Local large deviations for periodic infinite horizon Lorentz gases

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Abstract

We prove optimal local large deviations for the periodic infinite horizon Lorentz gas viewed as a $\mathbb{Z}^d$-cover ($d = 1, 2$) of a dispersing billiard. In addition to this specific example, we prove a general result for a class of nonuniformly hyperbolic dynamical systems and observables associated with central limit theorems with nonstandard normalisation.

1 Introduction

Local large deviations (LLD) for one dimensional i.i.d. random variables that do not satisfy the classical central limit theorem (with the standard normalisation) but are in the domain of a stable law were recently obtained by Caravenna & Doney [9, Theorem 1.1] and refined by Berger [6, Theorem 2.3]. Such results have been extended to multivariate i.i.d. random variables in the domain of the stable laws by Berger in [7]. Roughly speaking, an LLD measures the probability that the sum of the random variables assumes precise, but asymptotically large values. In the absence of second and even first moments, the proofs are considerably harder.

For dynamical systems, the first LLD results in the absence of the classical central limit theorem were obtained in [18]; they are as optimal as [6, Theorem 2.3]. The main shift in that paper is an analytic proof which overcomes the restriction of having independence. Although promising, the results in [18] are limited to the Gibbs Markov maps. The aim of this paper is to prove an optimal LLD estimate for *infinite* horizon periodic Lorentz maps, which were shown to satisfy a central limit theorem with nonstandard normalisation by Szász & Varjú [22]. A crucial new ingredient of the

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proofs of the present LLD results consists of a new operator renewal technique on the Young tower for the billiard map.

Periodic dispersing billiards and Lorentz gases were introduced into ergodic theory and studied by [21]. For a general reference, see [11]. We recall that the classical central limit theorem was proved in the finite horizon case by [8] and local, moderate and large deviations were recently obtained in Dolgopyat & Nándori [13]. In the same work [13] the authors designed a strategy to prove the local limit theorem and mixing properties for group extensions (such as $\mathbb{Z}^d$) of probability preserving flows by free flight functions with finite second moments. For a similar strategy but weaker results we refer to [3]. The strategy in [13] consists of the systematic use of local large and moderate deviations for the underlying probability preserving Poincaré map. Their result applies to the *finite* horizon Lorentz flow. In that case, both the free flight and the roof function are bounded. We believe that the LLD obtained in this paper (Theorem 1.1 below) can be used to prove the local limit theorem and mixing properties for the *infinite* horizon Lorentz flow.

A periodic Lorentz map $(\widetilde{T}, \widetilde{M}, \tilde{\mu})$ is a $\mathbb{Z}^d$-cover of a periodic dispersing billiard (T, M, μ). The notation for the dispersing billiard is recalled in Section 2. We consider the cases $d = 1$ (tubular billiard) and $d = 2$ (planar billiard). We are interested in the case of *infinite horizon* where the time between collisions for the billiard map is unbounded, subject to certain nondegeneracy conditions described in Section 2.

Let $\kappa : M \to \mathbb{Z}^d$ denote the cell-change function (discrete free flight function) between collisions, and define $\kappa_n = \sum_{j=0}^{n-1} \kappa \circ T^j$. For the Lorentz gas, geometrically $\kappa_n \in \mathbb{Z}^d$ denotes the cell in the infinite measure phase space $\widetilde{M}$ where the n'th collision takes place for initial conditions starting in the 0'th cell.

Set

$$a_n = \sqrt{n \log n}.$$

The central limit theorem with nonstandard normalisation proved in [22] says that $a_n^{-1}\kappa_n$ converges in distribution to a nondegenerate d-dimensional normal distribution. In fact, [22] proves a stronger result, namely the corresponding local limit theorem. Our main result is:[1]

Theorem 1.1 (LLD for the dispersing billiard) *There exists $C > 0$ such that*

$$\mu(\kappa_n = N) \le C \frac{n}{a_n^d} \frac{\log |N|}{1 + |N|^2} \quad \textit{for all } n \ge 1,\ N \in \mathbb{Z}^d.$$

Remark 1.2 Again, there is the geometric interpretation that $\mu(\kappa_n = N)$ represents the probability that an initial condition in the 0'th cell of $\widetilde{M}$ lies in the N'th cell after n collisions.

