Nonstandard functional central limit theorem for nonuniformly hyperbolic dynamical systems, including Bunimovich stadia

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Abstract

We consider a class of nonuniformly hyperbolic dynamical systems with a first return time satisfying a central limit theorem (CLT) with nonstandard normalisation $(n \log n)^{1/2}$. For such systems (both maps and flows) we show that it automatically follows that the functional central limit theorem or weak invariance principle (WIP) with normalisation $(n \log n)^{1/2}$ holds for Hölder observables.

Our approach streamlines certain arguments in the literature. Applications include various examples from billiards, geodesic flows and intermittent dynamical systems. In this way, we unify existing results as well as obtaining new results. In particular, we deduce the WIP with nonstandard normalisation for Bunimovich stadia as an immediate consequence of the corresponding CLT proved by Bálint & Gouëzel.

1 Introduction

Let (X, μ) be a probability space, $f : X \to X$ a measurable map, and $v : X \to \mathbb{R}$ an integrable observable. Let $a_n = n^{1/2}$ or $a_n = (n \log n)^{1/2}$. Define

$$W_n(t) = a_n^{-1} \sum_{j=0}^{nt-1} (v - \textstyle\int_X v \, d\mu) \circ f^j, \quad t = 0, \tfrac{1}{n}, \tfrac{2}{n}, \ldots, 1, \tag{1.1}$$

and linearly interpolate to obtain $W_n \in C[0,1]$. Let W denote unit Brownian motion. We say that v *satisfies a weak invariance principle (WIP) with variance* σ^2 *and*

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normalisation a_n if $W_n \to_\mu \sigma W$ in $C[0,1]$ as $n \to \infty$.[1] It is the *standard WIP* if $a_n = n^{1/2}$ and *nonstandard WIP* if $a_n = (n \log n)^{1/2}$. If $\sigma = 0$, the WIP is called *degenerate*.

Similarly, we say that v *satisfies a standard/nonstandard central limit theorem (CLT) with variance* σ^2 if $a_n^{-1} \sum_{j=0}^{n-1}(v - \int_X v\, d\mu) \circ f^j \to_\mu N(0,\sigma^2)$ as $n \to \infty$.

We note that the WIP is also known as the *functional central limit theorem*.

The standard CLT is well-known for a wide class of nonuniformly hyperbolic dynamical systems, going back to work of [37, 38] for Hölder observables of Anosov and Axiom A diffeomorphisms and flows. The standard CLT also holds for nonuniformly expanding/hyperbolic systems with summable decay of correlations modelled by Young towers [40, 41]. In such examples, the standard WIP also holds.

There are numerous examples, especially intermittent maps [36] and examples from dispersing billiards as mentioned below, where correlations decay at rate $1/n$ and the CLT still holds but with the nonstandard normalisation $(n \log n)^{1/2}$. In most, but not all, of these examples, the nonstandard WIP has also been shown to hold. An exception is the Bunimovich stadium [9], where the nonstandard CLT was proved by [5] but the nonstandard WIP was not previously proved.

Example 1.1 (Bunimovich stadia [9]) As described in more detail in Section 7, these are billiards in a convex domain enclosed by two semicircles and two parallel line segments tangent to the semicircles. We consider (dynamically) Hölder observables v, and let I_v denote the average of v over trajectories bouncing perpendicular to the straight edges. It was shown by Bálint & Gouëzel [5] that v satisfies a nonstandard CLT for $I_v \neq 0$ and a standard CLT for $I_v = 0$.

A consequence of the results in this paper is that for Bunimovich stadia the nonstandard WIP holds for (dynamically) Hölder observables v with $I_v \neq 0$. The analogous result for the billiard flow is also shown to hold.

In this paper, we give a unifying approach for nonuniformly hyperbolic systems modelled by Young towers, showing how to establish the nonstandard WIP as a consequence of the nonstandard CLT. In addition to the Bunimovich stadium example, we also prove the nonstandard WIP for a family of multidimensional nonuniformly expanding nonMarkovian nonconformal intermittent maps [17]. Existing examples that we recover include one-dimensional intermittent maps [16] and billiards with cusps [4].

This work was motivated by our study of certain geodesic flows on surfaces with nonpositive curvature [28]. For these examples, the nonstandard CLT follows from a general approach initiated by [4] in their study of billiards with cusps. In contrast, their proof of the nonstandard WIP relies on additional *ad hoc* arguments. We show here that the nonstandard WIP in [4] is immediate in such situations given the

[1]We write $\to_\mu$ to denote weak convergence with respect to a specific probability measure μ on the left-hand side. So $A_n \to_\mu A$ means that A_n is a family of random variables on a probability space (X,μ) and $A_n \to_w A$.

nonstandard CLT. In particular, the example-specific details in [4, Section 8] can be dispensed with. Hence we are able to prove the nonstandard WIP for the geodesic flow example in [28]. Moreover, our method is somewhat independent of how the nonstandard CLT is proved and hence we are able to cover the much more difficult situation of Bunimovich stadia.[2]

The remainder of this paper is organised as follows. Sections 2 and 3 are preliminary in nature, summarising how to induce limit laws for maps and flows and establishing a nonstandard WIP for Gibbs-Markov maps. In Section 4, we state and prove our main result on nonstandard limit laws for nonuniformly hyperbolic systems modelled by Young towers. Results for flows are given in Section 5. Sections 6 and 7 contain the applications to intermittent maps and billiards (while the discussion of geodesic flows on certain nonpositively curved surfaces is left to [28]).

Notation We use "big O" and $\ll$ notation interchangeably, writing $a_n = O(b_n)$ or $a_n \ll b_n$ if there are constants $C > 0$, $n_0 \geq 1$ such that $a_n \leq C b_n$ for all $n \geq n_0$. We write $a_n \approx b_n$ if $a_n \ll b_n$ and $b_n \ll a_n$. As usual, $a_n = o(b_n)$ means that $a_n/b_n \to 0$ and $a_n \sim b_n$ means that $a_n/b_n \to 1$. We denote the integer part of x by $[x]$.

2 Inducing statistical limit laws

In this section, we establish results for inducing the CLT/WIP with standard/nonstandard normalisation for discrete/continuous time. Many results of this type already exist in the literature [5, 10, 20, 22, 27, 32, 33, 34, 37, 44] but are formulated for slightly different situations.

We begin with an inducing theorem for flows in Section 2.1 before covering the simpler situation for maps in Section 2.2. In Section 2.3, we discuss how to verify certain hypotheses.

2.1 Inducing for flows

Let $f : X \to X$ be an ergodic measure-preserving transformation on a probability space (X, μ) and let $r : X \to \mathbb{R}^+$ be an integrable roof function. Define the suspension

$$X^r = \{(x, u) \in X \times [0, \infty) : 0 \leq u \leq r(x)\}/\sim, \qquad (x, r(x)) \sim (fx, 0),$$

and the suspension flow $g_t(x, u) = (x, u + t)$ computed modulo identifications. A g_t-invariant probability measure is given by $\mu^r = (\mu \times \text{Lebesgue})/\bar{r}$ where $\bar{r} = \int_X r\, d\mu$.

Let $v : X^r \to \mathbb{R}$ be an integrable observable with $\int_{X^r} v\, d\mu^r = 0$. Define $v^X : X \to \mathbb{R}$ and $Q : X^r \to \mathbb{R}$ by

$$v^X(x) = \int_0^{r(x)} v(x, u)\, du, \qquad Q(x, u) = \int_0^u v(x, s)\, ds,$$

[2]These extra difficulties are still present in the proof of the nonstandard CLT [5] but, by the method presented here, play no further role when passing to the nonstandard WIP.

as well as the Birkhoff integrals/sums

$$v_t = \int_0^t v \circ g_s \, ds : X^r \to \mathbb{R}, \qquad v_n^X = \sum_{j=0}^{n-1} v^X \circ f^j : X \to \mathbb{R}.$$

Also, define processes $W_n \in C[0,\infty)$ on X^r and $W_n^X \in D[0,\infty)$ on X, setting

$$W_n(t) = a_n^{-1} v_{nt} \,, \qquad W_n^X(t) = a_n^{-1} v_{[nt]}^X \,,$$

where $a_n > 0$ is a sequence satisfying $\lim_{n\to\infty} a_n = \infty$ and such that $\sup_{n\geq 1} a_{[\lambda n]}/a_a < \infty$ for all $\lambda > 0$. As in the introduction (Section 1), we say that v^X satisfies a WIP with variance σ^2 and normalisation a_n if $W_n^X \to_\mu \sigma W$. Similarly, we say that v satisfies a WIP with variance σ^2 and normalisation a_n if $W_n \to_{\mu^r} \sigma W$.

Theorem 2.1 *Suppose that:*

(I1) *v^X satisfies a WIP with variance σ^2 and normalisation a_n on (X, μ), and*

(I2) *$a_n^{-1} \max_{0\leq j\leq n} |v^X| \circ f^j \to_\mu 0$ as $n \to \infty$.*

Then v satisfies a CLT with variance $\bar{r}^{-1}\sigma^2$ and normalisation a_n on (X^r, μ^r). If moreover

(I3) *$a_n^{-1} \sup_{t\in[0,n]} |Q| \circ g_t \to_{\mu^r} 0$ as $n \to \infty$,*

then v satisfies a WIP with variance $\bar{r}^{-1}\sigma^2$ and normalisation a_n on (X^r, μ^r).

Remark 2.2 Condition (I3) provides control during excursions in X^r from X.

The remainder of this section is devoted to the proof of Theorem 2.1. It is convenient to work with the Skorohod spaces $D[0,T]$ and $D[0,\infty)$ of real-valued càdlàg functions (right-continuous $\psi(t^+) = \psi(t)$ with left-hand limits $\psi(t^-)$) on the respective interval, with the sup-norm topology in the case of $D[0,T]$ and the topology of uniform convergence on compact subsets in the case of $D[0,\infty)$. (Alternatively, one could work with the space of continuous functions, replacing certain piecewise constant functions by piecewise linear interpolants throughout.)

