and 4.5.

4 Equivariant bifurcation theory

Consider families of ODEs $\dot{x} = f(x, \lambda)$, with bifurcation parameter $\lambda \in \mathbb{R}$ and vector field $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ satisfying $f(0, 0) = 0$ and the Γ-equivariance condition

$$f(\gamma x, \lambda) = \gamma f(x, \lambda) \text{ for all } x \in \mathbb{R}^n,\ \lambda \in \mathbb{R},\ \gamma \in \Gamma.$$

A local bifurcation from the equilibrium $x = 0$ occurs if $(df)_{0,0}$ is nonhyperbolic. The center subspace E^c is the sum of generalized eigenspaces corresponding to eigenvalues on the imaginary axis, and is Γ-invariant. By center manifold theory, local dynamics ((x, λ) near $(0, 0)$) are captured by the center manifold W^c. After center manifold reduction (or Lyapunov-Schmidt reduction if the focus is on equilibria), it may be assumed that $\mathbb{R}^n = E^c$.

If $(df)_{0,0}$ possesses zero eigenvalues, then there is a *steady-state bifurcation*. Generically, $(df)_{0,0} = 0$ and E^c is absolutely irreducible. There are two subcases.

If Γ acts trivially on $\mathbb{R}^n$, then $n = 1$ and generically there is a *saddle-node* (or *limit point*) bifurcation where the zero sets of $f(x, \lambda)$ and $\pm x^2 \pm \lambda$ are diffeomorphic for (x, λ) near $(0, 0)$. Higher order degeneracies can be treated using singularity theory. The equilibria and their stability determines the local dynamics. All bifurcating equilibria have isotropy Γ, so there is no symmetry-breaking.

From now on, consider the remaining subcase where Γ acts absolutely irreducibly and nontrivially on $\mathbb{R}^n$. Then $\mathrm{Fix}\,\Gamma = \{0\}$, $f(0, \lambda) \equiv 0$, and $(df)_{0,\lambda} = c(\lambda) I_n$ where generically $c'(0) \neq 0$. Assume that $c'(0) > 0$, so the "trivial solution" $x = 0$ is asymptotically stable *subcritically* ($\lambda < 0$) and unstable *supercritically* ($\lambda > 0$). Bifurcating solutions lie outside $\mathrm{Fix}\,\Gamma$ and hence there is spontaneous symmetry-breaking.

4.1 Axial isotropy subgroups

The *Equivariant Branching Lemma* guarantees branches of equilibria with isotropy Σ for each axial isotropy subgroup. There are three associated branching patterns, see Figure 1.

<Figure 1 near here>

If $N(\Sigma)/\Sigma = \mathbb{Z}_2$, then f_Σ is odd. Generically $\partial_x^3 f_\Sigma(0,0) \neq 0$, since $(x_1^2 + \cdots + x_n^2)x$ is Γ-equivariant, and there are two branches of equilibria bifurcating supercritically or subcritically together, and lying on the same group orbit. The branches form a *symmetric pitchfork* whose direction of branching is determined by $\operatorname{sgn} \partial_x^3 f_\Sigma(0,0)$.

If $N(\Sigma)/\Sigma \cong \mathbf{1}$, then generically f_Σ is even. If all quadratic Γ-equivariant maps vanish on $\operatorname{Fix}\Sigma$, then the bifurcation is sub/supercritical depending on $\operatorname{sgn} \partial_x^3 f_\Sigma(0,0)$ but the branches lie on distinct group orbits. This is an *asymmetric pitchfork*.

If $\partial_x^2 f_\Sigma(0,0) \neq 0$, then the equilibria exist *transcritically*: for $\lambda < 0$ and $\lambda > 0$.

The natural actions of $\mathbb{D}_m$ on $\mathbb{R}^2$ are absolutely irreducible. The axial branches are symmetric pitchforks for $m \geq 4$ even, asymmetric pitchforks for $m \geq 5$ odd, and transcritical for $m = 3$.

The actions of $\mathbb{D}_m$, $m \geq 5$ odd, provide the simplest instances of *hidden symmetries*, where certain $N(\Sigma)/\Sigma$-equivariant mappings on $\operatorname{Fix}\Sigma$ do not extend to smooth Γ-equivariant mappings on $\mathbb{R}^n$.

4.2 Nonaxial maximal isotropy subgroups

For Σ a real maximal isotropy subgroup, $\dim \operatorname{Fix}\Sigma$ odd, there exist branches of equilibria with isotropy Σ. When $\dim \operatorname{Fix}\Sigma$ is even, there are examples where equilibria exist and examples where no equilibria exist. For Σ complex or quaternionic, there exist branches of rotating waves with isotropy Σ. In the quaternionic case, the rotating waves foliate the $\mathbf{SU}(2)$ group orbits according to the Hopf fibration.

