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Deterministic homogenization under optimal moment assumptions for fast-slow systems. Part 1

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Abstract. We consider deterministic homogenization (convergence to a stochastic differential equation) for multiscale systems of the form

$$x_{k+1} = x_k + n^{-1} a_n(x_k, y_k) + n^{-1/2} b_n(x_k, y_k), \quad y_{k+1} = T_n y_k,$$

where the fast dynamics is given by a family T_n of nonuniformly expanding maps. Part 1 builds on our recent work on martingale approximations for families of nonuniformly expanding maps. We prove an iterated weak invariance principle and establish optimal iterated moment bounds for such maps. (The iterated moment bounds are new even for a fixed nonuniformly expanding map T.) The homogenization results are a consequence of this together with parallel developments on rough path theory in Part 2 by Chevyrev, Friz, Korepanov, Melbourne and Zhang.

Résumé. Nous étudions un problème d'homogénéisation déterministe (avec convergence vers une équation différentielle stochastique) pour un système multi-échelle de la forme suivante :

$$x_{k+1} = x_k + n^{-1} a_n(x_k, y_k) + n^{-1/2} b_n(x_k, y_k), \quad y_{k+1} = T_n y_k,$$

où la dynamique rapide est donnée par une famille T_n de transformations non uniformément dilatantes. La partie 1 prolonge nos travaux récents sur l'approximation par des martingales pour des familles de transformations non uniformément dilatantes. Nous montrons un principe d'invariance faible itéré, et établissons des bornes optimales sur les moments dans ce cadre (ces bornes sont nouvelles même pour une transformation non uniformément dilatante T fixée). En combinant ceci et des développements parallèles sur la théorie des chemins rugueux par Chevyrev, Friz, Korepanov, Melbourne et Zhang, nous obtenons les résultats d'homogénéisation dans la partie 2.

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1. Introduction

Recently, there has been a great deal of interest in deterministic homogenization [5–9,12,16,17,20,24] whereby deterministic multiscale systems converge to a stochastic differential equation as the time-scale separation goes to infinity. A byproduct of this is a deeper understanding [16] of the correct interpretation of limiting stochastic integrals [28].

Using rough path theory [11,23], it was shown in [16,17] that homogenization reduces to proving certain statistical properties for the fast dynamics. These statistical properties take the form of an "iterated invariance principle" (iterated WIP) which gives the correct interpretation of the limiting stochastic integrals, and control of "iterated moments" which provides tightness in the rough path topology used for proving convergence. In particular, the homogenization question was settled in [16,17] for uniformly expanding/hyperbolic fast (Axiom A) dynamics and for nonuniformly expanding/hyperbolic fast dynamics modelled by Young towers with exponential tails [29]. The results in [16,17] also covered fast dynamics modelled by Young towers with polynomial tails [30] but the results were far from optimal. It turns out that advances on two separate fronts are required to obtain optimal results:

(i) Martingale methods for nonuniformly expanding maps modelled by Young towers, yielding optimal control of iterated moments;
(ii) Discrete-time rough path theory in p-variation topologies, relaxing the required control for ordinary and iterated moments.

These two directions rely on techniques in smooth ergodic theory and in stochastic analysis respectively, so the homogenization question divides naturally into two parts. This paper *Part 1* covers the ergodic-theoretical aspects required for (i), while the rough path aspects required for (ii) are dealt with in *Part 2* by Chevyrev *et al.* [4]. As we explain below, together these provide an optimal solution to the homogenization question when the fast dynamics is given by a nonuniformly expanding map or a family of such maps.

The homogenization question that we are interested in takes the following form. Let $T_n : \Lambda \to \Lambda$, $n \ge 1$, be a family of dynamical systems with ergodic invariant probability measures μ_n. Consider the fast-slow system

$$x_{k+1} = x_k + n^{-1} a_n(x_k, y_k) + n^{-1/2} b_n(x_k, y_k), \qquad y_{k+1} = T_n y_k, \tag{1.1}$$

where $x_k = x_k^{(n)}$ takes values in $\mathbb{R}^d$ with $x_0 \equiv \xi \in \mathbb{R}^d$, and y_k takes values in Λ. Our main assumption is that T_n is a uniform family of nonuniformly expanding maps of order $p > 2$ as in [20] (see Section 3 below for precise definitions). We impose mild regularity conditions on $a_n, b_n : \mathbb{R}^d \times \Lambda \to \mathbb{R}^d$ and require that $\int_\Lambda b_n(x, y)\, d\mu_n(y) = 0$ for all $x \in \mathbb{R}^d$, $n \ge 1$.

Define $\hat{x}_n(t) = x_{[nt]}^{(n)}$ and let λ_n be a family of probability measures on Λ. We regard $\hat{x}_n$ as a sequence of random variables on the probability spaces (Λ, λ_n) with values in the Skorohod space $D([0, 1], \mathbb{R}^d)$. The aim is to prove weak convergence, $\hat{x}_n \to_{\lambda_n} X$ as $n \to \infty$, where X is the solution to a stochastic differential equation.

Example 1.1. To fix ideas, we focus first on the case where $T_n \equiv T$ is a single nonuniformly expanding map. Pomeau-Manneville intermittent maps [27] provide the prototypical examples of such maps. We consider in particular the class of intermittent maps studied in [22], namely

$$T : [0, 1] \to [0, 1], \quad Tx = \begin{cases} x\bigl(1 + 2^\gamma x^\gamma\bigr) & x < \frac{1}{2}, \\ 2x - 1 & x > \frac{1}{2}. \end{cases} \tag{1.2}$$

Here $\gamma > 0$ is a parameter and there is a unique absolutely continuous invariant probability measure μ provided $\gamma < 1$. Moreover, the central limit theorem (CLT) holds for all Hölder observables $v : [0, 1] \to \mathbb{R}$, provided $\gamma < \frac{1}{2}$. By [13], the CLT fails for typical Hölder observables once $\gamma > \frac{1}{2}$. Even for $\gamma = \frac{1}{2}$, the CLT requires a nonstandard normalization. Hence it is natural to restrict here to the range $\gamma \in (0, \frac{1}{2})$. (The range $\gamma \in (\frac{1}{2}, 1)$ leads to superdiffusive phenomena [13, 26] and we refer to [3,12] for the homogenization theory for the corresponding fast-slow systems.)

The homogenization problem for fast-slow systems driven by such intermittent maps T (with $\lambda_n \equiv \mu$) was previously considered in [16] and then [5]. The techniques therein sufficed in the restricted range $\gamma \in (0, \frac{2}{5})$ and even then only in the special case $b(x, y) = h(x)v(y)$ where $h : \mathbb{R}^d \to \mathbb{R}^{d \times m}$, $v : \Lambda \to \mathbb{R}^d$. There are two additional steps, covered in Parts 1 and 2 respectively, that lead to homogenization in the full range $\gamma \in (0, \frac{1}{2})$ and for general b:

(i) As mentioned above, to obtain homogenization results it suffices to prove the iterated WIP and control of iterated moments. These statistical properties are formulated at the level of the map T for Hölder observables $v : [0, 1] \to \mathbb{R}^d$ with $\int v\, d\mu = 0$. The iterated WIP was already proved in [16] in the full range $\gamma \in (0, \frac{1}{2})$. Define

$$S_n v = \sum_{0 \le j < n} v \circ T^j, \qquad \mathbb{S}_n v = \sum_{0 \le i < j < n} \bigl(v \circ T^i\bigr) \otimes \bigl(v \circ T^j\bigr).$$

There are numerous methods for estimating ordinary moments $|S_n v|_{L^{2(p-1)}}$ for $p < 1/\gamma$. Estimates for iterated moments $|\mathbb{S}_n v|_{L^{2(p-1)/3}}$ were given in [16]. In Theorem 2.4 of the current paper, we estimate $|\mathbb{S}_n v|_{L^{p-1}}$; this is the first result giving optimal estimates for iterated moments. Using [5], we can then cover the full range $\gamma \in (0, \frac{1}{2})$ in the product case $b(x, y) = h(x)v(y)$.

(ii) The papers [16,17] use rough path theory in Hölder spaces. However, Hölder rough path theory requires control of the ordinary moments $|S_n v|_{L^{2q}}$ and the iterated moments $|\mathbb{S}_n v|_{L^q}$ for some $q > 3$. As shown in [25, Section 3], such control even for the ordinary moments requires $\gamma < \frac{1}{4}$. The papers by Chevyrev *et al.* [4,5] employ rough path theory in p-variation spaces and require iterated moment estimates only for $q > 1$. Whereas [5] is restricted to the product case $b(x, y) = h(x)v(y)$, Part 2 [4] covers general b following [17]. This method combined with the previous iterated WIP and iterated moment estimates in [16] covers the range $\gamma \in (0, \frac{2}{5})$ for general b.

Combining (i) and (ii), we cover the optimal range $\gamma \in (0, \frac{1}{2})$ for general b.

In addition, we obtain the homogenization result $\hat{x}_n \to_{\lambda_n} X$ for a larger class of measures including the natural choice $\lambda_n \equiv \text{Leb}$.

Returning to families of nonuniformly expanding maps, in [20] we considered intermittent maps $T_n : [0,1] \to [0,1]$, $n \in \mathbb{N} \cup \{\infty\}$, as in (1.2) with parameters γ_n such that $\lim_{n\to\infty} \gamma_n = \gamma_\infty$. Homogenization results with $\lambda_n = \mu_n$ and $\lambda_n \equiv \mu_\infty$ were obtained in [20] for a restricted class of fast-slow systems with $b_n(x,y) = h_n(x)v_n(y)$, h_n exact, for $\gamma_\infty \in (0, \frac{1}{2})$. (For such systems, rough path theory was not needed.) By the results in this paper, combined with those in Part 2, we treat general b_n, again in the full range $\gamma_\infty \in (0, \frac{1}{2})$. Moreover, we cover a larger class of measures including $\lambda_n \equiv \text{Leb}$.

The remainder of Part 1 is organized as follows. In Sections 2 and 3, we consider nonuniformly expanding maps (fixed, and in uniform families [20], respectively). In particular, we obtain optimal estimates for iterated moments in Theorem 2.4 and the iterated WIP for families in Theorem 3.4. In Section 4, we consider examples including the intermittent maps in Example 1.1. The theory is extended to families of nonuniformly expanding semiflows in Section 5.

We refer to Part 2 for the parallel developments in rough path theory and a precise statement and proof of homogenization for the fast-slow systems (1.1).

Notation. For $a, b \in \mathbb{R}^d$, we define the outer product $a \otimes b = ab^T \in \mathbb{R}^{d\times d}$. For $J \in \mathbb{R}^{m\times n}$, we use the norm $|J| = (\sum_{i=1}^m \sum_{j=1}^n J_{ij}^2)^{1/2}$. Then $|a \otimes b| \le |a||b|$ for $a, b \in \mathbb{R}^d$.

For real-valued functions f, g, the integral $\int f\,dg$ denotes the Itô integral (where defined). Similarly, for vector-valued functions, $\int f \otimes dg$ denotes matrices of Itô integrals.

We use "big O" and $\ll$ notation interchangeably, writing $a_n = O(b_n)$ or $a_n \ll b_n$ if there are constants $C > 0$, $n_0 \ge 1$ such that $a_n \le C b_n$ for all $n \ge n_0$. As usual, $a_n = o(b_n)$ means that $\lim_{n\to\infty} a_n/b_n = 0$.

Let $v : \Lambda \to \mathbb{R}$ be an observable on a metric space (Λ, d_Λ) and let $\eta \in (0,1]$. Recall that $v : \Lambda \to \mathbb{R}$ is η-Hölder observable, $v \in C^\eta(\Lambda)$, if $\|v\|_\eta = |v|_\infty + \sup_{x\ne y} |v(x) - v(y)|/d_\Lambda(x,y)^\eta < \infty$. where $|v|_\infty = \sup_\Lambda |v|$. For $v = (v_1, \dots, v_d) : \Lambda \to \mathbb{R}^d$, $d \ge 1$, we write $v \in C^\eta(\Lambda, \mathbb{R}^d)$ if $v_j \in C^\eta(\Lambda)$ for $j = 1, \dots, d$, and set $\|v\|_\eta = \sum_{j=1}^d \|v_j\|_\eta$.

2. Nonuniformly expanding maps

In this section, we recall and extend the results in [20] for nonuniformly expanding maps.

Let (Λ, d_Λ) be a bounded metric space with finite Borel measure ρ and let $T : \Lambda \to \Lambda$ be a nonsingular transformation. Let $Y \subset \Lambda$ be a subset of positive measure, and let α be an at most countable measurable partition of Y. We suppose that there is an integrable *return time* function $\tau : Y \to \mathbb{Z}^+$, constant on each a with value $\tau(a) \ge 1$, and constants $\beta > 1$, $\eta \in (0,1]$, $C_1 \ge 1$ such that for each $a \in \alpha$,

(1) $F = T^\tau$ restricts to a (measure-theoretic) bijection from a onto Y.
(2) $d_\Lambda(Fx, Fy) \ge \beta d_\Lambda(x,y)$ for all $x, y \in a$.
(3) $d_\Lambda(T^\ell x, T^\ell y) \le C_1 d_\Lambda(Fx, Fy)$ for all $x, y \in a$, $0 \le \ell < \tau(a)$.
(4) $\zeta_0 = \frac{d\rho|_Y}{d\rho|_Y \circ F}$ satisfies $|\log \zeta_0(x) - \log \zeta_0(y)| \le C_1 d_\Lambda(Fx, Fy)^\eta$ for all $x, y \in a$.

Such a dynamical system $T : \Lambda \to \Lambda$ is called *nonuniformly expanding*. (It is not required that τ is the first return time to Y.) We refer to the induced map $F = T^\tau : Y \to Y$ as a *uniformly expanding map*. There is a unique absolutely continuous F-invariant probability measure μ_Y on Y and $d\mu_Y/d\rho \in L^\infty$.

Define the (one-sided) Young tower map [30], $f_\Delta : \Delta \to \Delta$,

$$\Delta = \{(y,\ell) \in Y \times \mathbb{Z} : 0 \le \ell \le \tau(y) - 1\}, \qquad f_\Delta(y,\ell) = \begin{cases} (y, \ell+1), & \ell \le \tau(y) - 2, \\ (Fy, 0), & \ell = \tau(y) - 1. \end{cases}$$

The projection $\pi_\Delta : \Delta \to \Lambda$, $\pi_\Delta(y,\ell) = T^\ell y$, defines a semiconjugacy from f_Δ to T. Define the ergodic f_Δ-invariant probability measure $\mu_\Delta = \mu_Y \times \{\text{counting}\}/\int_Y \tau\,d\mu_Y$ on Δ. Then $\mu = (\pi_\Delta)_*\mu_\Delta$ is an absolutely continuous ergodic T-invariant probability measure on Λ.