Proposition 2.3 *Assume that condition* (I2) *holds. Then*

$$\sup_{t\in[0,T]} |W_n^X(t) \circ f - W_n^X(t)| \to_\mu 0 \quad \textit{for all } T > 0.$$

Proof Note that $W_n^X(t)\circ f - W_n^X(t) = a_n^{-1}(v^X\circ f^{[nt]} - v^X)$. Hence

$$\sup_{t\in[0,T]} |W_n^X(t)\circ f - W_n^X(t)| \le 2a_n^{-1} \max_{0\le j\le [nT]} |v^X|\circ f^j \to_\mu 0$$

where we used (I2) and that $a_{[nT]}/a_n$ is bounded. ∎

Define the *lap numbers* $N_t = N_t(x,u) = \max\{n\ge 0 : \sum_{j=0}^{n-1} r(f^j x) \le u+t\}$ on X^r for $t\ge 0$. Set

$$\psi_n(t) = \frac{N_{nt}}{n} : X^r \to \mathbb{R}\,, \qquad \bar\psi(t) = \frac{t}{\bar r} \in \mathbb{R}.$$

Also, define the processes $\widehat{W_n^X} \in D[0,\infty)$ on X^r by setting $\widehat{W_n^X}(x,u) = W_n^X(x)$.

Proposition 2.4 *Assume that conditions* (I1) *and* (I2) *hold. Then* $\widehat{W_n^X}\circ\psi_n \to_{\mu^r} \bar r^{-1/2}\sigma W$ *in* $D[0,\infty)$.

Proof First, we show that $\widehat{W_n^X} \to_{\mu^r} \sigma W$. By condition (I1), $W_n^X \to_\mu \sigma W$. Define the absolutely continuous probability measure $\widehat\mu$ on X by $d\widehat\mu/d\mu = \bar r^{-1} r$. By the ergodicity of μ, Proposition 2.3 and [45, Theorem 1], we can pass weak convergence of W_n^X from μ to $\widehat\mu$ yielding that $W_n^X \to_{\widehat\mu} \sigma W$. But

$$\mu^r(\widehat{W_n^X}\in E) = \bar r^{-1}\int_X r 1_{\{W_n^X\in E\}}\, d\mu = \widehat\mu(W_n^X\in E)$$

for all Borel sets $E\subset D[0,\infty)$, so $\widehat{W_n^X}\to_{\mu^r}\sigma W$.

Second, it follows from the definition of the lap number and the ergodicity of μ that $\lim_{t\to\infty} N_t(x,u)/t = 1/\bar r$ for μ-a.e. x and every u. Hence

$$\psi_n(t)(x,u) = N_{nt}(x,u)/n = tN_{nt}(x,u)/(nt) \to t/\bar r = \bar\psi(t)$$

for μ-a.e. x and every u, t as $n\to\infty$. It follows that $\sup_{t\in[0,T]}|\psi_n(t) - \bar\psi(t)| \to 0$ μ^r-a.e. Hence[3] $(\widehat{W_n^X}, \psi_n) \to_{\mu^r} (\sigma W, \bar\psi)$. By the continuous mapping theorem,

$$\widehat{W_n^X}\circ\psi_n \to_{\mu^r} \sigma W\circ\bar\psi = \bar r^{-1/2}\sigma W,$$

as required. ∎

Proof of Theorem 2.1 By the definition of lap number, we have the decomposition

$$v_t(x,u) = v^X_{N_t(x,u)}(x) + Q(g_t(x,u)) - Q(x,u).$$

[3] There is a technical issue since the sup-norm topology on càdlàg spaces is not separable, but this is easily resolved as in [20, Proposition A.4].

Hence,

$$W_n(t)(x,u) = a_n^{-1}\big(v^X_{N_{nt}(x,u)}(x) + Q(g_{nt}(x,u)) - Q(x,u)\big).$$

But $a_n^{-1} v^X_{N_{nt}(x,u)}(x) = \widehat{W_n^X}(x,u) \circ \psi_n(t)(x,u)$, so

$$W_n = \widehat{W_n^X} \circ \psi_n + F_n \quad \text{where} \quad F_n(t) = a_n^{-1}(Q \circ g_{nt} - Q). \tag{2.1}$$

Clearly, $F_n(1) \to_{\mu^r} 0$. Also, if condition (I3) holds, then $\sup_{t\in[0,T]} |F_n(t)| \to_{\mu^r} 0$ for all $T > 0$. Hence, the result follows from (2.1) and Proposition 2.4. ∎

2.2 Inducing for maps

A similar result holds for discrete suspensions (towers). Let (F, X, μ) be an ergodic measure-preserving transformation and $r : X \to \mathbb{Z}^+$ an integrable function. Define the tower $X^r = \{(x, \ell) \in X \times \mathbb{Z} : 0 \le \ell \le r(x)\}$ and tower map

$$f : X^r \to X^r, \qquad f(x,\ell) = \begin{cases} (x, \ell+1), & 0 \le \ell < r(x) - 1 \\ (Fx, 0), & \ell = r(x) - 1 \end{cases},$$

with ergodic invariant probability measure $\mu^r = (\mu \times \text{counting})/\bar r$ where $\bar r = \int_X r\, d\mu$.

Let $v : X^r \to \mathbb{R}$ be an integrable observable with $\int_{X^r} v\, d\mu^r = 0$. Define $v^X : X \to \mathbb{R}$ and $Q : X^r \to \mathbb{R}$ by

$$v^X(x) = \sum_{\ell=0}^{r(x)-1} v(x,\ell), \qquad Q(x,\ell) = \sum_{k=0}^{\ell} v(x,k).$$

Theorem 2.5 *The statement of Theorem* 2.1 *holds true in this context with condition* (I3) *taking the form*

(I3) $a_n^{-1} \max_{0 \le j \le n} |Q| \circ f^j \to_{\mu^r} 0$ *as* $n \to \infty$.

Proof The proof is identical to that for Theorem 2.1 with the obvious modifications (sums in place of integrals, etc). ∎

2.3 Verification of hypotheses

In the situation of maps in Subsection 2.2, condition (I3) simplifies considerably, as we now explain. Define $\widetilde{v}^X : X \to \mathbb{R}$ by

$$\widetilde{v}^X(x) = \max_{0 \le \ell \le r(x)} \Big| \sum_{k=0}^{\ell-1} v(x,k) \Big|.$$

Proposition 2.6 *Suppose that*

$$a_n^{-1} \max_{0\le j\le n} \widetilde{v}^X \circ F^j \to_\mu 0 \quad \text{as } n\to\infty. \tag{2.2}$$

Then (I2) *and* (I3) *hold.*

Proof Clearly, (2.2) implies (I2), so we focus on (I3). Note that $|Q(x,\ell)| \le \widetilde{v}^X(x)$ for all $(x,\ell)\in X^r$. Also, for any $(x,\ell)\in X^r$, $n\ge 1$, we can write $f^n(x,\ell)=(x',\ell')\in X^r$ where $x'=F^{n'}x$ for some $n'\le n$. Hence

$$a_n^{-1}\max_{j\le n}|Q\circ f^j| \le a_n^{-1}\max_{j\le n}\widetilde{v}^X\circ F^j. \tag{2.3}$$

Let $z_n = a_n^{-1}\max_{0\le j\le n}\widetilde{v}^X\circ F^j$ and define $\hat{z}_n(x,u)=z_n(x)$. Then for $\epsilon>0$, $K>0$:

$$\begin{aligned}\mu^r\Big(a_n^{-1}\max_{0\le j\le n}|Q\circ f^j|>\epsilon\Big) &\le \mu^r(\hat{z}_n>\epsilon) = \bar{r}^{-1}\int_X r1_{\{z_n>\epsilon\}}\,d\mu \\ &\le \bar{r}^{-1}K\mu(z_n>\epsilon)+\bar{r}^{-1}\mu(r>K).\end{aligned}$$

Since $r\in L^1(X)$ and $z_n\to_\mu 0$ by (2.2), condition (I3) then follows. ∎

Remark 2.7 Similarly, for flows we can define $\widetilde{v}^X : X\to\mathbb{R}$ by

$$\widetilde{v}^X(x) = \max_{0\le u\le r(x)}\Big|\int_0^u v(x,s)\,ds\Big|.$$

Suppose that

$$a_n^{-1}\max_{0\le j\le n}\widetilde{v}^X\circ f^j \to_\mu 0 \quad \text{as } n\to\infty.$$

Then certainly (I2) holds. Condition (I3) also holds provided $\inf(r)>0$. This extra condition is required for the step (2.3) in the proof of Proposition 2.6. It ensures that for any $(x,u)\in X^r$, $t>0$, we can write $g_t(x,u)=(x',u')\in X^r$ where $x'=f^nx$ for some $n\le 1+t/\inf(r)$.

Suppose that $a_n=n^{1/2}$. If v^X (resp. $\widetilde{v}^X$) is in $L^2(X)$, then condition (I2) (resp. (2.2)) is automatically satisfied. The next result is useful for verifying (I2) and (2.2) when $a_n=(n\log n)^{1/2}$.