4.3 Submaximal isotropy subgroups

It has been conjectured falsely that steady-state bifurcation leads generically to equilibria only with maximal isotropy. The simplest counterexample is the 24 element group $\Gamma = \mathbb{Z}_3 \ltimes \mathbb{Z}_2^3$ generated by

$$\rho = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \kappa = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(Alternatively, $\Gamma = \mathbf{T} \oplus \mathbb{Z}_2(-I_3)$ where $\mathbf{T} \subset \mathbf{SO}(3)$ is the tetrahedral group.)

The isotropy subgroup $\Sigma = \mathbb{Z}_2(\kappa)$ has two-dimensional fixed-point subspace $\mathrm{Fix}\,\Sigma = \{(x, y, 0)\}$. The only one-dimensional fixed-point spaces contained in $\mathrm{Fix}\,\Sigma$ are the x- and y-axes. The general Γ-equivariant vector field is

$$\dot{x} = g(x^2, y^2, z^2, \lambda)x, \quad \dot{y} = g(y^2, z^2, x^2, \lambda)y, \quad \dot{z} = g(z^2, x^2, y^2, \lambda)z.$$

After scaling,

$$g(x^2, y^2, z^2, \lambda) = \lambda - x^2 - ay^2 - bz^2 + o(x^2, y^2, z^2, \lambda). \tag{4.1}$$

Restricting to $\mathrm{Fix}\,\Sigma$ and dividing out the axial solutions $x = 0$ and $y = 0$ yields at lowest order the equations $\lambda = x^2 + ay^2 = y^2 + bx^2$. Submaximal solutions exist provided $\mathrm{sgn}(a - 1) = \mathrm{sgn}(b - 1)$.

In general, the existence of equilibria with submaximal isotropy must be treated on a case-by-case basis (for each absolutely irreducible representation of Γ and isotropy subgroup Σ).

4.4 Asymptotic stability

Subcritical and axial transcritical branches are automatically unstable. Moreover, the existence of a quadratic Γ-equivariant mapping $q : \mathbb{R}^n \to \mathbb{R}^n$ and $x \in \mathrm{Fix}\,\Sigma$ such that $(dq)_x$ has eigenvalues with nonzero real part guarantees that branches of equilibria with axial isotropy Σ are generically unstable (even when $q|_{\mathrm{Fix}\,\Sigma} \equiv 0$).

There are no general results for asymptotic stability, and calculations must be done on a case-by-case basis. (The remarks in Subsection 3.1 are useful here.)

4.5 Branching patterns and finite determinacy

The following notion of finite determinacy is based on equivariant transversality theory. Assume Γ acts absolutely irreducibly. Consider the set $\mathcal{F}$ of Γ-equivariant vector fields $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ satisfying $(df)_{0,0} = 0$. For an open dense subset of $\mathcal{F}$, branches of relative equilibria near $(0, 0)$ are normally hyperbolic. The collection of branches of relative equilibria, together with their isotropy type, direction of branching, and stability properties, is called a *branching pattern*. These persist under small perturbations and are *finitely determined*: there exists $q = q_\Gamma \geq 2$, and an open dense subset $\mathcal{U}(q) \subset \mathcal{F}$ such that the branching patterns of f and $f + g$ are identical for $f \in \mathcal{U}(q)$, $g \in \mathcal{F}$, provided $g(x, \lambda) = o(\|x\|^q)$.

Furthermore, branching patterns are *strongly finitely determined*: there exists $d \geq 2$ and an open dense subset $\mathcal{S}(d) \subset \mathcal{F}$ such that the branching patterns of f and $f + g$ are identical for $f \in \mathcal{S}(d)$ and *all* (not necessarily equivariant) g satisfying $g(x, \lambda) = o(\|x\|^d)$.

For example, consider the hyperoctahedral group $S_n \ltimes \mathbb{Z}_2^n$, $n \geq 1$. Here S_n acts by permutations of the coordinates $(x_1, \ldots, x_n)$ and $\mathbb{Z}_2^n$ consists of diagonal matrices with entries ± 1. Let $\Gamma = T \ltimes \mathbb{Z}_2^n$ where $T \subset S_n$ is a transitive subgroup. Then Γ acts absolutely irreducibly on $\mathbb{R}^n$ and is strongly 3-determined. Submaximal branches of equilibria exist except when $T = S_n$, $T = A_n$ and, if $n = 6$, $T = \mathbf{PGL}_2(\mathbb{F}_5)$.