Although we focus on the discrete free flight function $\kappa : M \to \mathbb{Z}^d$, our results apply immediately to the flight function $V : M \to \mathbb{R}^d$ given by the difference in $\mathbb{R}^d$

[1]We set $\log x = 1$ for $x \in [0, 2)$.

between consecutive collision points. Indeed, defining $V_n = \sum_{j=0}^{n-1} V \circ T^j$, it is evident that $|\kappa_n - V_n|$ is bounded by the diameter $\sqrt{d}$ of the cells (since V_n is the distance between successive collisions whereas κ_n is the distance between the centres of the corresponding cells). Hence for any $r > 0$ there exists $C > 0$ such that

$$\mu(V_n \in B_r(x)) \le C \frac{n}{a_n^d} \frac{\log|x|}{1+|x|^2} \quad \text{for all } n \ge 1,\ x \in \mathbb{R}^d.$$

Remark 1.3 The LLD bound for the dispersing billiard follows from a uniform version [19] of the local limit theorem [22] in the range $N \ll \sqrt{n \log n}$. Hence, the principal novelty of Theorem 1.1 lies in the range $N \gg \sqrt{n \log n}$. We note that, as in [18], the approach in this paper does not rely on the local limit theorem and extends to situations where the local limit theorem fails, see Theorem 7.1.

The approach in this paper, following [18], is Fourier analytic and relies on smoothness properties of the leading eigenvalues and their spectral projections for the appropriate transfer operator. We show how to obtain C^r control for all $r < 2$, going considerably beyond previous estimates of [4, 19]. The methods developed in Section 5 to obtain this control in the context of exponential Young towers are the main technical advance of this paper and should have other applications, not only to LLD.

In Section 2, we recall the setting for dispersing billiards. In Section 3, we prove Theorem 1.1 in the range $n \ll \log|N|$. Sections 4 to 6 treat the complementary range $\log|N| \le \epsilon_1 n$ where ϵ_1 is chosen sufficiently small. Key technical estimates are stated in Section 4 and proved in Section 5. In Section 6, we complete the proof of Theorem 1.1. In Section 7, we state and prove an abstract version, Theorem 7.1, of our main result, giving an LLD for a general class of nonuniformly hyperbolic systems modelled by Young towers with exponential tails.

Notation We use "big O" and $\ll$ notation interchangeably, writing $b_n = O(c_n)$ or $b_n \ll c_n$ if there are constants $C > 0$, $n_0 \ge 1$ such that $b_n \le C c_n$ for all $n \ge n_0$. As usual, $b_n = o(c_n)$ means that $\lim_{n\to\infty} b_n/c_n = 0$ and $b_n \sim c_n$ means that $\lim_{n\to\infty} b_n/c_n = 1$.

We write $B_r(x)$ to denote the open ball in $\mathbb{R}^d$ and $\mathbb{C}$ of radius r centred at x.

2 Setup

Define the d-torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. The $\mathbb{Z}^d$-periodic Lorentz gas describes the evolution of a point particle moving in the $\mathbb{Z}^d$-periodic domain $\widetilde{Q}$ contained either in the plane $\mathbb{R}^2$ (if $d = 2$) or in the tube $\mathbb{R} \times \mathbb{T}$ (if $d = 1$). The collisions are assumed to be elastic (equality of pre-collision and post-collision angles). The Lorentz gas map $\widetilde{T} : \widetilde{M} \to \widetilde{M}$ is the collision map on the two-dimensional phase space (position in $\partial\widetilde{Q}$ and unit velocity) given by $\widetilde{M} = \partial\widetilde{Q} \times (-\pi/2, \pi/2)$.

We assume that $\widetilde{Q}$ is the lifted domain[2] of $Q = \mathbb{T}^2 \setminus \Omega$, where $\Omega \subset \mathbb{T}^2$ is a finite union of convex obstacles (scatterers) with C^3 boundaries and nonvanishing curvature, and pairwise disjoint boundaries. The dispersing billiard $T : M \to M$ corresponding to the associated collision map is obtained from $\widetilde{T} : \widetilde{M} \to \widetilde{M}$ by quotienting. We denote by μ the unique ergodic T-invariant smooth probability measure on M.

The Lorentz gas map $\widetilde{T} : \widetilde{M} \to \widetilde{M}$ can be viewed as a $\mathbb{Z}^d$-cover of the dispersing billiard (T, M, μ) by the cell-change function $\kappa : M \to \mathbb{Z}^d$. We assume that κ is unbounded, so that we are in the case of *infinite* horizon. To avoid nondegeneracies in the case $d = 2$, we require that there exist at least two nonparallel collisionless trajectories in the interior of $\widetilde{Q}$. (For $d = 1$, we require that there exists a collisionless trajectory not orthogonal to the direction of the $\mathbb{Z}$-cover, which is equivalent to our assumption that $\kappa : M \to \mathbb{Z}$ is unbounded.) Under these conditions, [22] proved that κ satisfies a central limit theorem and local limit theorem with positive-definite covariance matrix $\Sigma \in \mathbb{R}^{d\times d}$ and nonstandard normalisation $a_n = \sqrt{n \log n}$.