Proposition 2.8 *Let $V : X\to\mathbb{R}$ be a measurable function satisfying $\mu(|V|>n) = O(n^{-2})$. Then $(n\log n)^{-1/2}\max_{0\le j\le n}|V|\circ F^j\to_\mu 0$ as $n\to\infty$.*

Proof Let $a_n=(n\log n)^{1/2}$, $q_n=(n\log\log n)^{1/2}$, and define

$$E_n=\{x\in X : |V(F^jx)|>q_n \text{ for some } 0\le j\le n\}. \tag{2.4}$$

Then

$$\mu(E_n) \le \sum_{j=0}^{n} \mu(|V \circ F^j| > q_n) = (n+1)\mu(|V| > q_n) \ll nq_n^{-2} = (\log\log n)^{-1}.$$

Setting $V_n = V1_{\{|V|\le q_n\}}$, we have $V_n \circ F^j = V \circ F^j$ for all $j \le n$ on E_n^c and hence it suffices to show that $a_n^{-1} \max_{0\le j\le n} |V_n| \circ F^j \to_\mu 0$. Now,

$$\int_X V_n^4 \, d\mu \le \sum_{k\le q_n+1} k^4 \mu(k-1 < |V| \le k) \ll \sum_{k\le q_n+1} k^3 \mu(|V| > k) \ll \sum_{k\le q_n+1} k \approx q_n^2,$$

so

$$\Big| \max_{0\le j\le n} |V_n| \circ F^j \Big|_4^4 \le n|V_n|_4^4 \ll nq_n^2 = n^2 \log\log n.$$

Hence

$$a_n^{-4}\Big| \max_{0\le j\le n} |V_n| \circ F^j \Big|_4^4 \ll \log\log n/(\log n)^2,$$

which implies the required convergence. ∎

3 Nonstandard WIP for Gibbs-Markov maps

In this section, we prove a nonstandard WIP for Gibbs-Markov maps. We would expect that this result is well-known to experts, but we could not find a convenient reference. For the nonstandard CLT, see [1, 23]. For our purposes in this paper, it suffices to consider piecewise constant observables, and we do so, but this assumption is easily relaxed. Similarly, we only consider the simplest tail conditions ((3.1) below) since this also suffices for our purposes.

Let (Z, μ) be a probability space with an at most countable measurable partition $\{Z_k : k \ge 1\}$ and let $F : Z \to Z$ be an ergodic measure-preserving map. Define the separation time $s(z, z')$ to be the least integer $n \ge 0$ such that $F^n z$ and $F^n z'$ lie in distinct partition elements. We assume that $s(z, z') = \infty$ if and only if $z = z'$; then $d_\theta(z, z') = \theta^{s(z,z')}$ is a metric for $\theta \in (0, 1)$.

We say that $F : Z \to Z$ is a *Gibbs-Markov map* if:

(i) $F : Z_k \to Z$ is a measure-theoretic bijection onto a union of partition elements for each $k \ge 1$;

(ii) $\inf_k \mu(FZ_k) > 0$;

(iii) there exists $\theta \in (0, 1)$ such that $\log \xi$ is d_θ-Lipschitz, where $\xi = d\mu/d\mu \circ F$.

(For standard facts about Gibbs-Markov maps, we refer to [2, 3].)

Let $V : Z \to \mathbb{R}$ be an observable that is in the nonstandard domain of the central limit theorem. In particular, we assume that there exists $\sigma > 0$ such that

$$\mu(|V| > n) \sim \sigma^2 n^{-2} \quad \text{as } n \to \infty. \tag{3.1}$$

(So $V \in L^p(Z)$ for all $p < 2$, but $V \notin L^2(Z)$.) We suppose that $\int_Z V\, d\mu = 0$.

As mentioned above, we suppose for simplicity that V is *piecewise constant* (constant on partition elements Z_k). In this section, we prove:

Theorem 3.1 *Let $F : Z \to Z$ be a mixing Gibbs-Markov map. Suppose that V is piecewise constant and satisfies* (3.1)*. Then V satisfies the nonstandard WIP with variance σ^2.*

Remark 3.2 If V is piecewise constant (say) and lies in L^2, then it is well-known that V satisfies a standard (possibly degenerate) WIP. (Again, finding a good reference seems hard, but (for example) this is a very special case of [20, Theorem 2.1] with $G = 1$ and $d = 1$. Conditions (i)–(iii) of [20, Theorem 2.1] reduce to integrability of the observable since V is piecewise constant and $G = 1$.)

For $u : Z \to \mathbb{R}$, define

$$\|u\|_\theta = |u|_\infty + |u|_\theta, \qquad |u|_\theta = \sup_{z \neq z',\, s(z,z') \geq 1} \frac{|u(z) - u(z')|}{d_\theta(z, z')}.$$

Let $L : L^1(Z) \to L^1(Z)$ be the transfer operator corresponding to (F, Z, μ), so $\int_Z Lu\, v\, d\mu = \int_Z u\, (v \circ F)\, d\mu$ for $u \in L^1(Z)$, $v \in L^\infty(Z)$. Since F is a mixing Gibbs-Markov map, there exist constants $\gamma \in (0, 1)$, $C_0 > 0$ such that

$$\|L^j u - \textstyle\int_Z u\, d\mu\|_\theta \leq C_0 \gamma^j \|u\|_\theta \quad \text{for all } u : Z \to \mathbb{R} \text{ continuous, } j \geq 0. \tag{3.2}$$

Let $q_n = (n \log\log n)^{1/2}$, and define $V_n = V 1_{\{|V| \leq q_n\}} - \int_Z V 1_{\{|V| \leq q_n\}}\, d\mu$. Write

$$V_n = m_n + \chi_n \circ F - \chi_n \quad \text{where} \quad \chi_n = \sum_{j=1}^{\infty} L^j V_n.$$

Proposition 3.3 *Suppose that $V \in L^1(Z)$ is piecewise constant. Then $\sup_{n \geq 1} \|\chi_n\|_\theta < \infty$.*

Proof We recall the pointwise formula $(LV_n)(z) = \sum_{k \geq 1} \xi(z_k) V_n(z_k)$, where the sum is over those k for which there is a preimage of z under F lying in Z_k, in which case $z_k \in Z_k$ is the unique such preimage. There is a constant $C_1 > 0$ such that

$$0 < \xi(z) \leq C_1 \mu(Z_k), \qquad |\xi(z) - \xi(z')| \leq C_1 \mu(Z_k) d_\theta(z, z'),$$

for all $z, z' \in Z_k$, $k \geq 1$. Hence, for $z \in Z$,

$$|(LV_n)(z)| \leq C_1 \textstyle\sum_k \mu(Z_k)|V_n(z_k)| = C_1 \textstyle\sum_k \mu(Z_k)|V_n|_{Z_k}| = C_1|V_n|_1 \leq 2C_1|V|_1.$$

Next, let $z, z' \in Z$ lying in a common partition element. Since F is Markov, we can match up preimages z_k, $z_k' \in Z_k$. It follows that

$$|(LV_n)(z) - (LV_n)(z')| \leq C_1 \textstyle\sum_k \mu(Z_k)|V_n(z_k)| d_\theta(z, z') \leq 2C_1|V|_1 \, d_\theta(z, z').$$

Hence $\|LV_n\|_\theta \leq 4C_1|V|_1$. By (3.2), we conclude that $\|\chi_n\|_\theta \leq C_0(1-\gamma)^{-1}\|LV_n\|_\theta \leq 4C_0C_1(1-\gamma)^{-1}|V|_1$. ∎

Corollary 3.4 *The following hold uniformly in $n \geq 1$ and $1 \leq p \leq \infty$:*

$$|m_n|_p = |V1_{\{|V| \leq q_n\}}|_p + O(1) \quad \textit{and} \quad |m_n|_\theta = O(1).$$

In particular, $|m_n|_2^2 \sim \sigma^2 \log n$, $|m_n|_4^4 \ll q_n^2$ and $\|m_n\|_\theta \ll q_n$.

Proof Note that

$$\big||m_n|_p - |V1_{\{|V| \leq q_n\}}|_p\big| \leq \left|\int_Z V1_{\{|V| \leq q_n\}} \, d\mu\right| + 2|\chi_n|_p \leq |V|_1 + 2|\chi_n|_\infty$$

and

$$|m_n|_\theta = |\chi_n \circ F - \chi_n|_\theta \leq (1 + \theta^{-1})|\chi_n|_\theta,$$

hence the first two estimates follow by Proposition 3.3.

Estimates for m_n thereby reduce to estimates for $V1_{\{|V| \leq q_n\}}$. For example,

$$|V1_{\{|V| \leq q_n\}}|_2^2 = 2\int_0^{q_n} t\mu(|V| \geq t) \, dt \sim 2\sigma^2 \log q_n \sim \sigma^2 \log n.$$

The calculation for $|V1_{\{|V| \leq q_n\}}|_4^4$ is similar, and the estimate for $|V1_{\{|V| \leq q_n\}}|_\infty$ (and hence $\|m_n\|_\theta$) is immediate. ∎

Corollary 3.5 *There exist $\gamma \in (0, 1)$ and $C > 0$ such that*

$$\left|\int_Z m_n^2 \, (m_n^2 \circ F^j) \, d\mu - \left(\int_Z m_n^2 \, d\mu\right)^2\right| \leq C\gamma^j q_n^2 \log n \quad \textit{for all } n, j \geq 1.$$

Proof By (3.2),

$$\left|\int_Z m_n^2 \, (m_n^2 \circ F^j) \, d\mu - \left(\int_Z m_n^2 \, d\mu\right)^2\right| = \left|\int_Z \left(m_n^2 - \int_Z m_n^2 \, d\mu\right) \, (m_n^2 \circ F^j) \, d\mu\right|$$

$$= \left|\int_Z L^j \left(m_n^2 - \int_Z m_n^2 \, d\mu\right) \, m_n^2 \, d\mu\right| \leq \left|L^j \left(m_n^2 - \int_Z m_n^2 \, d\mu\right)\right|_\infty |m_n^2|_1$$

$$\leq 2C_0\gamma^j \|m_n^2\|_\theta \, |m_n|_2^2 \leq 2C_0\gamma^j \|m_n\|_\theta^2 \, |m_n|_2^2.$$

By Corollary 3.4, $\|m_n\|_\theta \ll q_n$ and $|m_n|_2^2 \approx \log n$ and so the proof is complete. ∎

Remark 3.6 A more careful argument shows that $\|L(m_n^2)\|_\theta \ll q_n$ and hence the estimate in Corollary 3.5 can be improved to $C\gamma^j q_n \log n$. However, this refinement is not required here.