4.6 Dynamics

Absolutely irreducible representations have arbitrarily high dimension, so steady-state bifurcation leads to rich dynamics. The group $\Gamma = \mathbb{Z}_3 \ltimes \mathbb{Z}_2^3$ with $\operatorname{sgn}(a - 1) \neq (b - 1)$ and $a + b > 2$ in (4.1) yields asymptotically stable heteroclinic cycles with planar connections connecting equilibria in the x-, y- and z-axes. See Figure 2. In $\mathbb{R}^4$, there is the possibility of *instant chaos* where chaotic dynamics bifurcates directly from the equilibrium 0.

<Figure 2 near here>

In the absence of quadratic equivariants, the *invariant sphere theorem* gives an open set of equivariant vector fields for which an attracting normally hyperbolic flow-invariant $(n - 1)$-dimensional sphere bifurcates supercritically. This simplifies computations of nontrivial dynamics.

5 Hopf bifurcation and mode-interactions

5.1 Equivariant Hopf bifurcation

The setting is the same as in Section 4, except that $L = (df)_{0,0}$ has imaginary eigenvalues $\pm i\omega$ of algebraic and geometric multiplicity $n/2$. Generically, $\mathbb{R}^n = E^c$ is Γ-*simple*: either the direct sum of two isomorphic absolutely irreducible subspaces, or nonabsolutely irreducible.

By Birkhoff normal form theory (see below), for any $k \geq 1$ there is a Γ-equivariant change of coordinates after which $f(x, \lambda) = f_k(x, \lambda) + o(\|x\|^k)$ where f_k is $(\Gamma \times S^1)$-equivariant. Here $S^1 = \overline{\{\exp(tL) : t \in \mathbb{R}\}}$ acts freely on $\mathbb{R}^n$ and $\Gamma \times S^1$ acts *complex* irreducibly ($\mathcal{D} = \mathbb{C}$). Hence $\dim \mathrm{Fix}\, J$ is even for each isotropy subgroup $J \subset \Gamma \times S^1$, and $N(J)/J \cong S^1$ when J is maximal. The *Equivariant Hopf Theorem* guarantees, generically, branches of rotating waves with absolute period approximately $2\pi/\omega$ for each maximal isotropy subgroup J.

The notions of finite and strong finite determinacy extend to complex irreducible representations and the rotating waves persist as periodic solutions for the original Γ-equivariant vector field f. Define the spatial and spatiotemporal symmetry groups $\Delta \subset \Sigma \subset \Gamma$ as in Subsection 3.3. Then $J = \{(\sigma, \theta(\sigma)) : \sigma \in \Sigma\}$ is a *twisted subgroup*, with $\theta : \Sigma \to S^1$ a homomorphism and $\Delta = J \cap \Gamma = \ker \theta$.

In the non-symmetry-breaking case, where Γ acts trivially on $\mathbb{R}^2$, phase-amplitude reduction leads to $\mathbb{Z}_2$-equivariant amplitude equations on $\mathbb{R}$ and higher order degeneracies are amenable to $\mathbb{Z}_2$-equivariant singularity theory. Similar comments apply to $\mathbf{O}(2)$-equivariant Hopf bifurcation where the amplitude equations are $\mathbb{D}_4$-equivariant. The technique fails for general groups Γ.

5.2 Mode-interactions and Birkhoff normal form

Steady-state and Hopf bifurcations are *codimension one*: occurring generically in one-parameter families of Γ-equivariant vector fields. Multiparameter families may undergo higher codimension bifurcations called *mode-interactions*. Suppressing parameters, steady-state/steady-state bifurcation occurs when $\mathbb{R}^n = E^c = V_1 \oplus V_2$ where

V_1 and V_2 are absolutely irreducible and $L = (df)_0$ has zero eigenvalues. If V_1 and V_2 are nonisomorphic then $L = 0$, otherwise L is nilpotent and there is an equivariant *Takens-Bogdanov* bifurcation. Similarly, there are codimension two steady-state/Hopf and Hopf/Hopf bifurcations.

Write $L = S + N$ (uniquely) where S is semisimple, N is nilpotent, and $SN = NS$. Then $\overline{\{\exp tS : t \in \mathbb{R}\}}$ is a torus $\mathbb{T}^p$ where $p \geq 0$ is the number of rationally independent eigenvalues for L.

For each $k \geq 1$, there is a Γ-equivariant degree k polynomial change of coordinates $P : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfying $P(0) = 0$, $(dP)_0 = I$ transforming f to *Birkhoff normal form* $f_k + o(\|x\|^k)$ where f_k is $(\Gamma \times \mathbb{T}^p)$-equivariant.

If $N \neq 0$, then $\overline{\{\exp tN^T : t \in \mathbb{R}\}} \cong \mathbb{R}$ and f_k can be chosen so that the nonlinear terms are $(\Gamma \times \mathbb{T}^p \times \mathbb{R})$-equivariant. The linear terms are *not* $\mathbb{R}$-equivariant.