An important part of the proof of the results in [22] and of Theorem 1.1 is that (T, M, μ) is modelled by a two-sided Young tower (f, Δ, μ_Δ) with exponential tails [10, 23]. We briefly recall the notion of Young tower.[3]

Let (Y, μ_Y) be a probability space with an at most countable measurable partition α, and let $F : Y \to Y$ be an ergodic measure-preserving transformation. Define the *separation time* $s(y, y')$ to be the least integer $n \geq 0$ such that $F^n y$ and $F^n y'$ lie in distinct partition elements in α. It is assumed that the partition α separates trajectories, so $s(y, y') = \infty$ if and only if $y = y'$; then $d_\theta(y, y') = \theta^{s(y,y')}$ is a metric for $\theta \in (0, 1)$. We say that F is a *(full-branch) Gibbs-Markov map* if

- $F|_a : a \to Y$ is a measurable bijection for each $a \in \alpha$, and
- There are constants $C > 0$, $\theta \in (0, 1)$ such that $|\log \xi(y) - \log \xi(y')| \leq C d_\theta(y, y')$ for all $y, y' \in a$, $a \in \alpha$, where $\xi = \frac{d\mu_Y}{d\mu_Y \circ F} : Y \to \mathbb{R}$.

Let $F : Y \to Y$ be a Gibbs-Markov map and let $\sigma : Y \to \mathbb{Z}^+$ be constant on partition elements such that $\mu_Y(\sigma > n) = O(e^{-an})$ for some $a > 0$, We define the *one-sided Young tower with exponential tails* $\bar{\Delta} = Y^\sigma$ and *tower map* $\bar{f} : \bar{\Delta} \to \bar{\Delta}$ as follows:

$$\bar{\Delta} = \{(y, \ell) \in Y \times \mathbb{Z} : 0 \leq \ell \leq \sigma(y) - 1\}, \quad \bar{f}(y, \ell) = \begin{cases} (y, \ell + 1) & \ell \leq \sigma(y) - 2 \\ (Fy, 0) & \ell = \sigma(y) - 1 \end{cases}.$$

Let $\bar{\sigma} = \int_Y \sigma \, d\mu_Y$. Then $\bar{\mu}_\Delta = (\mu_Y \times \text{counting})/\bar{\sigma}$ is an ergodic $\bar{f}$-invariant probability measure on $\bar{\Delta}$.

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[2] by the canonical projection from $\mathbb{R}^2$ (if $d = 2$) or from $\mathbb{R} \times \mathbb{T}$ (if $d = 1$) onto $\mathbb{T}^2$

[3] We suppress many standard details about Young towers, mentioning only those aspects required for this paper. For instance, we suppress the fact that the projection $\bar{\pi} : \Delta \to \bar{\Delta}$ corresponds in practice to collapsing stable leaves.

We say that (T, M, μ) is modelled by a Young tower (f, Δ, μ_Δ) with exponential tails if there exist a one-sided Young tower $(\bar f, \bar\Delta, \bar\mu_\Delta)$ and measure-preserving semiconjugacies

$$\pi : \Delta \to M, \qquad \bar\pi : \Delta \to \bar\Delta.$$

Next, we recall some properties proved in [22] of the cell-change function $\kappa : M \to \mathbb{Z}^d$. First, there is a constant $C > 0$ such that $\mu(|\kappa| = n) \sim Cn^{-3}$. Second, κ lifts to a function $\hat\kappa = \kappa \circ \pi : \Delta \to \mathbb{Z}^d$ that is constant on $\bar\pi^{-1}(a \times \{\ell\})$ for each $a \in \alpha$, $\ell \in \{0, \dots, \sigma(a) - 1\}$. Hence $\hat\kappa$ projects to an observable $\bar\kappa : \bar\Delta \to \mathbb{Z}^d$ constant on the partition elements $a \times \{\ell\}$ of $\bar\Delta$. In particular, $\bar\mu_\Delta(|\bar\kappa| = n) = \mu(|\kappa| = n) \sim Cn^{-3}$.

Define

$$\psi : Y \to \mathbb{R}, \qquad \psi(y) = \textstyle\sum_{\ell=0}^{\sigma-1} |\hat\kappa(y, \ell)|.$$

Proposition 2.1 *There exists $C > 0$ such that*

$$\mu_Y(\psi > n) \le Cn^{-2} \quad \textit{for all } n \ge 1.$$

In particular, $\psi \in L^r(Y)$ for all $r < 2$.