Set $a_n = (n \log n)^{1/2}$.

Lemma 3.7 $\big|\sum_{j=0}^{[nt]-1} m_n^2 \circ F^j - a_n^2\sigma^2 t\big|_2^2 = o(a_n^4)$ *as* $n \to \infty$ *for all* $t \geq 0$.

Proof By Corollary 3.4, $\int_Z m_n^2 \, d\mu \sim \sigma^2 \log n$ so

$$\int_Z \sum_{j=0}^{[nt]-1} m_n^2 \circ F^j \, d\mu = [nt] \int_Z m_n^2 \, d\mu \sim \sigma^2 t \, n \log n = a_n^2 \sigma^2 t. \tag{3.3}$$

Next, again by Corollary 3.4, $\int_Z m_n^4 \, d\mu \ll q_n^2$ and so

$$\int_Z \sum_{j=0}^{[nt]-1} m_n^4 \circ F^j \, d\mu \ll n q_n^2 = n^2 \log\log n = o(a_n^4).$$

By Corollary 3.5, for $i < j$,

$$\begin{aligned} &\int_Z (m_n^2 \circ F^i)\,(m_n^2 \circ F^j)\, d\mu = \int_Z m_n^2 \,(m_n^2 \circ F^{j-i})\, d\mu \\ &= \Big(\int_Z m_n^2 \, d\mu\Big)^2 + O\big(\gamma^{j-i} q_n^2 \log n\big) \sim \sigma^4 \log^2 n + O\big(\gamma^{j-i} q_n^2 \log n\big). \end{aligned}$$

Hence

$$\begin{aligned} &\int_Z \Bigg(\sum_{j=0}^{[nt]-1} m_n^2 \circ F^j\Bigg)^2 d\mu = 2 \sum_{0 \leq i < j < nt} \int_Z (m_n^2 \circ F^i)\,(m_n^2 \circ F^j)\, d\mu + \int_Z \sum_{j=0}^{[nt]-1} m_n^4 \circ F^j \, d\mu \\ &\sim \sigma^4 (n^2 t^2 - nt) \log^2 n + O\Bigg(q_n^2 \log n \sum_{0 < r < nt} (nt - r)\gamma^r\Bigg) + o(a_n^4) \\ &\sim \sigma^4 n^2 t^2 \log^2 n + O\Big(n q_n^2 \log n\Big) + o(a_n^4) \sim a_n^4 \sigma^4 t^2. \end{aligned} \tag{3.4}$$

Using (3.3) and (3.4),

$$\begin{aligned} &\Bigg|\sum_{j=0}^{[nt]-1} m_n^2 \circ F^j - a_n^2 \sigma^2 t\Bigg|_2^2 = \int_Z \Bigg(\sum_{j=0}^{[nt]-1} m_n^2 \circ F^j - a_n^2\sigma^2 t\Bigg)^2 d\mu \\ &= \int_Z \Bigg(\sum_{j=0}^{[nt]-1} m_n^2 \circ F^j\Bigg)^2 d\mu - 2a_n^2\sigma^2 t \int_Z \sum_{j=0}^{[nt]-1} m_n^2 \circ F^j \, d\mu + a_n^4 \sigma^4 t^2 \\ &= a_n^4\sigma^4 t^2 - 2a_n^2\sigma^2 t \cdot a_n^2 \sigma^2 t + a_n^4 \sigma^4 t^2 + o(a_n^4) = o(a_n^4) \end{aligned}$$

as required. ∎

Let $(\widehat{F}, \widehat{Z}, \widehat{\mu})$ denote the *natural extension* (see e.g. [35]) of (F, Z, μ) with measure-preserving semiconjugacy $\widehat{\pi} : \widehat{Z} \to Z$. Let $\widehat{m}_n = m_n \circ \widehat{\pi}$. By construction, $m_n \in \ker L$. It follows as in [19] that $\{\widehat{m}_n \circ \widehat{F}^{-j} : j = 1, 2, \dots\}$ is a martingale difference sequence. (A detailed explanation can be found for example in [18, Remark 3.12].) Hence, for $t \geq 0$, we can define the martingale difference array $\{X_{n,j} : 1 \leq j \leq [nt]\}$ where $X_{n,j} = a_n^{-1} \widehat{m}_n \circ \widehat{F}^{-j}$. Define the processes $\widehat{M_n^-}(t) = \sum_{j=1}^{[nt]} X_{n,j}$ on $(\widehat{Z}, \widehat{\mu})$.

Lemma 3.8 *$\widehat{M_n^-} \to_{\widehat{\mu}} \sigma W$ in $D[0,\infty)$ as $n \to \infty$.*

Proof We verify the hypotheses of [31, Theorem 3.2]. Fix $t \geq 0$. By Corollary 3.4,

$$\Big| \max_{j \leq [nt]} |X_{n,j}| \Big|_1 \leq a_n^{-1} |m_n|_\infty \ll a_n^{-1} q_n \to 0.$$

By [31, Theorem 3.2], it therefore remains to show that $\sum\limits_{1 \leq j \leq [nt]} X_{n,j}^2 \to_{\widehat{\mu}} \sigma^2 t$. But

$$\begin{aligned} \sum_{1 \leq j \leq nt} X_{n,j}^2 = a_n^{-2} \sum_{j=1}^{[nt]} (\widehat{m}_n^2) \circ \widehat{F}^{-j} &= \left(a_n^{-2} \sum_{j=0}^{[nt]-1} (\widehat{m}_n^2) \circ \widehat{F}^j \right) \circ \widehat{F}^{-[nt]} \\ &= \left(a_n^{-2} \sum_{j=0}^{[nt]-1} m_n^2 \circ F^j \right) \circ \widehat{\pi} \circ \widehat{F}^{-[nt]}. \end{aligned}$$

Hence it suffices that $a_n^{-2} \sum_{j=0}^{[nt]-1} m_n^2 \circ F^j \to_\mu \sigma^2 t$ which follows from Lemma 3.7. ∎

Proof of Theorem 3.1 Define processes

$$M_n(t) = a_n^{-1} \sum_{j=0}^{[nt]-1} m_n \circ F^j \,, \qquad \widetilde{W}_n(t) = a_n^{-1} \sum_{j=0}^{[nt]-1} V_n \circ F^j$$

on (Z, μ). Let $g(u)(t) = u(1) - u(1-t)$ and note that $M_n \circ \widehat{\pi} \circ \widehat{F}^{-n} = g(\widehat{M_n^-})$ and $g(\sigma W) =_d \sigma W$. By Lemma 3.8 and the continuous mapping theorem[4],

$$M_n =_d M_n \circ \widehat{\pi} \circ \widehat{F}^{-n} = g(\widehat{M_n^-}) \to_{\widehat{\mu}} g(\sigma W) =_d \sigma W.$$

But $\widetilde{W}_n(t) = M_n(t) + a_n^{-1}(\chi_n \circ F^{[nt]} - \chi_n)$ where $\sup_n |\chi_n|_\infty < \infty$ by Proposition 3.3. Hence $\widetilde{W}_n \to_\mu \sigma W$.

[4]Technical issues about the domain and range of g can be dealt with either by linearly interpolating and passing to $C[0,1]$, or by proceeding as in [27, Proposition 4.9].

Finally, define E_n as in (2.4) (with X replaced by Z). As before, $\mu(E_n) \ll (\log\log n)^{-1}$. On E_n^c, we have $(V_n - V)\circ F^j = -\int V1_{\{|V|\le q_n\}}\,d\mu = \int V1_{\{|V|>q_n\}}\,d\mu$, $0\le j\le n$, so for W_n defined as in (1.1),

$$\sup_{t\in[0,1]} |W_n(t) - \widetilde{W}_n(t)|_\infty \le a_n^{-1} n \left|\int_Z V1_{\{|V|>q_n\}}\,d\mu\right| \ll n a_n^{-1} q_n^{-1} = (\log n \log\log n)^{-1/2}.$$

Hence $W_n \to_\mu \sigma W$. ∎

4 Nonstandard limit laws for Young towers

In this section, we consider nonstandard limit laws for a class of nonuniformly hyperbolic systems [40, 41]. Throughout, $a_n = (n\log n)^{1/2}$.

4.1 Exponential Young towers

We start with a Gibbs-Markov map as in Section 3, now denoted $(\bar F, \bar Z, \bar\mu_Z)$ with partition $\bar Z_k$, $k\ge 1$. We suppose moreover that $\bar F$ is full-branch[5], so $\bar F$ is a measure-theoretic bijection from $\bar Z_k$ onto $\bar Z$ for all k. Let $\tau : \bar Z \to \mathbb{Z}^+$ be a piecewise constant return time with $\bar\mu_Z(\tau > n) = O(e^{-cn})$ for some $c>0$. Define the tower $\bar\Delta = \{(z,\ell) : z\in\bar Z,\ 0\le \ell < \tau(z)\}$ and tower map

$$\bar f_\Delta : \bar\Delta \to \bar\Delta\,, \qquad \bar f_\Delta(z,\ell) = \begin{cases} (z,\ell+1), & 0\le \ell < \tau(z)-1 \\ (\bar F z, 0), & \ell = \tau(z)-1 \end{cases}.$$

An ergodic $\bar f_\Delta$-invariant probability measure is given by $\bar\mu_\Delta = (\bar\mu_Z \times \text{counting})/\bar\tau$ where $\bar\tau = \int_{\bar Z} \tau\, d\bar\mu_Z$. We call $(\bar\Delta, \bar\mu_\Delta)$ a *one-sided exponential Young tower*.