In this section, we work with a fixed nonuniformly expanding map $T : \Lambda \to \Lambda$ with induced map $F = T^\tau : Y \to Y$ where $\tau \in L^p(Y)$ for some $p \ge 2$,[1] and Young tower map $f_\Delta : \Delta \to \Delta$. The corresponding ergodic invariant proba-

[1] In [20], we considered this set up with $p \ge 1$. Since we have no new results for $p < 2$ beyond those already in [20], we restrict in this paper to the case $p \ge 2$.

bility measures are denoted μ, μ_Y and μ_Δ. Throughout, $|\ |_p$ denotes the L^p-norm on (Λ,μ), (Y,μ_Y) and (Δ,μ_Δ) as appropriate. Also, $\|\ \|_\eta$ denotes the Hölder norm on Λ and Y.

Although the map T is fixed, the dependence of various constants on T is important in later sections. To simplify the statement of results in this section, we denote by C various constants depending continuously on $\operatorname{diam}\Lambda$, C_1, β, η, p and $|\tau|_p$.

Let $L : L^1(\Delta)\to L^1(\Delta)$ and $P : L^1(Y)\to L^1(Y)$ denote the transfer operators corresponding to $f_\Delta:\Delta\to\Delta$ and $F:Y\to Y$. (So $\int_\Delta Lv\,w\,d\mu_\Delta=\int_\Delta v\,w\circ f_\Delta\,d\mu_\Delta$ for $v\in L^1(\Delta)$, $w\in L^\infty(\Delta)$, and $\int_Y Pv\,w\,d\mu_Y=\int_Y v\,w\circ F\,d\mu_Y$ for $v\in L^1(Y)$, $w\in L^\infty(Y)$.)

Let $\zeta=d\mu_Y/d\mu_Y\circ F$. Given $y\in Y$ and $a\in\alpha$, let y_a denote the unique $y_a\in a$ with $Fy_a=y$. Then we have the pointwise expression for L,

$$(Lv)(y,\ell)=\begin{cases}\sum_{a\in\alpha}\zeta(y_a)v\bigl(y_a,\tau(y_a)-1\bigr), & \ell=0,\\ v(y,\ell-1), & 1\le\ell\le\tau(y)-1.\end{cases}\tag{2.1}$$

2.1. *Martingale-coboundary decomposition*

Let $T:\Lambda\to\Lambda$ be a nonuniformly expanding map as above with return time $\tau\in L^p(Y)$, $p\ge 2$. Fix $d\ge 1$ and let $v\in C^\eta(\Lambda,\mathbb{R}^d)$ with $\int_\Lambda v\,d\mu=0$. Define the lifted observable $\phi=v\circ\pi_\Delta:\Delta\to\mathbb{R}^d$.

We recall the martingale-coboundary decomposition

$$\phi=m+\chi\circ f_\Delta-\chi,\quad m\in\ker L\tag{2.2}$$

from [20, Section 2.2], which is obtained as follows. First, define the *induced observable* $\phi':Y\to\mathbb{R}^d$ by $\phi'(y)=\sum_{\ell=0}^{\tau(y)-1}\phi(y,\ell)$. Next, define $\chi',m':Y\to\mathbb{R}^d$ by $\chi'=\sum_{k=1}^\infty P^k\phi'$ and $\phi'=m'+\chi'\circ F-\chi'$. Let

$$\chi(y,\ell)=\chi'(y)+\sum_{k=0}^{\ell-1}\phi(y,k)\quad\text{and}\quad m(y,\ell)=\begin{cases}0, & \ell\le\tau(y)-2,\\ m'(y), & \ell=\tau(y)-1.\end{cases}\tag{2.3}$$

By [20, Section 2.2], $\|\chi'\|_\eta\le C\|v\|_\eta$. Furthermore,

Proposition 2.1. *$|m|_p\le C\|v\|_\eta$, $|\chi|_{p-1}\le C\|v\|_\eta$ and for all $n\ge 1$, $q\ge p$,*

$$\Bigl|\max_{k\le n}\bigl|\chi\circ f_\Delta^k-\chi\bigr|\Bigr|_p\le C\|v\|_\eta\bigl(n^{1/q}+n^{1/p}|1_{\{\tau\ge n^{1/q}\}}\tau|_p\bigr).$$

(In particular, $|\max_{k\le n}|\chi\circ f_\Delta^k-\chi||_p\le C'\|v\|_\eta n^{1/p}$.)

Proof. See [20, Propositions 2.4 and 2.7]. □

Proposition 2.2. *$|L^n|m|^p|_\infty\le C\|v\|_\eta^p$ for all $n\ge 1$.*

Proof. Using (2.1) and the definition of m, we have

$$\bigl(L|m|^p\bigr)(y,\ell)=\begin{cases}\sum_{a\in\alpha}\zeta(y_a)\bigl|m'(y_a)\bigr|^p, & \ell=0,\\ 0, & 1\le\ell\le\tau(y)-1.\end{cases}$$

Note that $|m'|\le 2|\chi'|_\infty+|\phi'|\le 2|\chi'|_\infty+\tau|v|_\infty\ll\tau\|v\|_\eta$. Also $|1_a\zeta|_\infty\ll\mu_Y(a)$ (see for example [20, Proposition 2.2]). Hence

$$L|m|^p\ll\sum_{a\in\alpha}\mu_Y(a)\tau(a)^p\|v\|_\eta^p=|\tau|_p^p\|v\|_\eta^p\ll\|v\|_\eta^p.$$

Hence, $|L^n|m|^p|_\infty\le|L|m|^p|_\infty\ll\|v\|_\eta^p$ for all $n\ge 1$. □

Let $\breve\phi=UL(m\otimes m)-\int_\Delta m\otimes m\,d\mu_\Delta:\Delta\to\mathbb{R}^{d\times d}$ where U is the Koopman operator $U\phi=\phi\circ f_\Delta$.

Proposition 2.3. $|\max_{k\le n}|\sum_{j=0}^{k-1}\breve\phi\circ f_\Delta^j||_p\le Cn^{1/2}\|v\|_\eta^2$.

Proof. See [20, Corollary 3.2]. $\square$

2.2. *Moment estimates*

Given $v\in C^\eta(\Lambda,\mathbb{R}^d)$ with $\int_\Lambda v\,d\mu=0$, we define

$$S_n v=\sum_{0\le j<n} v\circ T^j,\qquad \mathbb{S}_n v=\sum_{0\le i<j<n}\big(v\circ T^i\big)\otimes\big(v\circ T^j\big). \tag{2.4}$$

The main result in this section is the estimate for iterated moments $|\max_{k\le n}|\mathbb{S}_k v||_{p-1}$ in the next theorem.

Theorem 2.4 (Iterated moments). *For all $n\ge 1$,*

$$\Big|\max_{k\le n}|S_k v|\Big|_{2(p-1)}\le Cn^{1/2}\|v\|_\eta,\qquad \Big|\max_{k\le n}|\mathbb{S}_k v|\Big|_{p-1}\le Cn\|v\|_\eta^2.$$

Proof. Since $p\ge 2$, the estimate for $S_n v$ is given in [20, Corollary 2.10]. It remains to prove the bound for $\mathbb{S}_n v$, equivalently $\mathbb{S}_n\phi=\sum_{0\le i<j<n}(\phi\circ f_\Delta^i)\otimes(\phi\circ f_\Delta^j)$. Using (2.2),

$$\mathbb{S}_n\phi=\sum_{0\le j<n}\big(\chi\circ f_\Delta^j-\chi\big)\otimes\big(\phi\circ f_\Delta^j\big)+\sum_{0\le i<j<n}\big(m\circ f_\Delta^i\big)\otimes\big(\phi\circ f_\Delta^j\big)=I_n+J_n.$$

By Proposition 2.1,

$$\Big|\max_{k\le n}|I_k|\Big|_{p-1}\le|\phi|_\infty\sum_{0\le j<n}\big|\chi\circ f_\Delta^j-\chi\big|_{p-1}\le 2n|v|_\infty|\chi|_{p-1}\ll n\|v\|_\eta^2.$$

Next, $J_n=\sum_{i=0}^{n-2}(m\circ f_\Delta^i)\otimes((\sum_{j=1}^{n-i-1}\phi\circ f_\Delta^j)\circ f_\Delta^i)=\sum_{\ell=2}^n X_{n,\ell}$, where

$$X_{n,\ell}=\left(m\otimes\sum_{j=1}^{\ell-1}\phi\circ f_\Delta^j\right)\circ f_\Delta^{n-\ell}=\big(m\otimes\big\{(S_{\ell-1}\phi)\circ f_\Delta\big\}\big)\circ f_\Delta^{n-\ell}.$$

Now, $|S_n\phi|_p\le|S_n\phi|_{2(p-1)}\ll n^{1/2}\|v\|_\eta$ since $p\ge 2$. Hence by Proposition 2.2,

$$\begin{aligned}|X_{n,\ell}|_p^p&\le\int_\Delta|m|^p\big|(S_{\ell-1}\phi)\circ f_\Delta\big|^p\,d\mu_\Delta=\int_\Delta L|m|^p|S_{\ell-1}\phi|^p\,d\mu_\Delta\\&\le\big|L|m|^p\big|_\infty|S_{\ell-1}\phi|_p^p\ll\ell^{p/2}\|v\|_\eta^{2p}\ll n^{p/2}\|v\|_\eta^{2p},\end{aligned}$$

so $|X_{n,\ell}|_p^2\ll n\|v\|_\eta^4$.

Let $\mathcal{M}$ denote the underlying σ-algebra on (Δ,μ_Δ) and define $\mathcal{G}_{n,\ell}=f_\Delta^{-(n-\ell)}\mathcal{M}$, $2\le\ell\le n$. Since $Lm=0$,

$$L^{n+1-\ell}X_{n,\ell}=L\big(m\otimes\big\{(S_{\ell-1}\phi)\circ f_\Delta\big\}\big)=Lm\otimes(S_{\ell-1}\phi)=0$$

for all ℓ. It follows (cf. [20, Proposition 2.9]) that $\{X_{n,\ell},\mathcal{G}_\ell;\ 2\le\ell\le n\}$ is a sequence of martingale differences. Working coordinatewise, by Burkholder's inequality [2],

$$\Big|\max_{k\le n}|J_k|\Big|_p^2\ll\left|\left(\sum_{\ell=2}^n X_{n,\ell}^2\right)^{1/2}\right|_p^2=\left|\sum_{\ell=2}^n X_{n,\ell}^2\right|_{p/2}\le\sum_{\ell=2}^n\big|X_{n,\ell}^2\big|_{p/2}=\sum_{\ell=2}^n|X_{n,\ell}|_p^2\ll n^2\|v\|_\eta^4,$$

and so $|\max_{k\le n}|J_k||_p\ll n\|v\|_\eta^2$. This completes the proof. $\square$

Moments on Δ. It is standard that the moment estimates for $v:\Lambda\to\mathbb{R}^d$ follow from corresponding estimates for lifted observables $\phi=v\circ\pi_\Delta:\Delta\to\mathbb{R}^d$. In Proposition 2.8, we need such an estimate for an observable on Δ that need not be the lift of an observable on Λ. Hence, we recall now how to derive moment estimates on Δ.

We define a metric on Δ based on the metric d_Λ on Y:

$$d_\Delta\big((y,\ell),\big(y',\ell'\big)\big)=\begin{cases} d_\Lambda\big(Fy,Fy'\big) & \ell=\ell' \text{ and } y,\,y' \text{ are in the same } a\in\alpha,\\ \operatorname{diam}\Lambda & \text{else.}\end{cases} \tag{2.5}$$

Remark 2.5. In (2.5), if we use a symbolic metric on Y in place of d_Λ, then d_Δ is the usual symbolic metric on Δ.

As usual, $\|\ \|_\eta$ denotes the Hölder norm on Δ. From the definition of nonuniformly expanding map, $d_\Lambda(T^\ell y,T^{\ell'}y')\le C_1 d_\Delta((y,\ell),(y',\ell'))$; hence if $v:\Lambda\to\mathbb{R}^d$ is Hölder then so is its lift $\phi=v\circ\pi:\Delta\to\mathbb{R}^d$. Moreover, f_Δ is itself a nonuniformly expanding map on (Δ,d_Δ) with the same constants as T, so Theorem 2.4 yields:

Lemma 2.6. *Let* $\phi:\Delta\to\mathbb{R}^d$ *with* $\|\phi\|_\eta<\infty$, *such that* $\int_\Delta\phi\,d\mu_\Delta=0$. *Define* $S_n\phi=\sum_{0\le j<n}\phi\circ f_\Delta^j$ *and* $\mathbb{S}_n\phi=\sum_{0\le i<j<n}(\phi\circ f_\Delta^i)\otimes(\phi\circ f_\Delta^j)$. *Then*

$$\Big|\max_{k\le n}|S_k\phi|\Big|_{2(p-1)}\le Cn^{1/2}\|\phi\|_\eta \qquad \textit{and} \qquad \Big|\max_{k\le n}|\mathbb{S}_k\phi|\Big|_{p-1}\le Cn\|\phi\|_\eta^2.$$

2.3. *Drift and diffusion coefficients*

Let S_nv, $\mathbb{S}_nv$ be as in (2.4) and define $\Sigma,\ E\in\mathbb{R}^{d\times d}$,

$$\Sigma=\lim_{n\to\infty}\frac1n\int_\Lambda S_nv\otimes S_nv\,d\mu,\qquad E=\lim_{n\to\infty}\frac1n\int_\Lambda\mathbb{S}_nv\,d\mu. \tag{2.6}$$

Proposition 2.7. *The limits in* (2.6) *exist and are given by*

$$\Sigma=\int_\Delta m\otimes m\,d\mu_\Delta,\qquad E=\int_\Delta\chi\otimes\phi\,d\mu_\Delta.$$

Moreover, for all $n\ge1$,

$$\bigg|\frac1n\int_\Lambda S_nv\otimes S_nv\,d\mu-\Sigma\bigg|\le C\|v\|_\eta^2n^{1/p-1/2},\qquad \bigg|\frac1n\int_\Lambda\mathbb{S}_nv\,d\mu-E\bigg|\le C\|v\|_\eta^2\big(n^{-1/2}+n^{-(p-2)}\big).$$

Proof. The limit for Σ is obtained in [20, Corollary 2.12]. The proof of [20, Corollary 2.12] contains the estimate

$$\bigg|\frac1n\int_\Lambda S_nv\otimes S_nv\,d\mu-\Sigma\bigg|\ll n^{-1/2}\|v\|_\eta\big|\chi\circ f_\Delta^n-\chi\big|_p,$$

so the convergence rate for Σ follows from Proposition 2.1.