Proof This is proved in [22]. The main step [22, Lemma 16] uses the bound $\mu(|\kappa| > n) = O(n^{-2})$ together with the structure of infinite horizon dispersing billiards (see also [12, Lemma 5.1]). The bound for $\mu_Y(\psi > n)$ then follows (see for instance [12, Section 2])). ∎

We end this subsection by recalling some results about transfer operators and perturbed transfer operators on the one-sided tower. Let $P : L^1(\bar\Delta) \to L^1(\bar\Delta)$ be the transfer operator for $(\bar f, \bar\Delta, \bar\mu_\Delta)$, so $\int_{\bar\Delta} Pv\, w\, d\bar\mu_\Delta = \int_{\bar\Delta} v\, w \circ \bar f\, d\bar\mu_\Delta$ for all $v \in L^1$, $w \in L^\infty$. By [4, Section 3.3], there is a Banach space $\mathcal{B}'$ containing 1 and dense in L^1 (called $\mathcal{H}$ in [4]) such that $P : \mathcal{B}' \to \mathcal{B}'$ is quasicompact. (The definition of $\mathcal{B}'$ is not used in this paper.) In particular, the intersection of the spectrum of $P : \mathcal{B}' \to \mathcal{B}'$ with the unit circle consists of finitely many eigenvalues $\lambda_0, \dots, \lambda_{q-1}$ of finite multiplicity and these are the q'th roots of unity $\lambda_k = e^{2\pi ik/q}$. By ergodicity, these eigenvalues are simple.

We consider the perturbed family of transfer operators

$$P_t : L^1(\bar\Delta) \to L^1(\bar\Delta), \quad P_t v = P(e^{it\cdot\bar\kappa} v),\ t \in \mathbb{R}^d,$$

where $\cdot$ denotes the standard scalar product on $\mathbb{R}^d$. Applying results of [17], it is shown in [4, Section 3.3.2] that there exists $\delta > 0$ so that $t \mapsto P_t : \mathcal{B}' \to L^3$ is continuous for $t \in B_\delta(0)$. Moreover, there are continuous families of simple isolated eigenvalues $t \mapsto \lambda_{k,t}$ for $P_t : \mathcal{B}' \to \mathcal{B}'$ with $\lambda_{k,0} = \lambda_k$ and $|\lambda_{k,t}| \le 1$. Let $t \mapsto \Pi_{k,t}$ denote the corresponding spectral projections on $\mathcal{B}'$. Then

$$P_t^n = \sum_{k=0}^{q-1} \lambda_{k,t}^n \Pi_{k,t} + Q_t^n, \tag{2.1}$$

where $Q_t = P_t\big(I - \Pi_{0,t} - \cdots - \Pi_{q-1,t}\big)$. By [17, Corollary 2], there exist $C > 0$ and $\gamma \in (0,1)$ such that

$$\sup_{t\in B_\delta(0)} \|Q_t^n\|_{\mathcal{B}'} \le C\gamma^n. \tag{2.2}$$

Finally, by [22],

$$1 - \lambda_{0,t} \sim \Sigma t \cdot t \,\log(1/|t|) \quad \text{as } t \to 0. \tag{2.3}$$

3 The range $n \ll \log|N|$.

In this section, we prove Theorem 1.1 in the range $n \ll \log|N|$. This estimate holds at the level of $T : M \to M$ and $\kappa : M \to \mathbb{Z}^d$ (without requiring consideration of Young towers). Recall that $d \in \{1, 2\}$.

Lemma 3.1 *Let $\omega > 0$, $q \ge 1$. There exists $C > 0$ such that*

$$\mu(\kappa_n = N) \le C\frac{1}{|N|^2\, n^q} \quad \textit{for all } n \ge 1,\ N \in \mathbb{Z}^d \textit{ with } n \le \omega\log|N|.$$

Proof We use $|x| = \max_{j=1,\dots,d}|x_j|$ so that $|\kappa|$ is integer-valued. Let

$$S_n = \sum_{j=0}^{n-1} |\kappa|\circ T^j, \qquad M_n = \max_{j=0,\dots,n-1} |\kappa|\circ T^j.$$

For $q \ge 1$, define $J_n = \#\{0 \le j \le n-1 : |\kappa|\circ T^j \ge |N|/n^q\}$. Since $q \ge 1$, the constraint $\kappa_n = N$ implies that $J_n \ge 1$. Let $\epsilon > 0$. We show that

$$\mu\big(M_n > |N|^{1+\epsilon}\big) \ll \frac{n}{|N|^{2+2\epsilon}}, \qquad \mu\big(\kappa_n = N,\ J_n = 1\big) \ll \frac{1}{|N|^2\, n^{q-2}},$$
$$\mu\big(S_n \ge |N|,\ M_n \le |N|^{1+\epsilon},\ J_n \ge 2\big) \ll \frac{n^{3q+1}}{|N|^{2+\frac{1}{45}}}.$$

Then $\mu(\kappa_n = N) \ll \frac{1}{|N|^2\, n^{q-2}}$ since $n \ll \log|N|$. The result follows since $q \ge 1$ is arbitrary.