Let (Z, d_Z) be a metric space with Borel probability measure μ_Z. Let $F : Z\to Z$ be an ergodic measure-preserving transformation and $\bar\pi : Z \to \bar Z$ a measure-preserving semiconjugacy. The return time $\tau : \bar Z \to \mathbb{Z}^+$ lifts to a return time on Z which we also denote by τ. Define the (two-sided) exponential tower $\Delta = \{(z,\ell) : z\in Z,\ 0\le \ell < \tau(z)\}$ and tower map

$$f_\Delta : \Delta \to \Delta\,, \qquad f_\Delta(z,\ell) = \begin{cases} (z,\ell+1), & 0\le \ell < \tau(z)-1 \\ (F z, 0), & \ell = \tau(z)-1 \end{cases},$$

with ergodic invariant probability measure $\mu_\Delta = (\mu_Z \times \text{counting})/\bar\tau$. We have partitions $\{Z_k\}$ of Z and $\{\Delta_{k,\ell}\}$ of Δ where $Z_k = \bar\pi^{-1}(\bar Z_k)$ and $\Delta_{k,\ell} = Z_k \times \{\ell\}$.

[5]The full-branch assumption is mainly for convenience and is satisfied in the applications. It is easy to weaken the assumption significantly but some care is needed since for instance the result of [23] used in Lemma 4.1 assumes a big image and preimage condition which is stronger than the big images condition in Section 3.

There are further properties of exponential Young towers which are only required in the proof of Theorem 4.2 below in an argument identical to [5, Lemma 5.3]. Hence, we refer to [5] for the statement of these properties.

An observable $K : \Delta \to \mathbb{R}$ is called *piecewise constant* if K is constant on each partition element $\Delta_{k,\ell}$.

Lemma 4.1 *Let $K : \Delta \to \mathbb{R}$ be piecewise constant and define*

$$K_\tau : Z \to \mathbb{R}\,, \qquad K_\tau(z) = \sum_{\ell=0}^{\tau(z)-1} K(z,\ell).$$

If K satisfies a nonstandard CLT with variance $\sigma^2 > 0$, then $\mu_Z(|K_\tau| > n) \sim \bar\tau\sigma^2 n^{-2}$ as $n \to \infty$.

Proof We follow the proof of [33, Lemma 5.1(c)]. Since τ and K_τ are piecewise constant, they are well-defined and piecewise constant on the domain $\bar Z$ of the Gibbs-Markov map $\bar F : \bar Z \to \bar Z$. In particular, we can reduce from Δ and Z to $\bar\Delta$ and $\bar Z$.

Since $\tau \in L^p(\bar Z)$ for all finite p, it follows from Remark 3.2 that τ satisfies a (possibly degenerate) standard CLT on $(\bar Z, \bar\mu_Z)$. In particular, $a_n^{-1}\sum_{j=0}^{n-1}(\tau - \bar\tau)\circ \bar F^j \to_{\bar\mu_Z} 0$. By assumption, K satisfies a nonstandard CLT on $(\bar\Delta, \bar\mu_\Delta)$ with variance $\sigma^2 > 0$. Inducing as in [33, Appendix A] (with $X = \bar\Delta$, $Y = \bar Z$, $b_n = a_n$, and $\alpha = 2$ in [33, Remark A.3]), K_τ satisfies a nonstandard CLT on $(\bar Z, \bar\mu_Z)$ with variance $\bar\tau\sigma^2 > 0$. Since K_τ is piecewise constant on $\bar Z$, it follows from [23] that $\bar\mu_Z(|K_\tau| > n) \sim \bar\tau\sigma^2 n^{-2}$. ∎

4.2 Subexponential Young towers

Now let $R : \Delta \to \mathbb{Z}^+$ be a distinguished integrable piecewise constant observable. Define $R_\tau : Z \to \mathbb{Z}^+$ as in Lemma 4.1. Notice that $\int_Z R_\tau \, d\mu_Z = \bar R\bar\tau$ where $\bar R = \int_\Delta R\, d\mu_\Delta$.

We use R_τ to define a new (subexponential) tower $\Gamma = \{(z,\ell) : z \in Z,\, 0 \le \ell < R_\tau(z)\}$ and tower map

$$f_\Gamma : \Gamma \to \Gamma\,, \qquad f_\Gamma(z,\ell) = \begin{cases} (z,\ell+1), & 0 \le \ell < R_\tau(z) - 1 \\ (Fz, 0), & \ell = R_\tau(z) - 1 \end{cases},$$

with ergodic f_Γ-invariant probability measure $\mu_\Gamma = (\mu_Z \times \text{counting})/\bar R\bar\tau$.

Let d_θ be the metric on $\bar Z$ defined in Section 3. An observable $v : \Gamma \to \mathbb{R}$ is *dynamically Hölder* if it is bounded and there is a constant $C > 0$ such that

$$|v(z,\ell) - v(z',\ell)| \le C\big(d_Z(z,z') + d_\theta(\bar\pi z, \bar\pi z')\big) \tag{4.1}$$

for all $z, z' \in Z_k$, $k \ge 1$, $0 \le \ell < R_\tau|Z_k$.

We can now state and prove the main theorem of this section.

Theorem 4.2 *Let $v : \Gamma \to \mathbb{R}$ be a dynamically Hölder observable with $\int_\Gamma v\, d\mu_\Gamma = 0$. Define*

$$V : \Delta \to \mathbb{R}\,, \qquad V(z, j) = \sum_{\ell=R_{j-1}(z)}^{R_j(z)-1} v(z, \ell),$$

where $R_j(z) = \sum_{k=0}^{j} R(z, k)$. Suppose that

$$V = K + H$$

where $K : \Delta \to \mathbb{R}$ is piecewise constant and $H \in L^{2+\epsilon}(\Delta)$ for some $\epsilon > 0$.

(a) *If R and K satisfy nonstandard CLTs with variances $\sigma_R^2 > 0$ and $\sigma_K^2 > 0$, then v satisfies a nonstandard WIP with variance $\bar{R}^{-1}\sigma_K^2$.*

(b) *If $K = 0$, then there exists $\widetilde{\sigma}^2 \ge 0$ such that v satisfies a standard CLT with variance $\widetilde{\sigma}^2$.*

Proof (a) Induce further to obtain observables V_τ, $\widetilde{V}_\tau : Z \to \mathbb{R}$,

$$V_\tau(z) = \sum_{\ell=0}^{R_\tau(z)-1} v(z, \ell) = \sum_{j=0}^{\tau(z)-1} V(z, j)\,, \qquad \widetilde{V}_\tau(z) = \max_{0 \le k \le R_\tau(z)} \left| \sum_{\ell=0}^{k-1} v(z, \ell) \right|.$$

To obtain the nonstandard WIP for v with variance $\bar{R}^{-1}\sigma^2$, it suffices to verify the hypotheses of Theorem 2.5 with Γ, Z, R_τ playing the roles of X^r, X, r. By Proposition 2.6, it suffices to prove:

(I1) V_τ satisfies a nonstandard WIP with variance $\bar{\tau}\sigma^2$;

(2.2) $a_n^{-1} \max_{0 \le j \le n} \widetilde{V}_\tau \circ F^j \to_{\mu_Z} 0$ as $n \to \infty$;

on the probability space (Z, μ_Z).

Now, $|\widetilde{V}_\tau| \le |v|_\infty R_\tau$. By Lemma 4.1,

$$\mu_Z(|\widetilde{V}_\tau| > n) \le \mu_Z(R_\tau > n/|v|_\infty) \ll n^{-2}.$$

Hence condition (2.2) follows from Proposition 2.8.

Write

$$V_\tau = K_\tau + H_\tau \quad \text{where} \quad H_\tau(z) = \sum_{\ell=0}^{\tau(z)-1} H(z, \ell).$$

By Lemma 4.1, $\mu_Z(|K_\tau| > n) \sim \bar{\tau}\sigma^2 n^{-2}$. Since K_τ is locally constant, it is well defined and locally constant on the Gibbs-Markov base $\bar{F} : \bar{Z} \to \bar{Z}$. By Theorem 3.1, K_τ

satisfies the nonstandard WIP with variance $\bar{\tau}\sigma^2 n^{-2}$. Hence, to verify condition (I1), it remains to show that the contribution from $\widehat{H}_\tau = H_\tau - \int_Z H_\tau \, d\mu_Z$ is negligible. This is mainly [5, Lemmas 5.3 and 5.4] (written out also in [33, Section 6]).

By assumption, $H \in L^{2+\epsilon}(\Delta)$ and $\tau : Z \to \mathbb{Z}^+$ has exponential tails. Hence $H_\tau \in L^{2+\epsilon_1}(Z)$ for any $\epsilon_1 \in (0, \epsilon)$. The Hölder constants of H_τ on partition elements Z_k are unbounded but are of order R_τ and so are integrable. By [5, Lemma 5.3], a Gordin-type argument [19] shows that $\widehat{H}_\tau = m + \chi \circ F - \chi$ where $m, \chi \in L^p(Z)$ for some $p > 2$ and $\{m \circ F^j : j \geq 0\}$ is a reverse martingale-difference sequence. By Doob's inequality, $\big|\max_{0\leq k\leq n} |\sum_{j=0}^{k-1} m \circ F^j|\big|_2 \leq 4\big|\sum_{j=0}^{n-1} m \circ F^j\big|_2 = 4n^{1/2}|m|_2$. Also, $\big|\max_{0\leq j\leq n} |\chi \circ F^j|\big|_p \leq n^{1/p}|\chi|_p$. Hence $a_n^{-1}\max_{0\leq j\leq n} |\widehat{H}_\tau \circ F^j| \to_{\mu_Z} 0$ demonstrating the negligibility of $\widehat{H}_\tau$.