Next, we note that $|\chi\otimes(n^{-1}S_n\phi)|_1\le|\chi|_1|v|_\infty<\infty$ since $\chi\in L^{p-1}\subset L^1$. Also $n^{-1}S_n\phi\to0$ almost surely by the pointwise ergodic theorem. Hence it follows from the dominated convergence theorem that

$$\lim_{n\to\infty}\frac1n\int_\Delta\chi\otimes S_n\phi\,d\mu_\Delta=0.$$

Since $Lm=0$, we have $\int_\Delta(m\circ f_\Delta^i)\otimes(\phi\circ f_\Delta^j)\,d\mu_\Delta=0$ for all $i<j$. Hence by (2.2),

$$\int_\Delta\mathbb{S}_n\phi\,d\mu_\Delta=\int_\Delta\sum_{j=1}^{n-1}(\chi\circ f_\Delta^j-\chi)\otimes(\phi\circ f_\Delta^j)\,d\mu_\Delta=n\int_\Delta\chi\otimes\phi\,d\mu_\Delta-\int_\Delta\chi\otimes S_n\phi\,d\mu_\Delta.$$

It follows that $E=\lim_{n\to\infty}\frac1n\int_\Delta\mathbb{S}_n\phi\,d\mu_\Delta=\int_\Delta\chi\otimes\phi\,d\mu_\Delta$.

To obtain the convergence rate for E, we may suppose without loss that $p \in (2, \frac{5}{2}]$. Write $(p-1)^{-1} + q^{-1} = 1$ where $q \in [3, \infty)$. It follows from Hölder's inequality and Proposition 2.1 that $|\chi \otimes S_n\phi|_1 \le |\chi|_{p-1}|S_n\phi|_q \ll \|v\|_\eta |S_n\phi|_q$. By Theorem 2.4,

$$\int_\Delta |S_n\phi|^q \, d\mu_\Delta \le |S_n\phi|_\infty^{q-2(p-1)} \int_\Delta |S_n\phi|^{2(p-1)} \, d\mu_\Delta \ll \|v\|_\eta^q n^{q-2(p-1)} n^{p-1} = \|v\|_\eta^q n^{q-(p-1)}.$$

Hence $|\chi \otimes S_n\phi|_1 \ll \|v\|_\eta^2 n^{1-(p-1)/q} = \|v\|_\eta^2 n^{3-p}$ and the result follows. $\square$

For later use, we record the following result:

Proposition 2.8. *For $n \ge 1$,*

$$\left| \max_{k \le n} \left| \sum_{j=0}^{k-1} \left((\chi \otimes \phi) \circ f_\Delta^j - E \right) \right| \right|_1 \le C \|v\|_\eta^2 \left(n^{3/4} + n \int_Y \tau^2 1_{\{\tau \ge n^{1/4}\}} \, d\mu_Y \right).$$

Proof. Fix $q > 0$, and define

$$\psi : \Delta \to \mathbb{R}^{d \times d}, \qquad \psi(y, \ell) = (\chi \otimes \phi)(y, \ell) 1_{\{\tau(y) \ge q\}}.$$

By (2.3), $|\chi \otimes \phi|(y, \ell) \le (|\chi'|_\infty + \ell |v|_\infty)|v|_\infty \ll \|v\|_\eta^2 \tau(y)$. Hence

$$|\psi|_1 \ll \|v\|_\eta^2 \int_\Delta \tau(y) 1_{\{\tau(y) \ge q\}} \, d\mu_\Delta(y, \ell) \le \|v\|_\eta^2 \int_Y \tau^2 1_{\{\tau \ge q\}} \, d\mu_Y. \tag{2.7}$$

Write $\chi \otimes \phi - E = U + V$ where

$$U = \psi - \int_\Delta \psi \, d\mu_\Delta, \qquad V = \chi \otimes \phi - \psi - \int_\Delta (\chi \otimes \phi - \psi) \, d\mu_\Delta.$$

By (2.7),

$$\left| \max_{k \le n} |S_k U| \right|_1 \le \left| \sum_{j < n} \left| U \circ f_\Delta^j \right| \right|_1 \le n|U|_1 \le 2n|\psi|_1 \ll n\|v\|_\eta^2 \int_Y \tau^2 1_{\{\tau \ge q\}} \, d\mu_Y.$$

Next, $V(y, \ell) = (\chi \otimes \phi)(y, \ell) 1_{\{\tau < q\}} - \int_\Delta (\chi \otimes \phi) 1_{\{\tau < q\}} \, d\mu_\Delta$. By (2.3),

$$\begin{aligned} \left|\chi(y, \ell) - \chi(y', \ell)\right| &\le \left|\chi'(y) - \chi'(y')\right| + \sum_{k=0}^{\ell-1} \left|\phi(y, k) - \phi(y', k)\right| \\ &\ll \|v\|_\eta d_\Lambda(y, y')^\eta + \|\phi\|_\eta \sum_{k=0}^{\ell-1} d_\Delta\big((y, k), (y', k)\big)^\eta \\ &\ll \|v\|_\eta \tau(y) d_\Delta\big((y, \ell), (y', \ell)\big)^\eta. \end{aligned}$$

Here we used that $\|\phi\|_\eta \ll \|v\|_\eta$ and $d_\Lambda(y, y') \le d_\Delta((y, \ell), (y', \ell))$.

A simpler calculation shows that $|\chi(y, \ell)| \ll \tau(y)\|v\|_\eta$. It follows that $\|V\|_\eta \ll q\|v\|_\eta^2$. By Lemma 2.6,

$$\left| \max_{k \le n} |S_k V| \right|_1 \ll \|V\|_\eta n^{1/2} \ll q\|v\|_\eta^2 n^{1/2}.$$

The result follows by taking $q = n^{1/4}$. $\square$

Remark 2.9. Nonuniformly expanding maps are mixing up to a finite cycle. When they are mixing (in particular, if $\gcd\{\tau(a) : a \in \alpha\} = 1$), then we have formulas of Green-Kubo type for Σ and E in (2.6), namely

$$\Sigma = \int_\Lambda v \otimes v \, d\mu + \sum_{n=1}^\infty \int_\Lambda \left(v \otimes (v \circ T^n) + (v \circ T^n) \otimes v \right) d\mu, \qquad E = \sum_{n=1}^\infty \int_\Lambda v \otimes (v \circ T^n) \, d\mu.$$

3. Families of nonuniformly expanding maps

In this section, we prove the iterated WIP and iterated moment estimates for uniform families of nonuniformly expanding maps.

3.1. *Iterated WIP and iterated moments*

Throughout, $T_n : \Lambda_n \to \Lambda_n$, $n \geq 1$, is a family of nonuniformly expanding maps as in Section 2 with absolutely continuous ergodic T_n-invariant probability measures μ_n. To each T_n there is associated an induced uniformly expanding map $F_n : Y_n \to Y_n$ with ergodic invariant probability measure μ_{Y_n} and a return time $\tau_n \in L^p(Y_n)$ where $p \geq 2$.

We assume that T_n is a *uniform family of order* $p \geq 2$ in the sense of [20]. This means that the expansion and distortion constants $C_1 \geq 1$, $\beta > 1$, $\eta \in (0, 1]$ for the induced maps F_n can be chosen independent of n and that the family $\{\tau_n^p\}$ is uniformly integrable on (Y_n, μ_{Y_n}), i.e. $\sup_n \int_{Y_n} \tau_n^p 1_{\{\tau_n \geq q\}}\, d\mu_{Y_n} \to 0$ as $q \to \infty$. Let $v_n : \Lambda_n \to \mathbb{R}^d$, $n \geq 1$, be a family of observables with $\sup_{n\geq 1} \|v_n\|_\eta < \infty$ and $\int_{\Lambda_n} v_n\, d\mu_n = 0$.

Let $f_{\Delta_n} : \Delta_n \to \Delta_n$ be the corresponding family of Young tower maps, with invariant probability measures $\mu_{\Delta,n}$ and semiconjugacies $\pi_{\Delta_n} : \Delta_n \to \Lambda_n$. In particular, $\mu_n = \pi_{\Delta_n *}\mu_{\Delta_n}$.

Define the lifted observables $\phi_n = v_n \circ \pi_{\Delta_n} : \Delta_n \to \mathbb{R}^d$. By Section 2, we have the martingale-coboundary decompositions

$$\phi_n = m_n + \chi_n \circ f_{\Delta_n} - \chi_n.$$

Proposition 3.1. *The family* $\{|m_n|^2;\ n \geq 1\}$ *is uniformly integrable on* (Λ_n, μ_n).

Proof. See [20, Proposition 4.3]. □

Abusing notation from Section 2 slightly, we define

$$S_k v_n = \sum_{0\leq j<k} v_n \circ T_n^j, \qquad \mathbb{S}_k v_n = \sum_{0\leq i<j<k} \big(v_n \circ T_n^i\big) \otimes \big(v_n \circ T_n^j\big).$$

By uniformity, the constants C in Section 2 can be chosen independently of n. Hence the next result is an immediate consequence of Theorem 2.4:

Corollary 3.2 (Iterated moments). *For all* $n \geq 1$,

$$\Big|\max_{k\leq n} |S_k v_n|\Big|_{L^{2(p-1)}(\mu_n)} \leq C n^{1/2} \|v_n\|_\eta, \qquad \Big|\max_{k\leq n} |\mathbb{S}_n v_n|\Big|_{L^{p-1}(\mu_n)} \leq C n \|v_n\|_\eta^2.$$

Write

$$\Sigma_n = \lim_{k\to\infty} \frac{1}{k} \int_{\Delta_n} S_k v_n \otimes S_k v_n\, d\mu_{\Delta_n}, \qquad E_n = \lim_{k\to\infty} \frac{1}{k} \int_{\Delta_n} \mathbb{S}_k v_n\, d\mu_{\Delta_n}. \tag{3.1}$$

Corollary 3.3. *The limits in* (3.1) *exist for each n and are given by*

$$\Sigma_n = \int_{\Delta_n} m_n \otimes m_n\, d\mu_{\Delta_n}, \qquad E_n = \int_{\Delta_n} \chi_n \otimes \phi_n\, d\mu_{\Delta_n}.$$

For $p > 2$, *the convergence is uniform in* n.

Proof. This follows from Proposition 2.7. □

Define $W_n \in D([0, 1], \mathbb{R}^d)$, $\mathbb{W}_n \in D([0, 1], \mathbb{R}^{d\times d})$ by

$$W_n(t) = \frac{1}{\sqrt{n}} \sum_{0\leq j<nt} v_n \circ T_n^j, \qquad \mathbb{W}_n(t) = \frac{1}{n} \sum_{0\leq i<j<nt} \big(v_n \circ T_n^i\big) \otimes \big(v_n \circ T_n^j\big).$$

We can now state the main result of this section.

Theorem 3.4 (Iterated WIP). *Suppose that* $\lim_{n\to\infty}\Sigma_n=\Sigma$ *and* $\lim_{n\to\infty}E_n=E$. *Then*

$$(W_n,\mathbb{W}_n)\to_{\mu_n}(W,\mathbb{W})\quad \text{as } n\to\infty \text{ in } D\big([0,1],\mathbb{R}^d\times\mathbb{R}^{d\times d}\big),$$

where W is d-dimensional Brownian motion with covariance matrix Σ and $\mathbb{W}(t)=\int_0^t W\otimes dW+Et$. (As always in this paper, $\int_0^t W\otimes dW$ denotes the Itô integral.)

To prove Theorem 3.4, it is equivalent to show that $(Q_n,\mathbb{Q}_n)\to_{\mu_{\Delta_n}}(W,\mathbb{W})$ where

$$Q_n(t)=\frac{1}{\sqrt{n}}\sum_{0\le j<nt}\phi_n\circ f_{\Delta_n}^j,\qquad \mathbb{Q}_n(t)=\frac{1}{n}\sum_{0\le i<j<nt}\big(\phi_n\circ f_{\Delta_n}^i\big)\otimes\big(\phi_n\circ f_{\Delta_n}^j\big).$$

Define also $\mathbb{M}_n(t)=\frac{1}{n}\sum_{0\le i<j<nt}(m_n\circ f_{\Delta_n}^i)\otimes(\phi_n\circ f_{\Delta_n}^j)$.

Lemma 3.5. *Suppose that* $\lim_{n\to\infty}\Sigma_n=\Sigma$. *Then* $(Q_n,\mathbb{M}_n)\to_{\mu_{\Delta_n}}(W,\mathbb{M})$ *in* $D([0,1],\mathbb{R}^d\times\mathbb{R}^{d\times d})$, *where* $\mathbb{M}(t)=\int_0^t W\otimes dW$.

Proof. We verify the hypotheses of Theorem A.1. Hypothesis (a) holds by Proposition 3.1. Next, by Proposition 2.1, writing $|\ |_2$ as shorthand for $|\ |_{L^2(\mu_{\Delta_n})}$ and $|\ |_{L^2(\mu_{Y_n})}$,

$$\Big|\max_{k\le n}\Big|\sum_{0\le j<k}(\phi_n-m_n)\circ f_{\Delta_n}^j\Big|\Big|_2=\Big|\max_{k\le n}\big|\chi_n\circ f_{\Delta_n}^k-\chi_n\big|\Big|_2\le C\|v_n\|_\eta\big(n^{1/4}+n^{1/2}|1_{\{\tau_n\ge n^{1/4}\}}\tau_n|_2\big).$$

Since the family $\{\tau_n^2\}$ is uniformly integrable, $n^{-1/2}|\max_{k\le n}|\sum_{0\le j<k}(\phi_n-m_n)\circ f_{\Delta_n}^j||_2\to 0$, verifying hypothesis (b).

Finally, by Proposition 2.3, for $t\in[0,1]$,

$$\Big|\sum_{0\le j<nt}U_nL_n(m_n\otimes m_n)\circ f_{\Delta_n}^j-[nt]\Sigma_n\Big|_2\le Cn^{1/2}\|v_n\|_\eta^2.$$

Hypothesis (c) follows. $\square$

Proof of Theorem 3.4. Write $\mathbb{Q}_n(t)-\mathbb{M}_n(t)=A_n(t)-B_n(t)$, where

$$A_n(t)=\frac{1}{n}\sum_{0\le j<nt}(\chi_n\otimes\phi_n)\circ f_{\Delta_n}^j,\qquad B_n(t)=\frac{1}{n}\chi_n\otimes\sum_{0\le j<nt}\phi_n\circ f_{\Delta_n}^j.$$

By Lemma 3.5, it suffices to show that $\sup_{t\in[0,1]}|A_n(t)-B_n(t)-tE_n|\to_{\mu_{\Delta_n}}0$.