First,

$$\mu\big(M_n > |N|^{1+\epsilon}\big) \le \sum_{j=0}^{n-1} \mu(|\kappa|\circ T^j > |N|^{1+\epsilon}) = n\mu(|\kappa| > |N|^{1+\epsilon}) \ll n/|N|^{2+2\epsilon}.$$

Second, if $J_n = 1$, then there exists $j \in \{0,\dots,n-1\}$ such that $|\kappa|\circ T^j \ge |N|/n^q$ and $\sum_{0\le i\le n-1,\, i\ne j} |\kappa|\circ T^i \le (n-1)|N|/n^q \le |N|/n^{q-1}$. Since $\kappa_n = N$, this means that $|\kappa|\circ T^j \in (|N|-|N|/n^{q-1}, |N|+|N|/n^{q-1})$. Hence

$$\begin{aligned}
\mu\big(\kappa_n = N,\ J_n = 1\big) &\le \sum_{j=0}^{n-1} \mu\left(\big||\kappa|\circ T^j - |N|\big| \le |N|/n^{q-1}\right) \\
&= n\mu\left(\big||\kappa| - |N|\big| \le |N|/n^{q-1}\right) \\
&= n \sum_{|p-|N|| \le |N|/n^{q-1}} \mu(|\kappa| = p) \ll n \sum_{|p-|N|| \le |N|/n^{q-1}} \frac{1}{p^3} \\
&\le n \cdot \Big(\frac{2|N|}{n^{q-1}} + 3\Big) \cdot \Big(\frac{2}{|N|}\Big)^3 \ll \frac{1}{|N|^2\, n^{q-2}}.
\end{aligned}$$

Finally, we estimate $K = \mu\big(S_n \ge |N|,\ M_n \le |N|^{1+\epsilon},\ J_n \ge 2\big)$. Since $J_n \ge 2$, there exist $0 \le i < j \le n-1$ such that $|\kappa|\circ T^i \ge |N|/n^q$ and $|\kappa|\circ T^j \ge |N|/n^q$. It follows that

$$\begin{aligned}
K &\le \sum_{0\le i<j\le n-1} \mu\big(|N|^{1+\epsilon} \ge |\kappa|\circ T^i \ge \tfrac{|N|}{n^q},\ |\kappa|\circ T^j \ge \tfrac{|N|}{n^q}\big) \\
&= \sum_{0\le i<j\le n-1} \mu\big(|N|^{1+\epsilon} \ge |\kappa| \ge \tfrac{|N|}{n^q},\ |\kappa|\circ T^{j-i} \ge \tfrac{|N|}{n^q}\big) \\
&\le n \sum_{1\le r\le n-1} \mu\big(|N|^{1+\epsilon} \ge |\kappa| \ge \tfrac{|N|}{n^q},\ |\kappa|\circ T^{r} \ge \tfrac{|N|}{n^q}\big) \\
&= n \sum_{|N|^{1+\epsilon} \ge p \ge |N|/n^q}\ \sum_{1\le r\le n-1} \mu\big(|\kappa| = p,\ |\kappa|\circ T^{r} \ge \tfrac{|N|}{n^q}\big).
\end{aligned}$$

Note that the constraints $p \le |N|^{1+\epsilon}$, $n \le \omega \log|N|$ imply that

$$\frac{|N|}{n^q} \gg \frac{|N|}{(\log|N|)^q} \ge \frac{p^{1/(1+\epsilon)}}{(\log p^{1/(1+\epsilon)})^q},$$

so there is a constant $c > 0$ such that $\frac{|N|}{n^q} \ge cp^{1/(1+2\epsilon)}$. Also, the constraints $p \ge |N|/n^q$, $n \le \omega\log|N|$ imply that

$$n \le \omega\log|N| \le \omega\log(n^q p) = q\omega\log n + \omega\log p,$$

so there is a constant $\omega' > 0$ such that $n \le \omega'\log p$. Hence, we can choose $\epsilon > 0$ so that

$$K \le n \sum_{p \ge |N|/n^q}\ \sum_{1\le r<\omega'\log p} \mu\big(|\kappa| = p,\ |\kappa|\circ T^r \ge cp^{4/5}\big).$$