(b) Define $V_\tau : Z \to \mathbb{R}$ as in part (a). We again apply Theorem 2.5 with Γ, Z, R_τ playing the roles of X^r, X, r. To obtain the standard CLT, we must show that

(I1) V_τ satisfies a standard WIP;

(I2) $n^{-1/2}\max_{0\leq j\leq n} |V_\tau| \circ F^j \to_{\mu_Z} 0$ as $n \to \infty$;

on the probability space (Z, μ_Z).

The argument applied above for H, but now applied to V, shows that $V_\tau = m + \chi \circ F - \chi$ where $m, \chi \in L^p(Z)$ for some $p > 2$ and $\{m \circ F^j : j \geq 0\}$ is a reverse martingale-difference sequence. Again, $\big|\max_{0\leq j\leq n} |\chi \circ F^j|\big|_p \leq n^{1/p}|\chi|_p$. Hence by the WIP for L^2 martingales [7], we obtain the standard WIP for V_τ with variance $\int_Z m^2 \, d\mu_Z$. This verifies condition (I1).

Since $V_\tau \in L^p$, $p > 2$, it follows that $\big|\max_{0\leq j\leq n} |V_\tau| \circ F^j\big|_p \ll n^{1/p}$, verifying condition (I2). ∎

Remark 4.3 The assumption that τ has exponential tails can be weakened significantly. Certainly we used that $\tau \in L^2(Z)$ in the proof of Lemma 4.1. In addition, we require that H and τ are sufficiently well-behaved that $H_\tau \in L^p(Z)$ for some $p > 2$.

Remark 4.4 Throughout, we have focused on the "critical" case where v is bounded while R and V have similar behaviour. However, we can consider unbounded observables v when the first return time R lies in $L^p(\Delta)$ for some $p > 2$. We still require that v satisfies (4.1) and the assumptions on $V = K + H$ are unchanged. The condition on R in Theorem 4.2(a) can be removed and we require instead that $R_\tau \in L^2(Z)$.

In this way, we can recover results in [16, 21] for unbounded observables of intermittent maps.

4.3 Underlying dynamical systems

Let X be a metric space with Borel probability μ and $f : X \to X$ a measure-preserving transformation. Fix a measurable subset $Y \subset X$ and let $R_0 : Y \to \mathbb{Z}^+$ be the first return time $R_0(y) = \inf\{n \geq 1 : f^n y \in Y\}$. Define the first return map

$$f_Y = f^{R_0} : Y \to Y\,, \qquad f_Y(y) = f^{R_0(y)}(y).$$

An f_Y-invariant probability measure on Y is obtained by normalising $\mu|_Y$.

We assume that (f_Y, Y, μ_Y) is modelled by a (two-sided) exponential Young tower. That is, there exists $(f_\Delta, \Delta, \mu_\Delta)$ built over a map $F : Z \to Z$ with Gibbs-Markov quotient $\bar{F} : \bar{Z} \to \bar{Z}$ and return time $\tau : Z \to \mathbb{Z}^+$ with exponential tails, exactly as in Subsection 4.1, such that Z is a Borel subset of Y and such that the semiconjugacy

$$\pi_\Delta : \Delta \to Y, \qquad \pi_\Delta(z, \ell) = f_Y^\ell z,$$

satisfies $\pi_{\Delta\,*}\mu_\Delta = \mu_Y$. We assume further that $R = R_0 \circ \pi_\Delta : \Delta \to \mathbb{Z}^+$ is piecewise constant.

By construction, Δ and R satisfy the assumptions in Subsection 4.2, so we can use $R_\tau : Z \to \mathbb{Z}^+$ to define a subexponential tower map $f_\Gamma : \Gamma \to \Gamma$. It is easily verified that

$$\pi_\Gamma : \Gamma \to X, \qquad \pi_\Gamma(z, \ell) = f^\ell z,$$

defines a measure-preserving semiconjugacy between $(f_\Gamma, \Gamma, \mu_\Gamma)$ and (f, X, μ). Hence we obtain the following immediate consequence of Theorem 4.2. We say that an observable $v_0 : X \to \mathbb{R}$ is *dynamically Hölder* if the lifted observable $v_0 \circ \pi_\Gamma : \Gamma \to \mathbb{R}$ is dynamically Hölder.

Corollary 4.5 *Let $v_0 : X \to \mathbb{R}$ be a dynamically Hölder observable with $\int_X v_0\, d\mu = 0$. Define $V : \Delta \to \mathbb{R}$ as in Theorem 4.2 with $v = v_0 \circ \pi_\Gamma$. Suppose that*

$$V = K + H$$

where $K : \Delta \to \mathbb{R}$ is piecewise constant and $H \in L^{2+\epsilon}(\Delta)$ for some $\epsilon > 0$.

(a) *If R and K satisfy nonstandard CLTs with variances $\sigma_R^2 > 0$ and $\sigma_K^2 > 0$, then v_0 satisfies a nonstandard WIP with variance $\bar{R}^{-1}\sigma_K^2$.*

(b) *If $K = 0$, then there exists $\widetilde{\sigma}^2 \geq 0$ such that v_0 satisfies a standard CLT with variance $\widetilde{\sigma}^2$.* ∎

Remark 4.6 The variance $\widetilde{\sigma}^2$ in Corollary 4.5(b) is typically nonzero in our applications, where "typically" is interpreted in the very strong sense that $\widetilde{\sigma}^2 = 0$ only within a closed subspace of infinite codimension amongst Hölder observables v_0. See for example the discussion in [24, End of Section 4].

Remark 4.7 In the applications to be considered in Sections 6 and 7, Hölder observables $v_0 : X \to \mathbb{R}$ are dynamically Hölder.

It is often the case that $H = O(R^{1-\delta})$ for some $\delta > 0$. Since $R \in L^p(\Delta)$ for all $p < 2$, the integrability assumption on H in Corollary 4.5 is automatic.

In the examples, the nonstandard CLT for K is degenerate if and only if $K = 0$. Hence there is a dichotomy whereby either part (a) or part (b) of Corollary 4.5 applies.

5 Limit laws for suspension flows

Let $(f_\Gamma, \Gamma, \mu_\Gamma)$, $(f_\Delta, \Delta, \mu_\Delta)$, R, etc, be as in Section 4 and let $h : \Gamma \to (0, \infty)$ be a dynamically Hölder (hence bounded) roof function. We form the suspension

$$\Gamma^h = \{(x, u) : x \in \Gamma, 0 \le u < h(x)\}/\sim, \quad (x, h(x)) \sim (f_\Gamma(x), 0),$$

and the suspension flow $g_t(x, u) = (x, u + t)$ computed modulo identifications, with invariant probability measure $\mu^h = (\mu_\Gamma \times \text{Lebesgue})/\bar{h}$ where $\bar{h} = \int_\Gamma h \, d\mu_\Gamma$.

Theorem 5.1 *Let $v : \Gamma^h \to \mathbb{R}$ be a bounded observable with $\int_{\Gamma^h} v \, d\mu^h = 0$. Define*

$$v_h : \Gamma \to \mathbb{R}, \qquad v_h(x) = \int_0^{h(x)} v(x, u) \, du,$$

and assume that v_h is dynamically Hölder. Define $V : \Delta \to \mathbb{R}$ as in Theorem 4.2 with v replaced by v_h. Suppose that

$$V = K + H$$

where K is piecewise constant and $H \in L^{2+\epsilon}(\Delta)$ for some $\epsilon > 0$.

(a) *If R and K satisfy nonstandard CLTs with variance $\sigma_R^2 > 0$ and $\sigma_K^2 > 0$, then v satisfies a nonstandard WIP with variance $\bar{h}^{-1}\bar{R}^{-1}\sigma_K^2$.*

(b) *If $K = 0$, then there exists $\widetilde{\sigma}^2 \ge 0$, typically nonzero, such that v satisfies a standard CLT with variance $\widetilde{\sigma}^2$.*

Proof (a) Define

$$Q : \Gamma^h \to \mathbb{R}, \qquad Q(x, u) = \int_0^u v(x, s) \, ds.$$

We apply Theorem 2.1 with Γ^h, Γ, h playing the roles of X^r, X, r. To obtain the nonstandard WIP with variance $\bar{h}^{-1}\bar{R}^{-1}\sigma^2$, we must show that

(I1) v_h satisfies a nonstandard WIP with variance $\bar{R}^{-1}\sigma^2$;

(I2) $(n \log n)^{-1/2} \max_{0 \le j \le n} |v_h| \circ f_\Gamma^j \to_{\mu_\Gamma} 0$ as $n \to \infty$;

(I3) $(n \log n)^{-1/2} \sup_{t \in [0,n]} |Q| \circ g_t \to_{\mu^h} 0$ as $n \to \infty$;

on the probability space (Γ, μ_Γ). Condition (I1) follows from Theorem 4.2. Conditions (I2) and (I3) are trivial since $|v_h|$, $|Q(x, u)| \le |v|_\infty |h|_\infty < \infty$.

(b) It is convenient to regard Γ^h as a suspension Z^φ over Z with unbounded roof function $\varphi(z) = \int_0^{R_\tau(z)} h(z,u)\,du$. Define the further induced observable

$$V_\tau : Z \to \mathbb{R}\,, \qquad V_\tau(z) = \int_0^{\varphi(z)} v(z,u)\,du = \sum_{\ell=0}^{\tau(z)-1} V(z,\ell).$$

We apply Theorem 2.1 with Γ^h, Z, φ playing the roles of X^r, X, r. To obtain the standard CLT, we must show that

(I1) V_τ satisfies a standard WIP;

(I2) $n^{-1/2} \max\limits_{0\le j\le n} |V_\tau| \circ F^j \to_{\mu_Z} 0$ as $n\to\infty$;

on the probability space (Z,μ_Z).