Write $|\ |_q=|\ |_{L^q(\mu_{\Delta_n})}$. Since the family $\{\tau_n^2\}$ is uniformly integrable, it follows from Proposition 2.8 that

$$\Big|\sup_{t\in[0,1]}\big|A_n(t)-tE_n\big|\Big|_1\ll\|v_n\|_\eta^2\bigg(n^{-1/4}+\int_{Y_n}\tau_n^2 1_{\{\tau_n\ge n^{1/4}\}}\,d\mu_{Y_n}\bigg)\to 0.$$

Next, $\sup_{t\in[0,1]}|B_n(t)|\le|\chi_n|B_n'$ where $B_n'=n^{-1}\max_{k\le n}|\sum_{j=0}^{k-1}\phi_n\circ f_{\Delta_n}^j|$. By Theorem 2.4,

$$\big|B_n'\big|_2\ll n^{-1/2}\|v_n\|_\eta\ll n^{-1/2},$$

so $B_n'\to_{\mu_{\Delta_n}}0$. Also, by Proposition 2.1, $|\chi_n|_1\ll\|v_n\|_\eta=O(1)$. It follows that $\sup_{t\in[0,1]}|B_n(t)|\to_{\mu_{\Delta_n}}0$. $\square$

Corollary 3.6. *Suppose that* $\lim_{n\to\infty}\Sigma_n=\Sigma$ *and* $\lim_{n\to\infty}E_n=E$. *Let λ_n be a family of probability measures on Λ_n absolutely continuous with respect to μ_n. Suppose that the densities $\rho_n=d\lambda_n/d\mu_n$ satisfy $\sup_n\int\rho_n^{1+\delta}\,d\mu_n<\infty$ for some $\delta>0$ and that* $\inf_{N\ge1}\limsup_{n\to\infty}\int|\frac{1}{N}\sum_{j=0}^{N-1}\rho_n\circ T_n^j-1|\,d\mu_n=0$.

Then $(W_n,\mathbb{W}_n)\to_{\lambda_n}(W,\mathbb{W})$ as $n\to\infty$ in $D([0,1],\mathbb{R}^d\times\mathbb{R}^{d\times d})$ where W is d-dimensional Brownian motion with covariance matrix Σ and $\mathbb{W}(t)=\int_0^t W\otimes dW+Et$.

Proof. We verify the conditions of Remark B.2 with $\mathcal{B} = D([0,1], \mathbb{R}^d \times \mathbb{R}^{d\times d})$ and $d_{\mathcal{B}}(u,v) = \sup_{t\in[0,1]} |u(t) - v(t)|$. The result then follows from Theorem 3.4.

Conditions (S1) and (S5) of Remark B.2 hold by assumption so it remains to verify (S4). Define the sequence of random variables

$$R_n : \Lambda \to D\big([0,1], \mathbb{R}^d \times \mathbb{R}^{d\times d}\big), \qquad R_n = (W_n, \mathbb{W}_n).$$

We have $\sup_{t\in[0,1]} |(W_n \circ T_n)(t) - W_n(t)| \le 2n^{-1/2}|v_n|_\infty$. Also,

$$(\mathbb{W}_n \circ T_n)(t) - \mathbb{W}_n(t) = n^{-1} \sum_{1\le i<nt} \big(v_n \circ T_n^i\big) \otimes \big(v_n \circ T_n^n\big) - n^{-1} \sum_{1\le j<nt} v_n \otimes \big(v_n \circ T_n^j\big).$$

Write $|\ |_1 = |\ |_{L^1(\mu_n)}$. By Corollary 3.2,

$$\Big|\sup_{[0,1]} |\mathbb{W}_n \circ T_n - \mathbb{W}_n|\Big|_1 \le 4n^{-1}|v_n|_\infty \Big|\max_{k\le n} |S_k v_n|\Big|_1 \ll n^{-1/2}\|v_n\|_\eta^2.$$

Hence

$$\big|d_{\mathcal{B}}(R_n \circ T_n, R_n)\big|_1 \ll n^{-1/2}\big(|v_n|_\infty + \|v_n\|_\eta^2\big),$$

verifying condition (S4). □

Remark 3.7. By Corollary 3.2, $|N^{-1}\sum_{j=0}^{k-1} \rho_n \circ T_n^j - 1| \ll N^{-1/2}\|\rho_n\|_\eta$. Hence a sufficient condition for the assumptions on ρ_n in Corollary 3.6 is that $\sup_n \|\rho_n\|_\eta < \infty$.

3.2. *Existence of limits for Σ_n and E_n*

Theorem 3.4 and Corollary 3.6 establish the iterated WIP subject to the existence of $\lim_{n\to\infty} \Sigma_n$ and $\lim_{n\to\infty} E_n$. In this subsection, we describe mild conditions under which these limits exist.

Let (Λ, d_Λ) be a bounded metric space with finite Borel measure ρ. We assume that T_n, $n \in \mathbb{N} \cup \{\infty\}$, is a uniform family as in Section 3.1 but now of order $p > 2$ and defined on the common space Λ. In particular, each T_n is a nonuniformly expanding map as in Section 2, with absolutely continuous ergodic T_n-invariant Borel probability measures μ_n. We suppose that μ_n is *statistically stable*: $\mu_n \to_w \mu_\infty$ as $n \to \infty$. Moreover, we require that

$$\int_\Lambda \big(v \circ T_\infty^j\big)\big(w \circ T_\infty^k\big)\,(d\mu_n - d\mu_\infty) \to 0 \quad \text{and} \quad T_n^j \to_{\mu_n} T_\infty^j \quad \text{as } n \to \infty \tag{3.2}$$

for all $j, k \ge 0$ and all $v, w \in C^\eta(\Lambda)$. (The second part of condition (3.2) means that $\mu_n\{y \in \Lambda : d_\Lambda(T_n^j y, T_\infty^j y) > a\} \to 0$ for all $a > 0$.)

Let $v_n \in C^\eta(\Lambda, \mathbb{R}^d)$, $n \in \mathbb{N} \cup \{\infty\}$, with $\int_\Lambda v_n\,d\mu_n = 0$. We assume that $\lim_{n\to\infty} \|v_n - v_\infty\|_\eta = 0$.

Lemma 3.8. *Define $S_n v_n = \sum_{0\le j<n} v_n \circ T_n^j$ and $\mathbb{S}_n v_n = \sum_{0\le i<j<n} (v_n \circ T_n^i) \otimes (v_n \circ T_n^j)$. Then the limits*

$$\Sigma_n = \lim_{n\to\infty} \frac{1}{n} \int_\Lambda S_n v_n \otimes S_n v_n\,d\mu_n, \qquad E_n = \lim_{n\to\infty} \frac{1}{n} \int_\Lambda \mathbb{S}_n v_n\,d\mu_n,$$

exist for all $n \in \mathbb{N} \cup \{\infty\}$, and $\lim_{n\to\infty} \Sigma_n = \Sigma_\infty$, $\lim_{n\to\infty} E_n = E_\infty$.

Proof. The limits Σ_n and E_n exist for n fixed by Corollary 3.3. We refer to [20, Proposition 7.6] for the proof that $\lim_{n\to\infty} \Sigma_n = \Sigma_\infty$. Here we show that $\lim_{n\to\infty} E_n = E_\infty$.

Write $J_{n,n} = \int_\Lambda \mathbb{S}_n v_n\,d\mu_n$. Let $\delta > 0$. By Corollary 3.3, there exists $N \ge 1$ such that $|N^{-1}J_{n,N} - E_n| < \delta$ for all $n \ge 1$. Hence,

$$|E_n - E_\infty| < 2\delta + N^{-1}|J_{n,N} - J_{0,N}|. \tag{3.3}$$

Next

$$J_{n,N} - J_{0,N} = \int_\Lambda (\mathbb{S}_N v_n - \mathbb{S}_N v_\infty)\,d\mu_n + \int_\Lambda \mathbb{S}_N v_\infty\,(d\mu_n - d\mu_\infty).$$

By condition (3.2), $\lim_{n\to\infty}\int_\Lambda \mathbb{S}_N v_\infty\,(d\mu_n - d\mu_\infty) = 0$. Also,

$$|\mathbb{S}_N v_n - \mathbb{S}_N v_\infty| \le \sum_{0\le i<j<N} \big|(v_n\circ T_n^i)\otimes(v_n\circ T_n^j) - (v_\infty\circ T_\infty^i)\otimes(v_\infty\circ T_\infty^j)\big| \le A_1 + A_2,$$

where

$$A_1 = \sum_{0\le i<j<N} \big|(v_n\circ T_n^i)\otimes(v_n\circ T_n^j) - (v_\infty\circ T_n^i)\otimes(v_\infty\circ T_n^j)\big|,$$

$$A_2 = \sum_{0\le i<j<N} \big|(v_\infty\circ T_n^i)\otimes(v_\infty\circ T_n^j) - (v_\infty\circ T_\infty^i)\otimes(v_\infty\circ T_\infty^j)\big|.$$

Now,

$$A_1 \le \sum_{0\le i<j<N} \big(|v_n|\circ T_n^i |v_n - v_\infty|\circ T_n^j + |v_n - v_\infty|\circ T_n^i |v_\infty|\circ T_n^j\big) \le N^2\big(|v_n|_\infty + |v_\infty|_\infty\big)|v_n - v_\infty|_\infty.$$

Also,

$$A_2 \le N|v_\infty|_\infty \sum_{0\le j<N} \big|v_\infty\circ T_n^j - v_\infty\circ T_\infty^j\big| \le N|v_\infty|_\infty |v_\infty|_\eta g_{n,N},$$

where $g_{n,N}(y) = \sum_{j=0}^{N-1} d_\Lambda(T_n^j y, T_\infty^j y)^\eta$. By the assumption on v_n and condition (3.2), we obtain that $\lim_{n\to\infty}|\mathbb{S}_N v_n - \mathbb{S}_N v_\infty|_{L^1(\mu_n)} = 0$. Hence $\lim_{n\to\infty} J_{n,N} = J_{0,N}$ and so $\limsup_{n\to\infty}|E_n - E_\infty| \le 2\delta$ by (3.3). Since δ is arbitrary, the result follows. □

3.3. *Auxiliary properties*

Our results so far in this section on the iterated WIP and control of iterated moments verify the main hypotheses required to apply rough path theory in Part 2. However, there remain two relatively minor hypotheses, Assumption 2.11 and Assumption 2.12(ii)(a) in [4] which we address now. We continue to assume the set up of Section 3.2 though we require weaker regularity assumptions on v_n: it suffices that $v_n \in L^\infty(\Lambda, \mathbb{R}^d)$, $n \ge 1$, and $v_\infty \in C^\eta(\Lambda, \mathbb{R}^d)$ with $\lim_{n\to\infty}|v_n - v_\infty|_\infty = 0$. Fix $t \in [0,1]$, and define $V_n = n^{-1}\sum_{j=0}^{[nt]-1} v_n\circ T_n^j$.

Proposition 3.9.

(a) $\lim_{n\to\infty}|V_n - t\int_\Lambda v_\infty\,d\mu_\infty|_{L^p(\mu_n)} = 0$.
(b) $\lim_{n\to\infty}\int_\Lambda v_n\otimes v_n\,d\mu_n = \int_\Lambda v_\infty\otimes v_\infty\,d\mu_\infty$.

Proof. (a) Define $U_n = n^{-1}\sum_{j=0}^{[nt]-1} v_\infty\circ T_n^j$. Then $|V_n - U_n|_\infty \le t|v_n - v_\infty|_\infty \to 0$. Since v_∞ is Hölder, it follows from Corollary 3.2 that $|U_n - t\int_\Lambda v_\infty\,d\mu_n|_{L^p(\mu_n)} \ll \|v_\infty\|_\eta n^{-1/2}$. By (3.2), $\int_\Lambda v_\infty\,d\mu_n \to \int_\Lambda v_\infty\,d\mu_\infty$ and the result follows.

(b) We have $\int_\Lambda |v_n\otimes v_n - v_\infty\otimes v_\infty|\,d\mu_n \le (|v_n|_\infty + |v_\infty|_\infty)|v_n - v_\infty|_\infty \to 0$. Also, $\int_\Lambda v_\infty\otimes v_\infty(d\mu_n - d\mu_\infty) \to 0$ by (3.2). □

4. Examples

In this section, we consider examples of nonuniformly expanding dynamics, including families of intermittent maps (1.2) discussed in the introduction, covered by the results in this paper.

4.1. *Application to intermittent maps*

Fix a family of intermittent maps $T_n : [0,1] \to [0,1]$, $n \in \mathbb{N}\cup\{\infty\}$, as in (1.2) with parameters $\gamma_n \in (0, \frac{1}{2})$ such that $\lim_{n\to\infty}\gamma_n = \gamma_\infty$. By [19, Example 5.1], T_n is a uniform family of nonuniformly expanding maps of order p for all $p \in (2, \gamma_\infty^{-1})$. By [1,18], μ_n is strongly statistically stable. That is, the densities $h_n = d\mu_n/d\mathrm{Leb}$ satisfy $\lim_{n\to\infty}\int_\Lambda |h_n - h_\infty|\,d\mathrm{Leb} = 0$. Using this property, conditions (3.2) are easily verified.

Hence our main results on control of iterated moments (Corollary 3.2) and the iterated WIP (Theorem 3.4 and Lemma 3.8) hold for families of intermittent maps T_n and Hölder observables $v_n : [0,1] \to \mathbb{R}^d$ with $\int v_n\,d\mu_n = 0$ and

$\lim_{n\to\infty}\|v_n - v_\infty\|_\eta = 0$. Also the auxiliary properties in Proposition 3.9 are satisfied. This leads via Part 2 to homogenization results $\hat{x}_n \to_{\mu_n} X$ for fast-slow systems (1.1). Since $\mu_n(\hat{x}_n \in B) - \mu_\infty(\hat{x}_n \in B) = \int_\Lambda 1_{\{\hat{x}_n\in B\}}(h_n - h_\infty)\,d\mathrm{Leb}$ for suitable subsets $B \subset D([0,1],\mathbb{R}^d)$, it follows from strong statistical stability that $\hat{x}_n \to_{\mu_\infty} X$.