By [22, Lemma 16] (see also [12, Lemma 5.1]), there is a constant $C > 0$ such that

$$\mu\big(|\kappa| = p,\ |\kappa|\circ T^r \ge cp^{4/5}\big) \le Cp^{-2/45}\mu(|\kappa| = p) \ll p^{-(3+2/45)}$$

for $1 \le r < \omega' \log p$. Hence taking $\eta = 1/45$,

$$K \ll n \sum_{p \ge |N|/n^q} (\log p)\, p^{-(3+2/45)} \ll n \sum_{p \ge |N|/n^q} p^{-(3+\eta)} \ll \frac{n^{1+q(2+\eta)}}{|N|^{2+\eta}} \le \frac{n^{3q+1}}{|N|^{2+\eta}}$$

completing the proof. ∎

4 Key estimates on the one-sided tower

To prove Theorem 1.1, it remains by Lemma 3.1 to consider the range $\log |N| \le \epsilon_1 n$ where ϵ_1 is chosen sufficiently small. Since $\mu(\kappa_n = N) = \bar\mu_\Delta(\bar\kappa_n = N)$, it suffices to work on the one-sided tower $\bar\Delta$. To simplify the notation, we write (f, Δ, μ_Δ) for the one-sided tower map, and $\kappa : \Delta \to \mathbb{Z}^d$ for the free flight function on the one-sided tower. Since the free flight function on M has mean zero (by time-reversibility of the billiard map), it follows that $\int_\Delta \kappa \, d\mu_\Delta = 0$.

To apply the method from [18], we require the following lemmas concerning the leading eigenvalues $\lambda_{k,t}$ for P_t and their corresponding spectral projections $\Pi_{k,t}$ in (2.1). As clarified in [19, Lemma 5.1], the derivative of P_t at $t = 0$ is not a bounded operator from $\mathcal{B}' \to L^1$. In Section 5, we work with the Banach space $\mathcal{B} \subset \mathcal{B}' \cap L^\infty$ consisting of dynamically Hölder observables and show that we have sufficient control on $\Pi_t : \mathcal{B} \to L^1$. Let $\partial_j = \partial_{t_j}$ for $j = 1, \dots, d$. For $t, h \in \mathbb{R}^d$, $b > 0$, set

$$M_b(t,h) = |h| L(h) \big\{ 1 + L(h)\, |t|^2 L(t) + |h|^{-b|t|^2 L(t)} L(h)^2\, |t|^4 L(t)^2 \big\}$$

where $L(t) = \log(1/|t|)$.

Lemma 4.1 *Let $j \in \{1, \dots, d\}$, $k \in \{0, \dots, q-1\}$. There exists $\delta > 0$ such that $t \mapsto \lambda_{k,t}$ and $t \mapsto \Pi_{k,t} : \mathcal{B} \to L^1$ are C^1 on $B_\delta(0)$. Moreover, $\partial_j \lambda_{k,0} = 0$.*

Furthermore, there exist $C > 0$, $\delta > 0$, $b > 0$ such that for all $t, h \in B_\delta(0)$,

$$|\partial_j \lambda_{k,t+h} - \partial_j \lambda_{k,t}| \le C M_b(t,h), \qquad \|\partial_j \Pi_{k,t+h} - \partial_j \Pi_{k,t}\|_{\mathcal{B} \mapsto L^1} \le C M_b(t,h).$$

Lemma 4.2 *$\lambda_{k,t} - \lambda_k \sim -\lambda_k\, \Sigma t \cdot t\, L(t)$ as $t \to 0$ for each $k = 0, \dots, q-1$.*

Corollary 4.3 *Let $\beta \ge 0$, $r \in \mathbb{R}$, $k = 0, \dots, q-1$. There exist $C > 0$, $\delta > 0$ such that*

$$\int_{B_{3\delta}(0)} |t|^\beta L(t)^r |\lambda_{k,t}|^n \, dt \le C \frac{(\log n)^r}{a_n^{d+\beta}} \quad \text{for all } n \ge 1.$$

Proof By Lemma 4.2, $|\lambda_{k,t}| - 1 \sim -\Sigma t \cdot t\, L(t)$ and hence $\log |\lambda_{k,t}| = -\Sigma t \cdot t\, L(t)(1 + o(1))$. Since Σ is positive-definite, there exists $c > 0$ such that $\log |\lambda_{k,t}| \le -c |t|^2 L(t)$. The result now follows from [18, Lemma 2.3]. (The argument in [18] uses that a_n satisfies $n \log a_n \sim a_n^2$, so $a_n \sim (\frac{1}{2} n \log n)^{1/2}$, which agrees with the definition of a_n used here up to an inconsequential constant factor.) ∎