Condition (I1) is verified by the argument in the proof of Theorem 4.2(b). Since $V_\tau \in L^p$, $p>2$, it follows that $\big|\max\limits_{0\le j\le n} |V_\tau|\circ F^j\big|_p \ll n^{1/p}$, verifying condition (I2). ∎

6 Applications to intermittent maps

In this section, we consider some intermittent map examples. In all the examples, $f : X \to X$ is a piecewise C^1 map with domain $X = [0,1]$ or $X \subset [0,1]\times\mathbb{T}$ and there is a unique absolutely continuous mixing f-invariant probability measure μ. There is a stipulated subset $Y \subset X$ with $\mu(Y) > 0$, first return time $R : Y \to \mathbb{Z}^+$ and first return map $f^R : Y \to \mathbb{R}$. We consider a Hölder observable $v : X \to \mathbb{R}$ with $\int_X v\,d\mu = 0$ and define the induced observable $V = \sum_{\ell=0}^{R-1} v\circ f^\ell : Y \to \mathbb{R}$. Without loss of generality we assume that v is C^η for some $\eta \in \big(0,\frac12\big)$.

Example 6.1 (The LSV map) Let $X = [0,1]$. The simplest example of an intermittent map $f : X \to X$ is the LSV map [29] given by

$$fx = \begin{cases} x(1+2^{1/\alpha}x^{1/\alpha}), & 0\le x<\frac12 \\ 2x-1, & \frac12 \le x \le 1. \end{cases}$$

There is a unique absolutely continuous f-invariant probability measure μ for all $\alpha > 1$. It is well-known that Hölder observables $v : X \to \mathbb{R}$ satisfy the standard WIP when $\alpha > 2$. See [34] for functional limit theorems when $\alpha \in (1,2)$. Here we focus on the case $\alpha = 2$. By [21] (see also [44]), v satisfies a nonstandard CLT if $v(0) \ne 0$ and a standard CLT otherwise. The nonstandard WIP for $v(0)\ne 0$ is proved in [16].

We now indicate how to recover the nonstandard WIP using the results in this paper. The situation is greatly simplified from Section 4: we can take $Y = \Delta = \bar\Delta = Z = \bar Z$ (and $\tau = 1$). The common space is denoted Y here and we take $Y = \big[\frac12, 1\big]$. By [29], $\mu(R=n) \sim \frac12\sigma_R^2 n^{-3}$ for some $\sigma_R^2 > 0$. It is standard (see [29] or [21, Proof of Theorem 1.3]) that f^R is a full-branch Gibbs-Markov map. We have a semiconjugacy $\pi_\Gamma : \Gamma \to X$ where $(f_\Gamma, \Gamma, \mu_\Gamma)$ is a one-sided Young tower with $\Gamma = \{(y, \ell) : y \in Y, 0 \le \ell < R(y)\}$.

A calculation [21, Proof of Theorem 1.3] shows that $V = K + H$ where $K = Rv(0)$ and $H = O(R^{1-2\eta})$. By [21], R satisfies a nonstandard CLT with variance σ_R^2. Hence, for $v(0) \neq 0$, we obtain the nonstandard WIP with variance $\bar{R}^{-1}v(0)^2\sigma_R^2$ by Corollary 4.5(a).

Example 6.2 (An example with two neutral fixed points) We consider an example studied in [15] with $X = [0, 1]$ and

$$fx = \begin{cases} x(1 + 3^{1/2}x^{1/2}), & x \in \left[0, \frac{1}{3}\right) \\ 3x - 1, & x \in \left[\frac{1}{3}, \frac{2}{3}\right) \\ 1 - (1 - x)(1 + 3^{1/2}(1 - x)^{1/2}), & x \in \left[\frac{2}{3}, 1\right] \end{cases}.$$

As in Example 6.1, we can take $Y = \Delta = \bar{\Delta} = Z = \bar{Z}$ (and $\tau = 1$), and we choose $Y = \left[\frac{1}{3}, \frac{2}{3}\right]$. By [15, Proof of Lemma 6.3], $\mu(R > n) \sim \sigma_R^2 n^{-2}$ for some $\sigma_R^2 > 0$ and by symmetry $\mu(R1_{(\frac{1}{3},\frac{1}{2})} > n) = \mu(R1_{(\frac{1}{2},\frac{2}{3})} > n) \sim \frac{1}{2}\sigma_R^2 n^{-2}$. The first return map $F = f^R : Y \to Y$ is a full-branch Gibbs-Markov map. Moreover, $V = K + H$ where $K = R1_{(\frac{1}{3},\frac{1}{2})}v(0) + R1_{(\frac{1}{2},\frac{2}{3})}v(1)$ and $H = O(R^{1-2\eta})$. It follows that $\mu(|K| > n) \sim \sigma_K^2 n^{-2}$ where $\sigma_K^2 = \frac{1}{2}\sigma_R^2(v(0)^2 + v(1)^2)$. If $\sigma_K^2 > 0$, then it is a consequence of Theorem 3.1 that K satisfies the nonstandard WIP with variance σ_K^2. By Corollary 4.5, we obtain a standard CLT if $v(0) = v(1) = 0$ and the nonstandard WIP with variance $\bar{R}^{-1}\sigma_K^2$ otherwise.

Example 6.3 (NonMarkovian examples) Zweimüller [42, 43] studied a class of nonMarkovian interval maps $f : X \to X$ with indifferent fixed points, called AFN maps. For definiteness, we focus on the example

$$fx = x + bx^{3/2} \bmod 1 \tag{6.1}$$

for $b \in [1, \infty)$. Note that f has $[b] + 1$ branches and there is a neutral fixed point at 0. When b is an integer, we can proceed as in Example 6.1 so we are particular interested in the case where b is not an integer and hence f is nonMarkovian. (The methods described here apply more generally to mixing AFN maps.) We only sketch the details since a more complicated example is treated in Example 6.4 below.

We choose $Y = [y_0, 1]$ where $y_0 < 1$ is maximal such that $fy_0 = 0$. It is easily checked that $\mu(R > n) \sim \sigma_R^2 n^{-2}$ for some $\sigma_R^2 > 0$. Although f^R is not Markov (hence not Gibbs-Markov) it is well-known that the transfer operator for f^R has a spectral gap on the space of bounded variation functions and hence R satisfies a nonstandard CLT [17, Remark C.2].

The same calculation as in Example 6.1 shows that $V = v(0)R + H$ where $H = O(R^{1-2\eta})$. This already yields the nonstandard CLT for v when $v(0) \neq 0$.

A standard technique [42, 43] for studying AFN maps is to reinduce the first return map $f^R : Y \to Y$ to obtain a Gibbs-Markov map $F : Z \to Z$. (See also [8].) We can then apply Corollary 4.5 to obtain the nonstandard WIP when $v(0) \neq 0$ and a typically nondegenerate standard CLT when $v(0) = 0$.

Example 6.4 (A multidimensional nonMarkovian example) To illustrate the generality of the techniques in this paper with regard to intermittent maps, we consider a family of multidimensional nonMarkovian nonconformal intermittent maps introduced by Eslami *et al.* [17]. Let $X_0 = [0,1] \times \mathbb{T}$ with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and define the map $f : X_0 \to X_0$ given by $f(x,\theta) = (f_1(x,\theta), f_2(\theta))$ where

$$f_1(x,\theta) = \begin{cases} x(1 + x^{1/\alpha}u(x,\theta)), & 0 \le x \le \frac{3}{4} \\ 4x - 3, , & \frac{3}{4} < x \le 1 \end{cases}, \qquad f_2(\theta) = 4\theta \bmod 1.$$

Here, $u : \left[0, \frac{3}{4}\right] \times \mathbb{T} \to (0,\infty)$ is C^2 with $u(0,\theta) \equiv c_0 > 0$ such that $x(1 + x^{1/\alpha}u(x,\theta)) \le 1$ on $\left[0, \frac{3}{4}\right] \times \mathbb{T}$. There is a neutral invariant circle $\{x = 0\}$ and we require that f is uniformly expanding outside any neighbourhood of this circle. There are further technical assumptions on u in [17] that we do not write down here. As shown in [17], f restricts to a mixing transformation on $X = f\left(\left[0, \frac{3}{4}\right] \times \mathbb{T}\right)$. In [17], the cases $\alpha \in (0,2)$ and $\alpha > 2$ are studied extensively, but the case $\alpha = 2$ is largely omitted since Theorem 3.1 was not available. Here, we cover this missing case.

Following [17], we take $Y = X \cap \left(\left[\frac{3}{4}, 1\right] \times \mathbb{T}\right)$. By [17, Proposition 4.11], $\mu(R = n) \sim \frac{1}{2}\sigma_R^2 n^{-3}$ for some $\sigma_R^2 > 0$. By [17, Corollary 4.10], the transfer operator for the first return map $f^R : Y \to Y$ has a spectral gap in a space of two-dimensional bounded variation functions, and hence we can apply [17, Remark C.2] to obtain a nonstandard CLT with variance σ_R^2 for R.

By the proof of [17, Theorem 1.3, Section 6], $V = K + H$, $K = I_v R$, where $I_v = \int_{\mathbb{T}} v(0,\theta)\, d\theta$ and $H = O(R^{1-2\eta})$. In this way, we already obtain that v satisfies the nonstandard CLT with variance $\bar{R}^{-1} I_v^2 \sigma_R^2$ if $I_v \neq 0$ and a typically nondegenerate standard CLT if $I_v = 0$.