In the remainder of this subsection, we show that all our results remain valid when μ_n is replaced by Lebesgue measure. (We continue to assume that the observables v_n are centered with respect to μ_n, so $\int v_n\,d\mu_n = 0$.)

The densities $h_n = d\mu_n/d\mathrm{Leb}$ are uniformly bounded below (see [22, Lemma 2.4] for explicit lower bounds). Hence it is immediate that the moment estimates in Corollary 3.2 hold with μ_n changed to Leb. Since Leb is not invariant, the following nonstationary version of the moment estimates is required in Part 2:

Proposition 4.1. *$|\sum_{\ell\le j<k} v_n\circ T_n^j|_{L^{2(p-1)}(\mathrm{Leb})} \le C(k-\ell)^{1/2}\|v_n\|_\eta$ and $|\sum_{\ell\le i<j<k}(v_n\circ T_n^i)\otimes(v_n\circ T_n^j)|_{L^{p-1}(\mathrm{Leb})} \le C(k-\ell)\|v_n\|_\eta^2$ for all $0\le \ell<k<n$.*

Proof. Since μ_n is T_n-invariant, it follows from Corollary 3.2 that

$$\Big|\sum_{\ell\le j<k} v_n\circ T_n^j\Big|_{L^{2(p-1)}(\mu_n)} = \Big|\sum_{0\le j<k-\ell} v_n\circ T_n^j\Big|_{L^{2(p-1)}(\mu_n)} \ll (k-\ell)^{1/2}\|v_n\|_\eta,$$

$$\Big|\sum_{\ell\le i<j<k}\big(v_n\circ T_n^i\big)\otimes\big(v_n\circ T_n^j\big)\Big|_{L^{p-1}(\mu_n)} = \Big|\sum_{0\le i<j<k-\ell}\big(v_n\circ T_n^i\big)\otimes\big(v_n\circ T_n^j\big)\Big|_{L^{p-1}(\mu_n)} \ll (k-\ell)\|v_n\|_\eta^2.$$

Now use that the densities h_n are uniformly bounded below. □

Next we turn to the iterated WIP. Defining Σ_n and E_n as in (3.1) for $n\in\mathbb{N}\cup\{\infty\}$, we already have that $\lim_{n\to\infty}\Sigma_n = \Sigma_\infty$ and $\lim_{n\to\infty}E_n = E_\infty$ by Lemma 3.8.

Proposition 4.2. *$(W_n,\mathbb{W}_n)\to_{\mathrm{Leb}}(W,\mathbb{W})$ as $n\to\infty$ in $D([0,1],\mathbb{R}^d\times\mathbb{R}^{d\times d})$, where W is d-dimensional Brownian motion with covariance matrix Σ_∞ and $\mathbb{W}(t)=\int_0^t W\otimes dW + E_\infty t$.*

Proof. By Theorem 3.4, $(W_n,\mathbb{W}_n)\to_{\mu_n}(W,\mathbb{W})$ as $n\to\infty$ in $D([0,1],\mathbb{R}^d\times\mathbb{R}^{d\times d})$. To pass from μ_n to Leb, we apply Corollary 3.6. Let $\rho_n = h_n^{-1} = d\mathrm{Leb}/d\mu_n$. Then $\sup_n|\rho_n|_\infty<\infty$. To deal with the remaining assumption in Corollary 3.6, write

$$\int_\Lambda\bigg|\frac{1}{N}\sum_{j=0}^{N-1}\rho_n\circ T_n^j - 1\bigg|\,d\mu_n \le I_1(N,n)+I_2(N,n)+I_3(N,n)+I_4(N)$$

where

$$I_1 = \int_\Lambda\frac{1}{N}\bigg|\sum_{j=0}^{N-1}\big(\rho_n\circ T_n^j - \rho_\infty\circ T_n^j\big)\bigg|\,d\mu_n,\quad I_2 = \int_\Lambda\frac{1}{N}\sum_{j=0}^{N-1}\rho_\infty\circ T_n^j|h_n-h_\infty|\,d\mathrm{Leb},$$

$$I_3 = \int_\Lambda\frac{1}{N}\bigg|\sum_{j=0}^{N-1}\big(\rho_\infty\circ T_n^j - \rho_\infty\circ T_\infty^j\big)\bigg|\,d\mu_\infty,\quad I_4 = \int_\Lambda\bigg|\frac{1}{N}\sum_{j=0}^{N-1}\rho_\infty\circ T_\infty^j - 1\bigg|\,d\mu_\infty.$$

Fix $N\ge 1$. By T_n-invariance of μ_n,

$$I_1(N,n)\le\int_\Lambda|\rho_n-\rho_\infty|\,d\mu_n = \int_\Lambda|h_n-h_\infty|\rho_\infty\,d\mathrm{Leb}\le|\rho_\infty|_\infty\int_\Lambda|h_n-h_\infty|\,d\mathrm{Leb},$$

and also

$$I_2(N,n)\le|\rho_\infty|_\infty\int_\Lambda|h_n-h_\infty|\,d\mathrm{Leb}.$$

By boundedness of ρ_∞ and strong statistical stability, $\lim_{n\to\infty}I_1(N,n)=\lim_{n\to\infty}I_2(N,n)=0$. By continuity of ρ_∞ and the dominated convergence theorem, $\lim_{n\to\infty}I_3(N,n)=0$. Hence for each fixed $N\ge1$,

$$\limsup_{n\to\infty}\int_\Lambda\bigg|\frac{1}{N}\sum_{j=0}^{N-1}\rho_n\circ T_n^j - 1\bigg|\,d\mu_n\le I_4(N).$$

By the mean ergodic theorem, $\lim_{N\to\infty} I_4(N)=0$. Hence $\limsup_{n\to\infty}\int_\Lambda |\frac{1}{N}\sum_{j=0}^{N-1}\rho_n\circ T_n^j-1|\,d\mu_n\to 0$ as $N\to\infty$. This verifies the final assumption in Corollary 3.6 completing the proof. □

Finally, we consider the analogue of Proposition 3.9 with μ_n replaced by Leb where appropriate. Again we can relax the assumptions on v_n; it suffices that $v_n\in L^\infty(\Lambda,\mathbb{R}^d)$, $n\ge 1$, and $v_\infty\in C^\eta(\Lambda,\mathbb{R}^d)$ with $\lim_{n\to\infty}|v_n-v_\infty|_\infty=0$.

Recall that $V_n=n^{-1}\sum_{j<[nt]} v_n\circ T_n^j$ where $t\in[0,1]$ is fixed.

Proposition 4.3.

(a) $\lim_{n\to\infty}|V_n-t\int_\Lambda v_\infty\,d\mu_\infty|_{L^p(\mathrm{Leb})}=0$.
(b) $\lim_{n\to\infty} n^{-1}\sum_{j<n}\int_\Lambda (v_n\otimes v_n)\circ T_n^j\,d\mathrm{Leb}=\int_\Lambda v_\infty\otimes v_\infty\,d\mu_\infty$.

Proof. (a) Using again that the densities $h_n=d\mu_n/d\mathrm{Leb}$ are uniformly bounded below,

$$\int_\Lambda\left|V_n-t\int_\Lambda v_\infty\,d\mu_\infty\right|d\mathrm{Leb}\ll\int_\Lambda\left|V_n-t\int_\Lambda v_\infty\,d\mu_\infty\right|d\mu_n\to 0$$

by Proposition 3.9(a).

(b) Set $w_n=v_n\otimes v_n-\int_\Lambda v_n\otimes v_n\,d\mu_n$. Then $w_n\in C^\eta(\Lambda,\mathbb{R}^{d\times d})$ with $\int_\Lambda w_n\,d\mu_n=0$ and

$$\begin{aligned}\left|n^{-1}\sum_{j<n}\int_\Lambda (v_n\otimes v_n)\circ T_n^j\,d\mathrm{Leb}-\int_\Lambda v_n\otimes v_n\,d\mu_n\right| &\le n^{-1}\int_\Lambda\left|\sum_{j<n} w_n\circ T_n^j\right|d\mathrm{Leb}\\ &\ll n^{-1}\int_\Lambda\left|\sum_{j<n} w_n\circ T_n^j\right|d\mu_n\ll n^{-1/2}\|w_n\|_\eta\to 0\end{aligned}$$

by Corollary 3.2. □

4.2. *Further examples*

In [20], the WIP and estimates of ordinary moments were obtained for many examples of nonuniformly expanding dynamics. We now obtain the corresponding iterated results.

Revisiting [19, Example 5.2] and [20, Example 4.10, Example 7.3], we consider families of quadratic maps $T_n:[-1,1]\to[-1,1]$, $n\in\mathbb{N}\cup\{\infty\}$, given by $T_n(x)=1-a_nx^2$, $a_n\in[0,2]$ with $\lim_{n\to\infty}a_n=a_\infty$. Fixing $b,c>0$ we assume that the Collet-Eckmann condition $|(T_n^k)'(1)|\ge ce^{bk}$ holds for all $k\ge 0$, $n\ge 1$.[2] The set of parameters such that this Collet-Eckmann condition holds has positive Lebesgue measure for b, c sufficiently small. Moreover T_n is a uniform family of nonuniformly expanding maps of order p (for any p) and satisfies strong statistical stability. Hence we obtain control of iterated moments (Corollary 3.2) and the iterated WIP (Theorem 3.4 and Lemma 3.8) for Hölder observables $v_n:[-1,1]\to\mathbb{R}^d$ with $\int_\Lambda v_n\,d\mu_n=0$ and $\lim_{n\to\infty}|v_n-v_\infty|_\infty=0$.

Revisiting [19, Example 5.4] and [20, Example 4.11, Example 7.3], we consider families of Viana maps $T_n:S^1\times\mathbb{R}\to S^1\times\mathbb{R}$, $n\in\mathbb{N}\cup\{\infty\}$. Again, we obtain control of iterated moments and the iterated WIP.

In both sets of examples, we obtain homogenization results $\hat{x}_n\to_{\mu_n}X$ by Part 2 and $\hat{x}_n\to_{\mu_\infty}X$ by strong statistical stability as explained at the beginning of Section 4.1.

5. Families of nonuniformly expanding semiflows

In this section, we consider uniform families of nonuniformly expanding semiflows. These are modelled as suspensions over uniform families of nonuniformly expanding maps. In keeping with the program of [24], no mixing assumptions are imposed on the semiflows.

Specifically, let $T_n:\Lambda_n\to\Lambda_n$, $n\ge 1$, be a uniform family of nonuniformly expanding maps of order $p\ge 2$ as in Section 3 with ergodic invariant probability measures μ_{Λ_n}. Let $h_n:\Lambda_n\to\mathbb{R}^+$ be a family of roof functions satisfying $\sup_n\|h_n\|_\eta<\infty$ and $\inf_n\inf h_n>0$. For each $n\ge 1$, define the suspension

$$\Omega_n=\Lambda_n^{h_n}=\{(x,u)\in\Lambda_n\times\mathbb{R}:0\le u\le h_n(x)\}/\sim,\quad (x,h_n(x))\sim(T_nx,0).$$

[2] There is a typo in [20, Example 4.10] where ce^{bn} should be ce^{bk}.

The suspension flow $g_{n,t}:\Omega_n\to\Omega_n$ is given by $g_{n,t}(x,u)=(x,u+t)$ computed modulo identifications. Let $\bar h_n=\int_{\Lambda_n} h_n\,d\mu_{\Lambda_n}$. Then $\mu_n=\mu_{\Lambda_n}^{h_n}=(\mu_{\Lambda_n}\times\text{Lebesgue})/\bar h_n$ is an ergodic $g_{n,t}$-invariant probability measure on Ω_n. We call $g_{n,t}:\Omega_n\to\Omega_n$ a *uniform family of nonuniformly expanding semiflows of order* p.

To simplify the statement of results in this section, we denote by C various constants depending continuously on the data associated with $T_n:\Lambda_n\to\Lambda_n$ as well as $\sup_n\|h_n\|_\eta$.

For $v:\Omega_n\to\mathbb{R}^d$, define

$$\|v\|_\eta=|v|_\infty+\sup_{(x,u)\neq(x',u)\in\Omega_n}\frac{|v(x,u)-v(x',u)|}{d_{\Lambda_n}(x,x')^\eta}.$$

5.1. *Moment estimates*

As in Sections 2 and 3, for uniform moment estimates it suffices to consider a fixed uniformly expanding semiflow $g_t:\Omega\to\Omega$. The main result in this section, Theorem 5.3, establishes the desired moment estimates. We also collect together some other results that fit best into the fixed semiflow setting.

Given $v:\Omega\to\mathbb{R}^d$, define the *induced observable*

$$\tilde v:\Lambda\to\mathbb{R},\qquad \tilde v(x)=\int_0^{h(x)}v(x,u)\,du.$$

Proposition 5.1. $|\tilde v|_\infty\le|h|_\infty|v|_\infty$ *and* $\|\tilde v\|_\eta\le\|h\|_\eta\|v\|_\eta$.

Proof. The estimate for $|\tilde v|_\infty$ is immediate. Also, for $x,x'\in\Lambda$ with $h(x)\le h(x')$,

$$\begin{aligned}\left|\tilde v(x)-\tilde v(x')\right|&\le\left|h(x)-h(x')\right||v|_\infty+\int_0^{h(x)}\left|v(x,u)-v(x',u)\right|du\\&\le\|h\|_\eta|v|_\infty d_\Lambda(x,x')+\int_0^{h(x)}\|v\|_\eta\,d_\Lambda(x,x')\,du\le\|h\|_\eta\|v\|_\eta d_\Lambda(x,x'),\end{aligned}$$

completing the proof. □

Define

$$S_t=\int_0^t v\circ g_s\,ds,\qquad \mathbb{S}_t=\int_0^t\int_0^s(v\circ g_r)\otimes(v\circ g_s)\,dr\,ds$$

on Ω. Also, for the induced observable $\tilde v:\Lambda\to\mathbb{R}^d$, define

$$\tilde S_n(x,u)=\sum_{0\le j<n}\tilde v(T^jx),\qquad \widetilde{\mathbb{S}}_n(x,u)=\sum_{0\le i<j<n}\tilde v(T^ix)\otimes\tilde v(T^jx),\quad (x,u)\in\Omega.$$

We introduce the *lap number* $N(t):\Omega\to\mathbb{N}$, $t\ge0$,

$$N(t)(x,u)=\max\left\{n\ge0:\sum_{j=0}^{n-1}h(T^jx)\le u+t\right\}.$$

Also, define

$$H:\Omega\to\mathbb{R}^d,\qquad H(x,u)=\int_0^u v(x,s)\,ds.$$

Proposition 5.2. *For all* $t\ge0$,

$$|S_t-\tilde S_{N(t)}|_\infty\le2|h|_\infty|v|_\infty,$$

$$\left|\mathbb{S}_t-\widetilde{\mathbb{S}}_{N(t)}-\int_0^t(H\otimes v)\circ g_s\,ds\right|\le2|h|_\infty|v|_\infty|\tilde S_{N(t)}|+2|h|_\infty^2|v|_\infty^2.$$

Proof. We use formal calculations from the proof of [16, Proposition 7.5], focusing on the precise estimates.