5 Proof of Lemmas 4.1 and 4.2

This section contains the proof of the key estimates Lemma 4.1 and 4.2 concerning the leading eigenvalues $\lambda_{k,t}$ and spectral projections $\Pi_{k,t}$ for the perturbed transfer operator P_t. This represents the main technical advance of this paper. The methods of [17] give log-Lipschitz control which is insufficient for our purposes. In [19], it was shown how to get almost C^2 control at $t = 0$; here we show how to get almost C^2 control in a full neighbourhood of 0. Our method is to consider leading eigenvalues $\tau_{k,t}$ and spectral projections $\pi_{k,t}$ at the level of the base Y of the tower Δ. The uniformity of the dynamics on Y enables strong control on $\tau_{k,t}$ and $\pi_{k,t}$ and this control lifts via the operator renewal theory of [14, 15, 20] to the Young tower Δ. Using [17], we are able to identify the lifted quantities with $\lambda_{k,t}$ and $\Pi_{k,t}$ thereby transferring the required regularity properties.

In Subsection 5.1, we consider estimates for renewal operators on the base Y of the tower. Significantly more refined estimates are obtained in Subsection 5.2. These estimates enable us in 5.3 to obtain the required strong control on $\tau_{k,t}$ and $\pi_{k,t}$. In Subsection 5.4, we show how to transfer this control to λ_t and Π_t.

We continue to work on the one-sided tower Δ. Fix $\theta \in (0,1)$ and recall the definition of the metric d_θ on Y from Section 2. We define the Banach space $\mathcal{B} = \mathcal{B}(\Delta)$ of dynamically Hölder observables $v : \Delta \to \mathbb{R}$ with $\|v\|_{\mathcal{B}} < \infty$, where

$$\|v\|_{\mathcal{B}} = \sup_{(y,\ell)\in\Delta} |v(y,\ell)| + \sup_{(y,\ell)\neq(y',\ell)} \frac{|v(y,\ell) - v(y',\ell)|}{d_\theta(y,y')}.$$

In this section, we often write $\mathcal{B}(\Delta)$ and $L^1(\Delta)$ for the function spaces on the Young tower Δ, to distinguish them from related function spaces defined on the base Y.

5.1 Renewal operators

Let $R : L^1(Y) \to L^1(Y)$ denote the transfer operator corresponding to the Gibbs-Markov map $F : Y \to Y$, so $\int_Y Rv\, w\, d\mu_Y = \int_Y v\, w \circ F\, d\mu_Y$ for all $v \in L^1$, $w \in L^\infty$. For $y \in Y$ and $a \in \alpha$, let y_a denote the unique preimage $y_a \in a$ such that $Fy_a = y$. Recall that $(Rv)(y) = \sum_a \xi(y_a)v(y_a)$ and that there is a constant $C > 0$ such that

$$0 < \xi(y_a) \le C\mu_Y(a), \qquad |\xi(y_a) - \xi(y'_a)| \le C\mu_Y(a)d_\theta(y,y'), \tag{5.1}$$

for all $y, y' \in Y$, $a \in \alpha$. (Standard references for properties of the transfer operator R for a Gibbs-Markov map include [1, 2].)

Define the Banach space $\mathcal{B}_1(Y)$ of observables $v : Y \to \mathbb{R}$ with $\|v\|_{\mathcal{B}_1(Y)} < \infty$ where $\|v\|_{\mathcal{B}_1(Y)} = \|v\|_\infty + \sup_{y\neq y'} |v(y) - v(y')|/d_\theta(y,y')$.

Proposition 5.1 *There exists $C > 0$ such that $\|R(uv)\|_{\mathcal{B}_1(Y)} \le C\|u\|_1\|v\|_{\mathcal{B}_1(Y)}$ for all $u \in L^1(Y)$ constant on partition elements and all $v \in \mathcal{B}_1(Y)$.*

Proof Since u is constant on partition elements, we write $u(a) = u|_a$. By (5.1),

$$\|R(uv)\|_\infty \ll \textstyle\sum_a \mu_Y(a)|u(a)|\,d\mu_Y\,\|v\|_\infty = \|u\|_1\|v\|_\infty.$$

Next, let $y, y' \in Y$. Then $(R(uv))(y) - (R(uv))(y') = I_1 + I_2$ where

$$I_1 = \textstyle\sum_a (\xi(y_a) - \xi(y'_a))u(a)v(y_a), \qquad I_2 = \textstyle\sum_a \xi(y'_a)u(a)(v(y_a) - v(y'_a)).$$