By [17, Lemma 3.1], we can reinduce the first return map $f^R : Y \to Y$ to obtain a Gibbs-Markov map $F : Z \to Z$ such that the return time for returns of f^R to Z has exponential tails. Hence we are in the situation of Section 4. By Corollary 4.5, we obtain the nonstandard WIP if $I_v \neq 0$.

7 Applications to dispersing billiards

In this section, we provide details and proofs for the billiard examples mentioned in Section 1. For background material on billiards, we refer to [12]. The billiard domain, denoted by Q, is a compact connected subset of $\mathbb{R}^2$ or $\mathbb{T}^2$ with piecewise smooth boundary and the billiard flow f_t is defined on $Q \times S^1$. Fix a point $q \in Q$ and a unit vector $\psi \in S^1$. Then q moves in straight lines with unit speed in direction ψ

until reflecting (angle of reflection equalling the angle of incidence) off the boundary ∂Q. This defines a volume-preserving flow. A natural Poincaré section is given by $X = \partial Q \times [-\pi/2, \pi/2]$ corresponding to collisions with ∂Q (with outgoing velocities in $[-\pi/2, \pi/2]$). The Poincaré map $f : X \to X$ is called the *collision map* or the *billiard map*. It preserves a probability measure μ, equivalent to Lebesgue, called Liouville measure.

A general framework introduced by [30] and explored further in [14] is to model a suitable first return map by an exponential Young tower Δ as described in Section 4. More precisely, one chooses[6] a positive measure set $Y \subset X$ with first return time $R : Y \to \mathbb{Z}^+$ and first return map $f_Y = f^R : Y \to Y$. Then it is shown that there is an exponential tower $f_\Delta : \Delta \to \Delta$ and a measure-preserving semiconjugacy $\pi_\Delta : \Delta \to Y$ such that (dynamically) Hölder observables $V : Y \to \mathbb{R}$ lift to dynamically Hölder observables $V \circ \pi_\Delta : \Delta \to \mathbb{R}$.

With this structure in place, we can now construct a (nonexponential) Young tower $f_\Gamma : \Gamma \to \Gamma$ and a measure-preserving semiconjugacy $\pi_\Gamma : \Gamma \to X$ such that (dynamically) Hölder observables $v : X \to \mathbb{R}$ lift to dynamically Hölder observables $v \circ \pi_\Gamma : \Gamma \to \mathbb{R}$. In this way, statistical limit laws for the billiard map $f : X \to X$ reduce to statistical limit laws on Γ as described in Section 4.

If moreover the flow time between collisions in X is Hölder and bounded below, then these statistical limit laws for $f : X \to X$ lift to statistical limit laws for the billiard flow f_t as described in Section 5.

Example 7.1 (Billiards with cusps) These are billiard domains $Q \subset \mathbb{R}^2$ where ∂Q is a simple closed curve consisting of finitely many convex inwards C^3 curves with nonvanishing curvature such that the interior angles at corner points are zero. By [13, Theorem 1.1], the billiard map $f : X \to X$ falls into the framework of Section 4. In [4], it is shown that there is a constant $I_v \in \mathbb{R}$, given explicitly in terms of the values of the observable v near the cusp, such that a nondegenerate WIP holds for $I_v \neq 0$ and a standard CLT holds for $I_v = 0$.

Our methods give a more streamlined approach which leads to exactly the same results as in [4]. The main step is a nonstandard CLT for R which is established in [4, Eq (2.5)]. Let $v : X \to \mathbb{R}$ be a Hölder observable and define the first return observable $V = \sum_{\ell=0}^{R-1} v \circ f^\ell : Y \to \mathbb{R}$. When there is a single cusp, $V = K + H$ where $K = I_v R$, and $H = O(R^{1-\delta})$ for some $\delta > 0$. (Such a decomposition is implicit in [4, Eq. (6.16)] and nearby calculations. A similar decomposition for billiards with flat cusps is computed explicitly in [26, End of Section 6] and [33, Proposition 8.1].) Hence the hypotheses of Theorem 4.2 are satisfied and all the remaining results in [4] are seen to hold independently of the details of the billiard model.

When there are several cusps, the situation is similar with K piecewise constant

[6] Roughly speaking, Y is chosen to be a subset of X bounded away from the regions where hyperbolicity is expected to break down, e.g. for billiards with cusps, Y excludes a neighbourhood of each cusp.

(at the level of the tower Δ) equal to a constant multiple of R in a neighbourhood of each cusp. (See [25] for an explicit calculation in the situation of several flat cusps.)

The roof function for the billiard flow is not bounded below and it can be shown that the flow mixes faster than the billiard map (at least superpolynomially quickly [6]). As a consequence of this, the standard CLT and WIP (and moreover the almost sure invariance principle) hold for Hölder observables of the billiard flow [6]. The variance is typically nonzero.

Example 7.2 (Bunimovich stadia [9]) These are convex billiard domains $Q \subset \mathbb{R}^2$ where ∂Q is a simple closed curve consisting of two semicircles C_1, C_2 of radius 1 and two parallel line segments S_1, S_2 of length L tangent to the semicircles.

By Markarian [30], the billiard map $f : X \to X$ falls within the Young tower framework of Section 4. Let $\widehat{C}_j = C_j \times (-\pi/2, \pi/2) \subset X$, $j = 1, 2$, and set

$$Y = \big(\widehat{C}_1 \setminus f\widehat{C}_1\big) \cup \big(\widehat{C}_2 \setminus f\widehat{C}_2\big).$$

Then [30] verifies the Chernov axioms [11] showing that $f_Y = f^R : Y \to Y$ is modelled by a Young tower with exponential tails.

The first return time $R : Y \to \mathbb{Z}^+$ decomposes into $R = R_{\text{slide}} + R_{\text{bounce}}$ as follows. Suppose $x \in \widehat{C}_1 \setminus f\widehat{C}_1$. The trajectory of x slides along the semicircle C_1 for $R_{\text{slide}} \geq 0$ iterates and then bounces between the line segments S_1 and S_2 for $R_{\text{slide}} \geq 0$ iterates before returning to Y. Similarly for $x \in \widehat{C}_2 \setminus f\widehat{C}_2$. Elementary geometric arguments [5, 30] show that $\mu(R_{\text{bounce}} = n) \sim cn^{-3}$ for some $c > 0$ and $\mu(R_{\text{slide}} = n) = O(n^{-4})$. In particular, $R \in L^p(Y)$ for all $p < 2$ and $R_{\text{slide}} \in L^p(Y)$ for all $p < 3$.

Let $v : X \to \mathbb{R}$ be a dynamically Hölder observable and let $V = \sum_{\ell=0}^{R-1} v \circ f^j$. Define

$$I_v = \frac{1}{2L} \int_{S_1 \cup S_2} v(q, 0)\, dq\,;$$

this is the average of v over trajectories bouncing perpendicular to the straight edges.

Lemma 7.3 *$V = I_v R + H$ where $H \in L^{2+\epsilon}(Y)$ for some $\epsilon > 0$.*

Proof This is [5, Line 1 of proof of Lemma 5.3]. In more detail, write $V = V_{\text{slide}} + V_{\text{bounce}}$ where

$$V_{\text{slide}} = \sum_{\ell=0}^{R_{\text{slide}}-1} v \circ f^\ell, \quad V_{\text{bounce}} = \sum_{\ell=R_{\text{slide}}}^{R-1} v \circ f^\ell.$$

Using the Hölder continuity of v, we can view V_{bounce} as a Riemann sum approximating the integral I_v to obtain

$$V_{\text{bounce}} = I_v R_{\text{bounce}} + O(R_{\text{bounce}}^{1-\delta}) = I_v R + O(R_{\text{slide}}) + O(R^{1-\delta}),$$

for some $\delta > 0$. Clearly, $|V_{\rm slide}| \le |v|_\infty R_{\rm slide}$, so

$$V = I_v R + H\,, \qquad |H| \ll R_{\rm slide} + R^{1-\delta}.$$

This completes the proof. ∎

By the proof of [5, Theorem 1.1] (see in particular [5, page 504, line 11]), the first return time $R : Y \to \mathbb{Z}^+$ (denoted there by φ_+) satisfies a nonstandard CLT with variance $\sigma_R^2 > 0$. Since $K = I_v R$ is a scalar multiple of R, it is immediate that the hypotheses of Theorem 4.2 are satisfied. We conclude that if $I_v \ne 0$, then v satisfies a nonstandard WIP with variance $\sigma^2 = \bar{R}^{-1} I_v^2 \sigma_R^2$; and if $I_v = 0$, then v satisfies a typically nondegenerate standard CLT.

Next, we consider the billiard flow with roof function h. Given a Hölder observable, we define $v_h : X \to \mathbb{R}$ and $V : Y \to \mathbb{R}$ as in Theorem 5.1. Then $V = K + H$ with $K = J_v R$ and $H = L^{2+\epsilon}(Y)$ for some $\epsilon > 0$ where

$$J_v = \frac{1}{2L}\int_{S_1\cup S_2} v_h(q,0)\,dq = \frac{1}{2L}\int_{(S_1\cup S_2)\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]} v\,d\,\mathrm{Leb}\,.$$

Hence we obtain the nonstandard WIP ($J_v \ne 0$) and the standard CLT ($J_v = 0$) for the billiard flow by Theorem 5.1.

The CLTs are not new, so our contribution beyond [5] is the nonstandard WIP for the billiard map and flow when $I_v \ne 0$.

Remark 7.4 A further example of fundamental importance is the planar infinite horizon Lorentz gas, for which the nonstandard WIP was studied extensively by [39]. In some sense, the situation is simpler than the other billiards examples, but the observable of interest is the displacement function which is vector-valued. This raises extra technical issues which we do not address here. (Our methods easily yield the nonstandard WIP for components of the displacement function.)

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