First, $S_t = \tilde{S}_{N(t)} + H \circ g_t - H$. Hence $|S_t - \tilde{S}_{N(t)}|_\infty \le 2|H|_\infty \le 2|h|_\infty |v|_\infty$.

Second, writing $T_n = \inf\{t' \ge 0 : N(t') = n\}$, we observe that

$$\widetilde{\mathbb{S}}_{N(t)} = \int_0^{T_{N(t)}} \tilde{S}_{N(s)} \otimes (v \circ g_s)\, ds = \int_0^t \tilde{S}_{N(s)} \otimes (v \circ g_s)\, ds - \tilde{S}_{N(t)} \otimes (H \circ g_t).$$

Hence

$$\mathbb{S}_t = \int_0^t \tilde{S}_{N(s)} \otimes (v \circ g_s)\, ds + \int_0^t (H \otimes v) \circ g_s\, ds - H \otimes \int_0^t v \circ g_s\, ds = \widetilde{\mathbb{S}}_{N(t)} + \int_0^t (H \otimes v) \circ g_s\, ds + K(t),$$

where $K(t) = \tilde{S}_{N(t)} \otimes (H \circ g_t) - H \otimes S_t$. We have

$$\big|K(t)\big| \le |\tilde{S}_{N(t)}||H|_\infty + |H|_\infty |S_t| \le |\tilde{S}_{N(t)}||h|_\infty |v|_\infty + |h|_\infty |v|_\infty \big(|\tilde{S}_{N(t)}| + 2|h|_\infty |v|_\infty\big).$$

The result follows. □

Theorem 5.3 (Iterated moments). *For all $t_1 \ge 0$,*

$$\Big|\sup_{t\in[0,t_1]} |S_t|\Big|_{L^{2(p-1)}(\Omega)} \le C t_1^{1/2} \|v\|_\eta, \qquad \Big|\sup_{t\in[0,t_1]} |\mathbb{S}_t|\Big|_{L^{p-1}(\Omega)} \le C t_1 \|v\|_\eta^2.$$

Proof. The estimates are trivial for $t \in [0,1]$ (since $t \le t^{1/2}$) so we restrict to $t_1 \ge 1$, $t \in [1, t_1]$.

Since $\inf h > 0$, it is immediate ([16, Proposition 7.4]) that

$$\big|N(t)\big|_\infty \le C_0 t \quad \text{for all } t \ge 1, \tag{5.1}$$

where $C_0 = (\inf h)^{-1} + 1$.

By (5.1),

$$\int_\Omega \sup_{1\le t\le t_1} |\tilde{S}_{N(t)}|^{2(p-1)}\, d\mu \le \int_\Omega \max_{k\le C_0 t_1} |\tilde{S}_k|^{2(p-1)}\, d\mu \le \bar{h}^{-1}|h|_\infty \int_\Lambda \max_{k\le C_0 t_1} \Big|\sum_{0\le j<k} \tilde{v} \circ T^j\Big|^{2(p-1)} d\mu_\Lambda.$$

Hence by Theorem 2.4,

$$\Big|\sup_{1\le t\le t_1} |\tilde{S}_{N(t)}|\Big|_{L^{2(p-1)}(\Omega)} \ll \Big|\max_{k\le C_0 t_1} \Big|\sum_{0\le j<k} \tilde{v} \circ T^j\Big|\Big|_{L^{2(p-1)}(\Lambda)} \ll t_1^{1/2} \|\tilde{v}\|_\eta \ll t_1^{1/2} \|v\|_\eta.$$

Similarly, $|\sup_{1\le t\le t_1} |\widetilde{\mathbb{S}}_{N(t)}||_{L^{p-1}(\Omega)} \ll t_1 \|v\|_\eta^2$. Also, $\int_0^{t_1} |(H\otimes v)\circ g_s|\, ds \le t_1 |h|_\infty |v|_\infty^2$ so the result follows from Proposition 5.2. □

Corollary 5.4. *For all $t_1 \ge 0$,*

$$\Big|\sup_{t\in[0,t_1]} \Big|\int_0^t (H \otimes v) \circ g_s\, ds - t\int_\Omega H \otimes v\, d\mu\Big|\Big|_{L^{2(p-1)}(\Omega)} \le C t_1^{1/2} \|v\|_\eta^2.$$

Proof. Using the S_t estimate in Theorem 5.3 with v replaced by $H \otimes v - \int_\Omega H \otimes v\, d\mu$, we obtain $|\sup_{t\in[0,t_1]} |\int_0^t (H \otimes v) \circ g_s\, ds - t\int_\Omega H\otimes v\, d\mu||_{2(p-1)} \ll t_1^{1/2} \|H \otimes v\|_\eta$. In addition, $\|H \otimes v\|_\eta \le \|H\|_\eta \|v\|_\eta \le |h|_\infty \|v\|_\eta^2$. □

Lemma 5.5. *For all $s \in [0, \inf h]$, $n \ge 1$,*

$$\Big|\sup_{t\in[0,1]} |\tilde{S}_{[nt]} \circ g_s - \tilde{S}_{[nt]}|\Big|_\infty \le 2|h|_\infty |v|_\infty, \qquad \Big|\sup_{t\in[0,1]} |\widetilde{\mathbb{S}}_{[nt]} \circ g_s - \widetilde{\mathbb{S}}_{[nt]}|\Big|_{L^{2(p-1)}(\Omega)} \le C n^{1/2} \|v\|_\eta^2.$$

Proof. The random variable $N(s)(x,u)$ lies in $\{0,1\}$ due to the restriction on s. If $N(s)(x,u)=0$, then $g_s(x,u)=(x,u+s)$. Now, $\tilde{S}_n$ and $\tilde{\mathbb{S}}_n$ are independent of u, and so $\tilde{S}_{[nt]}\circ g_s\equiv\tilde{S}_{[nt]}$ and $\tilde{\mathbb{S}}_{[nt]}\circ g_s\equiv\tilde{\mathbb{S}}_{[nt]}$ for all n, t and all s, x, u with $N(s)(x,u)=0$.

Hence we may suppose for the remainder of the proof that $N(s)\equiv 1$ in which case $g_s(x,u)=(fx,u+s-h(x))$. Then,

$$\left|\tilde{S}_{[nt]}\circ g_s(x,u)-\tilde{S}_{[nt]}(x,u)\right|=\left|\sum_{0\le j<[nt]}\big(\tilde{v}\big(T^{j+1}x\big)-\tilde{v}\big(T^jx\big)\big)\right|\le 2|\tilde{v}|_\infty\le 2|h|_\infty|v|_\infty.$$

Next,

$$\begin{aligned}\left|\tilde{\mathbb{S}}_{[nt]}\circ g_s(x,u)-\tilde{\mathbb{S}}_{[nt]}(x,u)\right|&=\left|\sum_{0\le i<j<[nt]}\big(\tilde{v}\big(T^{i+1}x\big)\otimes\tilde{v}\big(T^{j+1}x\big)-\tilde{v}\big(T^ix\big)\otimes\tilde{v}\big(T^jx\big)\big)\right|\\&=\left|\sum_{1\le i<[nt]}\tilde{v}\big(T^ix\big)\otimes\tilde{v}\big(T^{[nt]}x\big)-\sum_{1\le j<[nt]}\tilde{v}\otimes\tilde{v}\big(T^jx\big)\right|\\&\le 2|\tilde{v}|_\infty\left|\Big(\sum_{0\le j<[nt]-1}\tilde{v}\circ T^j\Big)(fx)\right|.\end{aligned}$$

Hence $|\sup_{t\in[0,1]}|\tilde{\mathbb{S}}_{[nt]}\circ g_s-\tilde{\mathbb{S}}_{[nt]}||_{2(p-1)}\ll n^{1/2}\|\tilde{v}\|_\eta^2\ll n^{1/2}\|v\|_\eta^2$ by Theorem 2.4. □

Proposition 5.6. $|\sup_{t\le t_1}|N(t)-t/\bar{h}||_{L^{2(p-1)}(\Omega)}\le Ct_1^{1/2}$ *for* $t_1\ge 1$.

Proof. Let $S_kh=\sum_{j=0}^{k-1}h\circ T^j$. By definition of $N(t)$,

$$S_{N(t)(x,u)}h(x)\le u+t<S_{N(t)(x,u)+1}h(x).$$

Hence $-S_{N(t)(x,u)}h(x)-|h|_\infty<-t\le -S_{N(t)(x,u)}h(x)+|h|_\infty$, so

$$\left|N(t)(x,u)-t/\bar{h}\right|\le\left\{\left|S_{N(t)(x,u)}h(x)-N(t)(x,u)\bar{h}\right|+2|h|_\infty\right\}/\bar{h}.$$

By (5.1), for all $(x,u)\in\Omega$,

$$\sup_{t\le t_1}\left|N(t)(x,u)-t/\bar{h}\right|\ll\max_{k\le C_0t_1}\left|S_kh(x)-k\bar{h}\right|+1.$$

Hence by Theorem 2.4,

$$\left|\sup_{t\le t_1}\left|N(t)-t/\bar{h}\right|\right|_{L^{2(p-1)}(\Omega)}\ll\left|\max_{k\le C_0t_1}|S_kh-k\bar{h}|\right|_{L^{2(p-1)}(\Lambda)}\ll t_1^{1/2},$$

as required. □

5.2. *Iterated weak invariance principle*

Let $g_{n,t}:\Omega_n\to\Omega_n$ be a uniform family of nonuniformly expanding semiflows of order $p\ge 2$. Let $v_n:\Omega_n\to\mathbb{R}^d$, $n\ge 1$, be a family of observables with $\sup_{n\ge 1}\|v_n\|_\eta<\infty$ and $\int_{\Omega_n}v_n\,d\mu_n=0$. The corresponding family of induced observables $\tilde{v}_n:\Lambda_n\to\mathbb{R}^d$ satisfies $\sup_{n\ge 1}\|\tilde{v}_n\|_\eta<\infty$ and $\int_{\Lambda_n}v_n\,d\mu_{\Lambda_n}=0$. Define Σ_n and E_n in terms of $\tilde{v}_n$ as in (3.1). Also define $H_n(x,u)=\int_0^u v_n(x,s)\,ds$.

In this section, we prove an iterated WIP for the processes $W_n\in C([0,1],\mathbb{R}^d)$ and $\mathbb{W}_n\in C([0,1],\mathbb{R}^{d\times d})$ on Ω_n given by

$$W_n(t)=\frac{1}{\sqrt{n}}\int_0^{nt}v_n\circ g_{n,s}\,ds,\qquad \mathbb{W}_n(t)=\int_0^t W_n(s)\otimes dW_n(s).$$

First, we consider the processes $\tilde{W}_n\in D([0,1],\mathbb{R}^d)$, $\tilde{\mathbb{W}}_n\in D([0,1],\mathbb{R}^{d\times d})$

$$\tilde{W}_n(t)(x,u)=\frac{1}{\sqrt{n}}\sum_{j=0}^{[nt]-1}\tilde{v}_n\big(T_n^jx\big),\qquad \tilde{\mathbb{W}}_n(t)(x,u)=\frac{1}{n}\sum_{0\le i<j<[nt]}\tilde{v}_n\big(T_n^ix\big)\otimes\tilde{v}_n\big(T_n^jx\big),$$

defined on Ω_n. Let $N_n(t)$ denote the lap numbers corresponding to the semiflows $g_{n,t}$ on Ω_n. Also define

$$\gamma_n \in D\big([0,1],\mathbb{R}\big), \qquad \gamma_n(t) = n^{-1}N_n(nt).$$

Proposition 5.7. *Suppose that* $E_n \to E$, $\Sigma_n \to \Sigma$, $\bar{h}_n \to \bar{h}$. *Then*

$$(\widetilde{W}_n, \widetilde{\mathbb{W}}_n) \circ \gamma_n \to_{\mu_n} \big(\bar{h}^{-1/2}\widetilde{W}, \bar{h}^{-1}\widetilde{\mathbb{W}}\big) \quad \text{in } D\big([0,1], \mathbb{R}^d \times \mathbb{R}^{d\times d}\big),$$

where $\widetilde{W}$ *is a* d*-dimensional Brownian motion with covariance matrix* Σ *and* $\widetilde{\mathbb{W}}(t) = \int_0^t \widetilde{W} \otimes d\widetilde{W} + Et$.

Proof. Choose $c_0 > 0$ such that $h_n \ge c_0$ for all n. Let λ_n be the sequence of probability measures on Ω_n supported on $\Lambda_n \times [0, c_0]$ with density $\rho_n = d\lambda_n/d\mu_n = 1_{\Lambda_n\times[0,c_0]}/c_0$.

The process $(\widetilde{W}_n, \widetilde{\mathbb{W}}_n)$ on (Ω_n, λ_n) has the same distribution as the process $(\widetilde{W}_n, \widetilde{\mathbb{W}}_n)|_{\Lambda_n}$ on $(\Lambda_n, \mu_{\Lambda_n})$, so by Theorem 3.4, $(\widetilde{W}_n, \widetilde{\mathbb{W}}_n) \to_{\lambda_n} (\widetilde{W}, \widetilde{\mathbb{W}})$.

By Lemma 5.5, for each $s \in [0, c_0]$,

$$\Big|\sup_{t\in[0,1]} |\widetilde{W}_{[nt]} \circ g_{n,s} - \widetilde{W}_{[nt]}|\Big|_\infty \ll n^{-1/2}, \qquad \Big|\sup_{t\in[0,1]} |\widetilde{\mathbb{W}}_{[nt]} \circ g_{n,s} - \widetilde{\mathbb{W}}_{[nt]}|\Big|_{L^1(\mu_n)} \ll n^{-1/2}.$$

Also, $\|\rho_n\|_\eta = |\rho_n|_\infty \le 1/c_0$ so by Theorem 5.3 with $v = \rho_n - 1$,

$$\left|\int_0^{t_1} \rho_n \circ g_{n,t}\, dt - t_1\right|_{L^2(\mu_n)} \ll t_1^{1/2}.$$

We have verified the assumptions of Lemma B.1, and it follows that $(\widetilde{W}_n, \widetilde{\mathbb{W}}_n) \to_{\mu_n} (\widetilde{W}, \widetilde{\mathbb{W}})$.