The study of mode-interactions proceeds by first analyzing $(\Gamma \times \mathbb{T}^p)$-equivariant normal forms, then considering exponentially small effects of the Γ-equivariant tail. Versions of the equivariant branching lemma and equivariant Hopf theorem establish existence of certain solutions. There are numerous examples of robust heteroclinic cycles connecting (relative) equilibria and periodic solutions, symmetric chaos, and symmetry-increasing bifurcations.

6 Bifurcations from relative equilibria and periodic solutions

Using the skew product (3.1), bifurcations from a relative equilibrium with isotropy Σ for a Γ-equivariant vector field reduce to bifurcations from a fully symmetric equilibrium for a Σ-equivariant vector field h coupled with Γ drifts. If h possesses (relative) equilibria or periodic solutions, then the drift is determined generically as in Subsections 3.2 and 3.4. Nevertheless, solving the drift equation can be useful for understanding behavior in physical space. This is facilitated by making equivariant polynomial changes of coordinates $(\gamma Q(x), P(x))$ putting h into Birkhoff normal form and simplifying ξ.

Bifurcations from (relative) periodic solutions also reduce, mainly, to bifurcations from equilibria (with enlarged symmetry group). By Subsection 3.4, it suffices to consider bifurcations from isolated periodic solutions $P = \{x(t)\}$ with spatial symmetry Δ and spatiotemporal symmetry Σ. Write $x(T) = \sigma x(0)$ where T is the relative period and σ is chosen so that the automorphism $\delta \mapsto \sigma^{-1}\delta\sigma$, $\delta \in \Delta$, has finite order k. Form the semidirect product $\Delta \rtimes \mathbb{Z}_{2k}$ by adjoining to Δ an element τ of order $2k$ such that $\tau^{-1}\delta\tau = \sigma^{-1}\delta\sigma$, for $\delta \in \Delta$. Codimension one bifurcations from P are in one-to-one correspondence (modulo tail terms) with bifurcations from fully symmetric equilibria for a $(\Delta \rtimes \mathbb{Z}_{2k})$-equivariant vector field. In particular, period-preserving and period-doubling bifurcations from P reduce to steady-state bifurcations, and Naimark-Sacker bifurcations reduce to Hopf bifurcations. This framework incorporates issues such as *suppression of period-doubling*. Similar results hold for higher codimension bifurcations.

The skew products (3.1) and (3.2) are valid for proper actions of certain noncompact Lie groups Γ provided the spatial symmetries are compact, leading to explanations of spiral and scroll wave phenomena in excitable media.

When the spatial symmetry group is noncompact, E^c may be infinite-dimensional and center manifold reduction may break down due to continuous spectrum issues. For Euclidean symmetry, there is a theory of modulation or *Ginzburg-Landau* equations.

Further Reading

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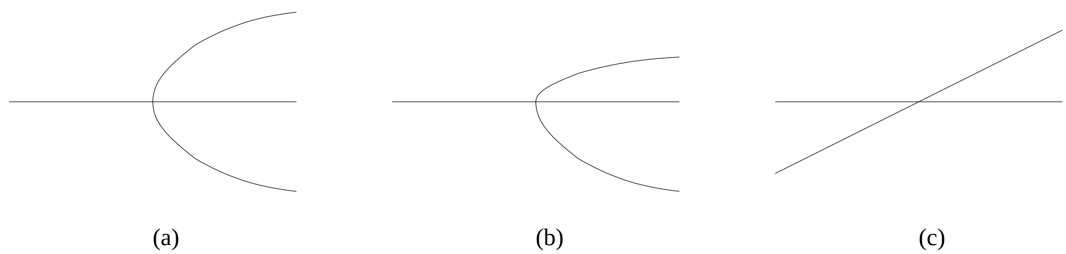

Figure 1: Axial branches: (a) supercritical symmetric pitchfork, (b) supercritical asymmetric pitchfork, (c) transcritical branches.

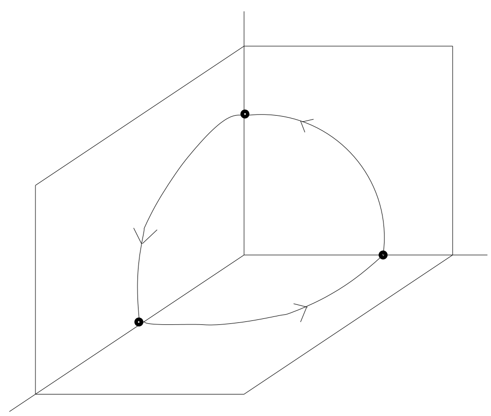

Figure 2: Robust heteroclinic cycle for the group $\Gamma = \mathbb{Z}_3 \ltimes \mathbb{Z}_2^3$.