Let $\gamma(t) = t\bar{h}^{-1}$. By Proposition 5.6,

$$\Big|\sup_{t\le 1}\big|\gamma_n(t) - \gamma(t)\big|\Big|_{L^1(\mu_n)} = n^{-1}\Big|\sup_{t\le 1}\big|N_n(nt) - nt\bar{h}^{-1}\big|\Big|_{L^1(\mu_n)} = O\big(n^{-1/2}\big).$$

Since γ is not random it follows that $(\widetilde{W}_n, \widetilde{\mathbb{W}}_n, \gamma_n) \to_{\mu_n} (\widetilde{W}, \widetilde{\mathbb{W}}, \gamma)$. By the continuous mapping theorem,

$$(\widetilde{W}_n, \widetilde{\mathbb{W}}_n) \circ \gamma_n \to_{\mu_n} (\widetilde{W}, \widetilde{\mathbb{W}}) \circ \gamma = \big(\bar{h}^{-1/2}\widetilde{W}, \bar{h}^{-1}\widetilde{\mathbb{W}}\big),$$

as required. □

Theorem 5.8 (Iterated WIP). *Suppose that* $\lim_{n\to\infty} \Sigma_n = \Sigma$, $\lim_{n\to\infty} E_n = E$, $\lim_{n\to\infty} \bar{h}_n = \bar{h}$ *and* $\lim_{n\to\infty} \int_{\Omega_n} H_n \otimes v_n\, d\mu_n = E'$. *Then*

$$(W_n, \mathbb{W}_n) \to_{\mu_n} (W, \mathbb{W}) \quad \text{in } D\big([0,1], \mathbb{R}^d \times \mathbb{R}^{d\times d}\big),$$

where W *is a* d*-dimensional Brownian motion with covariance matrix* $\bar{h}^{-1}\Sigma$ *and* $\mathbb{W}(t) = \int_0^t W \otimes dW + (\bar{h}^{-1}E + E')t$.

Proof. By Proposition 5.7, it suffices to show that $\sup_{t\le 1} |W_n(t) - \widetilde{W}_n(\gamma_n(t))| \to_{\mu_n} 0$ and $\sup_{t\le 1} |\mathbb{W}_n(t) - \widetilde{\mathbb{W}}_n(t) \circ \gamma_n - tE'| \to_{\mu_n} 0$.

First, note by Proposition 5.2 that

$$\big|W_n(t) - \widetilde{W}_n\big(\gamma_n(t)\big)\big|(x,u) = n^{-1/2}\left|\int_0^{nt} v_n\big(g_{n,s}(x,u)\big)\, ds - \sum_{0\le j<N_n(nt)} \tilde{v}_n\big(T_n^j x\big)\right| \le 2n^{-1/2}|h_n|_\infty |v_n|_\infty,$$

so $|\sup_{t\in[0,1]} |W_n(t) - \widetilde{W}_n(\gamma_n(t))||_\infty \to 0$.

Similarly, by Proposition 5.2,

$$n\left\{\left|\mathbb{W}_n(t) - \widetilde{\mathbb{W}}_n\big(\gamma_n(t)\big) - n^{-1}\int_0^{nt} (H_n \otimes v_n) \circ g_{n,s}\, ds\right|\right\}(x,u)$$

$$\le 2|h_n|_\infty |v_n|_\infty \left|\sum_{0\le j<N_n(t)} \tilde{v}_n\big(T_n^j x\big)\right| + 3|h_n|_\infty^2 |v_n|_\infty^2 \ll \left|\sum_{0\le j<N_n(t)} \tilde{v}_n\big(T_n^j x\big)\right| + 1.$$

By (5.1) and Theorem 2.4,

$$\Big|\sup_{t\in[0,1]}\Big|\sum_{0\le j<N_n(t)}\tilde v_n\circ T_n^j\Big|\Big|_{L^2(\Omega_n)}\ll\Big|\max_{k\le C_0 n}\Big|\sum_{0\le j<k}\tilde v_n\circ f_{\Delta_n}^j\Big|\Big|_{L^2(\Lambda_n)}\ll n^{1/2}\|\tilde v_n\|_\eta\ll n^{1/2},$$

and so

$$\Big|\sup_{t\in[0,1]}\Big|\mathbb{W}_n(t)-\widetilde{\mathbb{W}}_n\big(\gamma_n(t)\big)-n^{-1}\int_0^{nt}(H_n\otimes v_n)\circ g_{n,s}\,ds\Big|\Big|_{L^2(\Omega_n)}\to 0.$$

Also, by Corollary 5.4,

$$n^{-1}\Big|\sup_{t\in[0,1]}\Big|\int_0^{nt}(H_n\otimes v_n)\circ g_{n,s}\,ds-nt\int_{\Omega_n}H_n\otimes v_n\,d\mu_n\Big|\Big|_{L^2(\Omega_n)}\to 0.$$

Hence $|\sup_{t\in[0,1]}|\mathbb{W}_n(t)-\widetilde{\mathbb{W}}_n(\gamma_n(t))-tE'||_{L^2(\Omega_n)}\to 0$ and the proof is complete. □

5.3. *Application to intermittent semiflows*

Let $\Lambda=[0,1]$. Fix a family of intermittent maps $T_n:\Lambda\to\Lambda$, $n\in\mathbb{N}\cup\{\infty\}$, as in (1.2) with parameters $\gamma_n\in(0,\frac12)$ and absolutely continuous invariant probability measures denoted $\tilde\mu_n$. Suppose that $\lim_{n\to\infty}\gamma_n=\gamma_\infty$. Again, T_n is a uniform family of order p for all $p\in(2,\gamma_\infty^{-1})$ and the absolutely continuous invariant probability measures, denoted here by $\tilde\mu_n$, are strongly statistically stable.

Fix $\eta>0$ and let $h_n:\Lambda\to\mathbb{R}^+$ be a family of roof functions satisfying $\sup_n\|h_n\|_\eta<\infty$ and $\inf_n\inf h_n>0$. Define the corresponding uniform family of nonuniformly expanding semiflows $g_{n,t}:\Omega_n\to\Omega_n$ with ergodic invariant probability measures $\mu_n=(\tilde\mu_n\times\text{Lebesgue})/\bar h_n$ where $\bar h_n=\int_\Lambda h_n\,d\tilde\mu_n$.

Theorem 5.9. *Let $v_n:\Omega_n\to\mathbb{R}^d$, $n\ge 1$, with $\sup_n\|v_n\|_\eta<\infty$ and $\int_{\Omega_n}v_n\,d\mu_n=0$. Then there is a constant $C>0$ such that*

$$\Big|\sup_{t\in[0,t_1]}\Big|\int_0^t v_n\circ g_{n,s}\,ds\Big|\Big|_{L^{2(p-1)}(\Omega_n)}\le Ct_1^{1/2},$$

$$\Big|\sup_{t\in[0,t_1]}\Big|\int_0^t\int_0^s(v_n\circ g_{n,r})\otimes(v_n\circ g_{n,s})\,dr\,ds\Big|\Big|_{L^{p-1}(\Omega_n)}\le Ct_1,$$

for all $t_1\ge 0$, $n\ge 1$.

Proof. This is immediate from Theorem 5.3. □

Theorem 5.10. *Let $v_n:\Omega_n\to\mathbb{R}^d$, $n\in\mathbb{N}\cup\{\infty\}$, with $\sup_n\|v_n\|_\eta<\infty$ and $\int_{\Omega_n}v_n\,d\mu_n=0$. Suppose that $\lim_{n\to\infty}\sup_{x\in\Lambda,\,u\in[0,h_\infty(x)]\cap[0,h_n(x)]}|v_n(x,u)-v_\infty(x,u)|=0$ and $\lim_{n\to\infty}|h_n-h_\infty|_\infty=0$.*

(a) *Define*

$$S_n=\sum_{0\le j<n}\tilde v\circ T_\infty^j,\qquad \mathbb{S}_n=\sum_{0\le i<j<n}\big(\tilde v\circ T_\infty^i\big)\otimes\big(\tilde v\circ T_\infty^j\big).$$

where $\tilde v(x)=\int_0^{h_\infty(x)}v_\infty(x,u)\,du$. Then the limits

$$\Sigma_\infty=\lim_{n\to\infty}\frac1n\int_\Lambda S_n\otimes S_n\,d\tilde\mu_\infty,\qquad E_\infty=\lim_{n\to\infty}\frac1n\int_\Lambda\mathbb{S}_n\,d\tilde\mu_\infty.$$

exist.

(b) *Set $E'=\int_{\Omega_\infty}H_\infty\otimes v_\infty\,d\mu_\infty$ where $H_\infty(x,u)=\int_0^u v_\infty(x,u)\,du$. Define*

$$W_n(t)=n^{1/2}\int_0^{n^{-1}t}v_n\circ g_{n,s}\,ds,\qquad \mathbb{W}_n(t)=\int_0^t W_n(s)\otimes dW_n(s).$$

Then

$$(W_n, \mathbb{W}_n) \to_{\mu_n} (W, \mathbb{W}) \quad \text{in } D\big([0,1], \mathbb{R}^d \times \mathbb{R}^{d\times d}\big),$$

where W is a d-dimensional Brownian motion with covariance matrix $\bar{h}_\infty^{-1}\Sigma_\infty$ and $\mathbb{W}(t) = \int_0^t W \otimes dW + (\bar{h}_\infty^{-1} E_\infty + E')t$.

Proof. Part (a) follows from Lemma 3.8.

To prove part (b), we verify the hypotheses of Theorem 5.8. Since $|h_n - h_\infty|_\infty \to 0$ it follows from statistical stability that $\bar{h}_n \to \bar{h}_\infty$.

Let $H_n(x,u) = \int_0^u v_n(x,s)\,ds$. It is easy to see that $\int_0^{h_n(x)} (H_n \otimes v_n)(x,u)\,du \to \int_0^{h_\infty(x)} (H_\infty \otimes v_\infty)(x,u)\,du$ uniformly in x, so again by statistical stability $\int_{\Omega_n} H_n \otimes v_n \, d\mu_n \to E'$.

Finally, defining Σ_n and E_n using T_n, v_n and h_n in place of T_∞, v_∞ and h_∞, we have that $\Sigma_n \to \Sigma_\infty$ and $E_n \to E_\infty$ by Lemma 3.8. □

Appendix A: Iterated WIP for martingale difference arrays

In this appendix, we recast a classical iterated WIP of [15,21] into a form that is convenient for ergodic stationary martingale difference arrays of the type commonly encountered in the deterministic setting.

Let $\{(\Delta_n, \mathcal{M}_n, \mu_n)\}$ be a sequence of probability spaces. Suppose that $T_n : \Delta_n \to \Delta_n$ is a sequence of measure-preserving transformations with transfer operators L_n and Koopman operators U_n. Suppose that $\phi_n, m_n : \Delta_n \to \mathbb{R}^d$ lie in $L^2(\Delta_n)$ and that $\int_{\Delta_n} \phi_n \, d\mu_n = \int_{\Delta_n} m_n \, d\mu_n = 0$ and $m_n \in \ker L_n$.

Define the sequence of processes

$$\Phi_n : \Delta_n \to D\big([0,\infty), \mathbb{R}^d\big), \qquad \mathbb{M}_n : \Delta_n \to D\big([0,\infty), \mathbb{R}^{d\times d}\big),$$

by

$$\Phi_n(t) = \frac{1}{\sqrt{n}} \sum_{0 \le j < nt} \phi_n \circ f_{\Delta_n}^j, \qquad \mathbb{M}_n(t) = \frac{1}{n} \sum_{0 \le i < j < nt} \big(m_n \circ f_{\Delta_n}^i\big) \otimes \big(\phi_n \circ f_{\Delta_n}^j\big), \quad t \ge 0.$$

Theorem A.1. *Suppose that*:

(a) *the family $\{|m_n|^2, n \ge 1\}$ is uniformly integrable*;
(b) $\frac{1}{\sqrt{n}} \max_{k \le nt_1} |\sum_{j=0}^k (\phi_n - m_n) \circ f_{\Delta_n}^j| \to_{\mu_n} 0$ *as $n \to \infty$ for all $t_1 > 0$*;
(c) *there exists a constant matrix $\Sigma \in \mathbb{R}^{d\times d}$ such that for each $t > 0$*,

$$\frac{1}{n} \sum_{j=0}^{[nt]-1} \big\{U_n L_n(m_n \otimes m_n)\big\} \circ f_{\Delta_n}^j \to_{\mu_n} t\Sigma \quad \text{as } n \to \infty.$$

Then $(\Phi_n, \mathbb{M}_n) \to_{\mu_n} (W, \mathbb{M})$ in $D([0,\infty), \mathbb{R}^d \times \mathbb{R}^{d\times d})$ where W is a d-dimensional Brownian motion with covariance Σ and $\mathbb{M}(t) = \int_0^t W \otimes dW$.

Proof. It suffices to prove that $(\Phi_n, \mathbb{M}_n) \to_{\mu_n} (W, \mathbb{M})$ in $D([0,t_1], \mathbb{R}^d \times \mathbb{R}^{d\times d})$ for each fixed integer $t_1 \ge 1$.

Define $X_{n,j} = n^{-1/2} \phi_n \circ f_{\Delta_n}^{nt_1 - j}$, $Y_{n,j} = n^{-1/2} m_n \circ f_{\Delta_n}^{nt_1 - j}$, and

$$X_n(t) = \sum_{1 \le j \le nt} X_{n,j}, \qquad Y_n(t) = \sum_{1 \le j \le nt} Y_{n,j}, \qquad \mathbb{Y}_n(t) = \sum_{1 \le i < j \le nt} X_{n,i} \otimes Y_{n,j},$$

for $t \in [0, t_1]$.

By the arguments in the proof of [20, Theorem A.1], $\{Y_{n,j};\ 1 \le j \le nt_1\}$ is a martingale difference array with respect to the filtration $\mathcal{G}_{n,j} = T_n^{-(nt_1-j)} \mathcal{M}_n$ and $Y_n \to_{\mu_n} W$ in $D([0,t_1], \mathbb{R}^d)$. Moreover, X_n is adapted (i.e. $X_{n,j}$ is $\mathcal{G}_{n,j}$-measurable for all j, n) and $X_n = Y_n + Z_n$ where

$$\big|Z_n(t)\big| = \frac{1}{\sqrt{n}} \left| \sum_{j=1}^{[nt]} (\phi_n - m_n) \circ f_{\Delta_n}^{nt_1 - j} \right| \le \frac{2}{\sqrt{n}} \max_{k \le nt_1} \left| \sum_{j=0}^{k} (\phi_n - m_n) \circ f_{\Delta_n}^j \right|,$$

so $\sup_{t\le t_1}|Z_n(t)|\to_{\mu_n} 0$ by assumption (b). It follows easily that $(X_n,Y_n)\to_{\mu_n}(W,W)$ in $D([0,t_1],\mathbb{R}^d\times\mathbb{R}^d)$.

Also $\int_{\Delta_n}|Y_n(t)|^2\,d\mu_n = n^{-1}[nt]\int_{\Delta_n}|m_n|^2\,d\mu_n\le t_1|m_n|_2^2$ which is bounded by assumption (a), so condition C2.2(i) in [21, Theorem 2.2] is trivially satisfied. Applying [21, Theorem 2.2] (or alternatively [15]) we deduce that $(X_n,Y_n,\mathbb{Y}_n)\to_{\mu_n}(W,W,\mathbb{M})$ in $D([0,t_1],\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^{d\times d})$.

Next, let $\widetilde{D}$ denote càglàd functions. Adapting [16], we define $g: D([0,t_1],\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^{d\times d})\to\widetilde{D}([0,t_1],\mathbb{R}^d\times\mathbb{R}^{d\times d})$,

$$g(r,u,v)(t)=\big(r(t_1)-r(t_1-t)\,,\,\big\{v(t_1)-v(t_1-t)-r(t_1-t)\otimes\big(u(t_1)-u(t_1-t)\big)\big\}^*\big),$$

where * denotes matrix transpose.

We claim that

$$(\Phi_n,\mathbb{M}_n)=g(X_n,Y_n,\mathbb{Y}_n)+F_n\quad\text{where}\ \sup_{t\in[0,t_1]}\big|F_n(t)\big|\to_{\mu_n}0.$$

Suppose that the claim is true. By the continuous mapping theorem, $g(X_n,Y_n,\mathbb{Y}_n)\to_{\mu_n}g(W,W,\mathbb{M})$ in $\widetilde{D}([0,t_1],\mathbb{R}^d\times\mathbb{R}^{d\times d})$. Using the fact that the limiting process has continuous sample paths, it follows (see [16, Proposition 4.9]) that $(\Phi_n,\mathbb{M}_n)\to_{\mu_n}g(W,W,\mathbb{M})$ in $D([0,t_1],\mathbb{R}^d\times\mathbb{R}^{d\times d})$. By [16, Lemma 4.11], the processes $g(W,W,\mathbb{M})$ and $(W,\mathbb{M})$ are equal in distribution so $(\Phi_n,\mathbb{M}_n)\to_{\mu_n}(W,\mathbb{M})$ in $D([0,t_1],\mathbb{R}^d\times\mathbb{R}^{d\times d})$.

It remains to prove the claim. Write $g=(g^1,g^2)$ where $g^1: D([0,t_1],\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^{d\times d})\to\widetilde{D}([0,t_1],\mathbb{R}^d)$ and $g^2: D([0,t_1],\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^{d\times d})\to\widetilde{D}([0,t_1],\mathbb{R}^{d\times d})$.

First,

$$\begin{aligned}\Phi_n(t)&=\frac{1}{\sqrt{n}}\sum_{j=0}^{[nt]-1}\phi_n\circ f_{\Delta_n}^j=\sum_{j=0}^{[nt]-1}X_{n,nt_1-j}=\sum_{j=nt_1-[nt]+1}^{nt_1}X_{n,j}=\sum_{j=[n(t_1-t)]+1}^{nt_1}X_{n,j}+F_n^1(t)\\&=X_n(t_1)-X_n(t_1-t)+F_n^1(t)=g^1(X_n,Y_n,\mathbb{Y}_n)(t)+F_n^1(t),\end{aligned}$$

where $F_n^1(t)$ is either 0 or $-X_{n,[n(t_1-t)]+1}$. In particular,

$$\big|F_n^1(t)\big|\le n^{-1/2}\max_{i\le nt_1}\big|\phi_n\circ f_{\Delta_n}^i\big|.\tag{A.1}$$

Second,

$$\begin{aligned}\mathbb{M}_n(t)&=\frac{1}{n}\sum_{0\le i<j<nt}\big(m_n\circ f_{\Delta_n}^i\big)\otimes\big(\phi_n\circ f_{\Delta_n}^j\big)=\sum_{0\le i<j<nt}Y_{n,nt_1-i}\otimes X_{n,nt_1-j}\\&=\sum_{nt_1-[nt]<j<i\le nt_1}Y_{n,i}\otimes X_{n,j}=\sum_{nt_1-[nt]<i<j\le nt_1}(X_{n,i}\otimes Y_{n,j})^*\\&=\sum_{[n(t_1-t)]<i<j\le nt_1}(X_{n,i}\otimes Y_{n,j})^*+F_n^2(t)^*\\&=\big\{\mathbb{Y}_n(t_1)-\mathbb{Y}_n(t_1-t)-X_n(t_1-t)\otimes\big(Y_n(t_1)-Y_n(t_1-t)\big)\big\}^*+F_n^2(t)^*\\&=g^2(X_n,Y_n,\mathbb{Y}_n)(t)+F_n^2(t)^*,\end{aligned}$$

where $F_n^2(t)$ is either 0 or $-\sum_{[n(t_1-t)]+1<j\le nt_1}X_{n,[n(t_1-t)]+1}\otimes Y_{n,j}$. In particular, by Burkholder's inequality and assumption (a),

$$\begin{aligned}\big|F_n^2(t)\big|&\le n^{-1}\max_{i\le nt_1}\big|\phi_n\circ f_{\Delta_n}^i\big|\max_{q\le nt_1}\Big|\sum_{0\le j\le q}m_n\circ f_{\Delta_n}^j\Big|\ll n^{-1}\max_{i\le nt_1}\big|\phi_n\circ f_{\Delta_n}^i\big|t_1|m_n|_2\\&\ll n^{-1}\max_{i\le nt_1}\big|\phi_n\circ f_{\Delta_n}^i\big|.\end{aligned}\tag{A.2}$$

By (A.1) and (A.2), it remains to show that $n^{-1}\max_{i\le nt_1}|\phi_n\circ f_{\Delta_n}^i|\to_{\mu_n}0$. Note that

$$\big|\phi_n\circ f_{\Delta_n}^i\big|\le\big|m_n\circ f_{\Delta_n}^i\big|+\Big|\sum_{j=0}^{i}(\phi_n-m_n)\circ f_{\Delta_n}^j\Big|+\Big|\sum_{j=0}^{i-1}(\phi_n-m_n)\circ f_{\Delta_n}^j\Big|,$$

so

$$\max_{i\le nt_1}\big|\phi_n\circ f_{\Delta_n}^i\big|\le\max_{i\le nt_1}\big|m_n\circ f_{\Delta_n}^i\big|+2\max_{i\le nt_1}\bigg|\sum_{j=0}^{i}(\phi_n-m_n)\circ f_{\Delta_n}^j\bigg|.$$

Now for any $s>0$,

$$n^{-1}\max_{j\le nt_1}\big|m_n\circ f_{\Delta_n}^j\big|^2\le s+n^{-1}\max_{j\le nt_1}\big(|m_n|^2 1_{\{n^{-1}|m_n|^2>s\}}\big)\circ f_{\Delta_n}^j\le s+n^{-1}\sum_{j=0}^{nt_1}\big(|m_n|^2 1_{\{n^{-1}|m_n|^2>s\}}\big)\circ f_{\Delta_n}^j.$$

Hence

$$n^{-1}\Big|\max_{j\le nt_1}\big|m_n\circ f_{\Delta_n}^j\big|\Big|_2^2=n^{-1}\big|\max_{j\le nt_1}\big|m_n\circ f_{\Delta_n}^j\big|^2\big|_1\le s+t_1\big||m_n|^2 1_{\{n^{-1}|m_n|^2>s\}}\big|_1.$$

Since $s>0$ is arbitrary, it follows from assumption (a) that $\lim_{n\to\infty}n^{-1/2}|\max_{j\le nt_1}|m_n\circ f_{\Delta_n}^j||_2=0$. Combining this with assumption (b), $n^{-1/2}\max_{i\le nt_1}|\phi_n\circ f_{\Delta_n}^i|\to_{\mu_n}0$ as required. □

Appendix B: Strong distributional convergence for families

In this appendix, we formulate a result on strong distributional convergence [10,31] in the context of families of dynamical systems.

Let (Ω_n,μ_n), $n\ge 1$, be a sequence of probability spaces with measure-preserving semiflows $g_{n,t}:\Omega_n\to\Omega_n$. Suppose that λ_n is a sequence of probability measures on Ω_n such that $\lambda_n\ll\mu_n$. Define $\rho_n=d\lambda_n/d\mu_n$.

Lemma B.1. *Suppose that R_n is a sequence of random elements on Ω_n taking values in the metric space $(\mathcal{B},d_{\mathcal{B}})$ and that R is a random element of $\mathcal{B}$. Suppose moreover that*

(S1) $\sup_n\int\rho_n^{1+\delta}\,d\mu_n<\infty$ *for some* $\delta>0$;
(S2) $d_{\mathcal{B}}(R_n\circ g_{n,t},R_n)\to_{\mu_n}0$ *as* $n\to\infty$ *for each* $t\ge 0$ (*equivalently, for all* $t\in[0,t_0]$ *for some fixed* $t_0>0$);
(S3) $\inf_{t_1>0}\limsup_{n\to\infty}\int|\frac{1}{t_1}\int_0^{t_1}\rho_n\circ g_{n,t}\,dt\,-\,1|\,d\mu_n=0$.

Then $R_n\to_{\mu_n}R$ if and only if $R_n\to_{\lambda_n}R$.

Proof. The proof follows [14, Theorem 4,1]. Let $\mathrm{Lip}_{\mathcal{B}}$ denote the space of Lipschitz bounded functions $\psi:\mathcal{B}\to\mathbb{R}$. Define $A_n(\psi,w)=\int\psi\circ R_n\,w\,d\mu_n$ for $\psi\in\mathrm{Lip}_{\mathcal{B}}$ and $w:\mathcal{B}\to\mathbb{R}$ integrable. Note that $|A_n(\psi,w)|\le|\psi|_\infty|w|_1$ for all n.

Now $R_n\to_{\mu_n}R$ if and only if $\lim_{n\to\infty}A_n(\psi,1)=\mathbb{E}(\psi(R))$ for every $\psi\in\mathrm{Lip}_{\mathcal{B}}$. Similarly $R_n\to_{\lambda_n}R$ if and only if $\lim_{n\to\infty}A_n(\psi,\rho_n)=\mathbb{E}(\psi(R))$ for every $\psi\in\mathrm{Lip}_{\mathcal{B}}$. Hence it is enough to show that for every $\psi\in\mathrm{Lip}_{\mathcal{B}}$

$$\lim_{n\to\infty}\big(A_n(\psi,\rho_n)-A_n(\psi,1)\big)=0.$$

Fix $t\ge 0$. Since μ_n is $g_{n,t}$-invariant,

$$A_n(\psi,\rho_n\circ g_{n,t})-A_n(\psi,\rho_n)=\int(\psi\circ R_n-\psi\circ R_n\circ g_{n,t})\,\rho_n\circ g_{n,t}\,d\mu_n.$$

By (S1) and $g_{n,t}$-invariance, $\sup_{n,t}\int\rho_n^{1+\delta}\circ g_{n,t}\,d\mu_n<\infty$. Hence by Hölder's inequality,

$$\big|A_n(\psi,\rho_n\circ g_{n,t})-A_n(\psi,\rho_n)\big|\ll\bigg(\int|\psi\circ R_n-\psi\circ R_n\circ g_{n,t}|^q\,d\mu_n\bigg)^{1/q}$$

where q is the conjugate exponent to $1+\delta$. Now $|\psi\circ R_n-\psi\circ R_n\circ g_{n,t}|\le 2|\psi|_\infty$ and

$$|\psi\circ R_n-\psi\circ R_n\circ g_{n,t}|\le\mathrm{Lip}\,\psi\,d_{\mathcal{B}}(R_n,R_n\circ g_{n,t})\to_{\mu_n}0,$$

by (S2). Hence $\lim_{n\to\infty}(A_n(\psi,\rho_n\circ g_{n,t})-A_n(\psi,\rho_n))=0$ for each $t\ge 0$. Denote $U_{n,t_1}=t_1^{-1}\int_0^{t_1}\rho_n\circ g_{n,t}\,dt$. Then

$$\lim_{n\to\infty}A_n(\psi,U_{n,t_1}-\rho_n)=0,\tag{B.1}$$

for each fixed $t_1 > 0$. Now,

$$\begin{aligned}\big|A_n(\psi,\rho_n) - A_n(\psi,1)\big| &\le \big|A_n(\psi,\rho_n - U_{n,t_1})\big| + \big|A_n(\psi,U_{n,t_1} - 1)\big| \\ &\le \big|A_n(\psi,\rho_n - U_{n,t_1})\big| + |\psi|_\infty \int |U_{n,t_1} - 1|\,d\mu_n.\end{aligned}$$

By (B.1),

$$\limsup_{n\to\infty}\big|A_n(\psi,\rho_n) - A_n(\psi,1)\big| \le |\psi|_\infty \limsup_{n\to\infty}\int |U_{n,t_1} - 1|\,d\mu_n,$$

and the result follows from (S3). □

Remark B.2. The discrete-time version of Lemma B.1 takes the following form. Let (Λ_n,μ_n), $n \ge 1$, be a sequence of probability spaces with measure-preserving maps $T_n : \Lambda_n \to \Lambda_n$. Suppose that λ_n is a sequence of probability measures on Λ_n such that $\lambda_n \ll \mu_n$. Define $\rho_n = d\lambda_n/d\mu_n$. Suppose that R_n is a sequence of random elements on Λ_n taking values in the metric space $(\mathcal{B},d)$ and that R is a random element of $\mathcal{B}$. We continue to assume (S1). Suppose moreover that

(S4) $d_{\mathcal{B}}(R_n \circ T_n, R_n) \to_{\mu_n} 0$ as $n \to \infty$;
(S5) $\inf_{N\ge 1}\limsup_{n\to\infty}\int |\frac{1}{N}\sum_{j=0}^{N-1}\rho_n \circ T_n^j - 1|\,d\mu_n = 0$.

Then $R_n \to_{\mu_n} R$ if and only if $R_n \to_{\lambda_n} R$.

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