By (5.1),

$$\begin{aligned}|I_1| &\ll \textstyle\sum_a \mu_Y(a)d_\theta(y,y')|u(a)|\|v\|_\infty = \|u\|_1\|v\|_\infty\, d_\theta(y,y'),\\ |I_2| &\ll \textstyle\sum_a \mu_Y(a)|u(a)|\|v\|_{\mathcal{B}_1(Y)}\, d_\theta(y_a,y'_a) \le \|u\|_1\|v\|_{\mathcal{B}_1(Y)}\, d_\theta(y,y').\end{aligned}$$

Hence $|(R(uv))(y) - (R(uv))(y')| \ll \|u\|_1\|v\|_{\mathcal{B}_1(Y)}\, d_\theta(y,y')$, and the result follows. ∎

For $z \in \mathbb{C}$ with $|z| \le 1$ and $t \in \mathbb{R}^d$, define

$$\widehat{R}(z,t) : L^1(Y) \to L^1(Y), \qquad \widehat{R}(z,t)v = R(e^{it\cdot\kappa_\sigma} z^\sigma v) = \sum_{n=1}^\infty z^n R_{t,n} v$$

where $R_{t,n}v = R(1_{\{\sigma=n\}}e^{it\cdot\kappa_\sigma}v)$ and $\kappa_\sigma(y) = \sum_{\ell=0}^{\sigma(y)-1}\kappa(y,\ell)$.

We now show that $z \mapsto \widehat{R}(z,t)$ extends analytically to a neighbourhood of the unit disk when restricted to $\mathcal{B}_1(Y)$, and we obtain properties of this extension. Recall that $\psi(y) = \sum_{\ell=0}^{\sigma(y)-1}|\kappa(y,\ell)|$. By Proposition 2.1, κ_σ, $\psi \in L^r(Y)$ for all $r < 2$.

Proposition 5.2 *There exists $\delta > 0$ such that, regarded as operators on $\mathcal{B}_1(Y)$,*

(a) $z \mapsto \widehat{R}(z,t)$ is analytic on $B_{1+\delta}(0)$ for all $t \in \mathbb{R}^d$;

(b) $(z,t) \mapsto (\partial_z^m \widehat{R})(z,t)$ is C^1 on $B_{1+\delta}(0) \times \mathbb{R}^d$ for all $m \ge 0$;

(c) $z \mapsto (\partial_j \widehat{R})(z,t)$ is C^1 on $B_{1+\delta}(0)$ uniformly in $t \in \mathbb{R}^d$ for $j = 1,\dots,d$.

Proof It suffices to show that there exist $a > 0$, $C > 0$ such that

$$\|R_{t,n}\|_{\mathcal{B}_1(Y)} \le Ce^{-an}, \qquad \|\partial_j R_{t,n}\|_{\mathcal{B}_1(Y)} \le Ce^{-an},$$

for all $t \in \mathbb{R}^d$, $j = 1,\dots,d$, $n \ge 1$.

Since $\kappa_\sigma \in L^r(Y)$ for all $r < 2$ and σ has exponential tails, there exists $a > 0$ such that $\|1_{\{\sigma=n\}}\kappa_\sigma\|_1 \ll e^{-an}$.

Note that σ and κ_σ are constant on partition elements. By Proposition 5.1, $\|R_{t,n}\|_{\mathcal{B}_1(Y)} \ll \|1_{\{\sigma=n\}}\|_1$. Also, $\|\partial_j R_{t,n}\|_{\mathcal{B}_1(Y)} \ll \|1_{\{\sigma=n\}}\kappa_\sigma\|_1$ completing the proof. ∎

For $z \in \mathbb{C}$ with $|z| \le 1$ and $t \in \mathbb{R}^d$, define

$$\widehat{A}(z,t) : L^1(Y) \to L^1(\Delta), \qquad \widehat{A}(z,t)v = \sum_{n=1}^\infty z^n A_{t,n} v$$

where $(A_{t,n}v)(y,\ell) = 1_{\{\ell=n\}}(P_t^n v)(y,\ell) = 1_{\{\ell=n\}}e^{it\cdot\kappa_n(y,0)}v(y)$.

Proposition 5.3 *There exists $\delta > 0$ such that regarded as operators from $L^\infty(Y)$ to $L^1(\Delta)$,*

(a) $z \mapsto \widehat{A}(z,t)$ is analytic on $B_{1+\delta}(0)$ for all $t \in \mathbb{R}^d$;

(b) $(z,t) \mapsto (\partial_z \widehat{A})(z,t)$ is C^1 on $B_{1+\delta}(0) \times \mathbb{R}^d$.

Proof Let $\|\ \|$ denote $\|\ \|_{L^\infty(Y) \mapsto L^1(\Delta)}$. As in the proof of Proposition 5.2, it suffices to obtain exponential estimates for $\|A_{t,n}\|$ and $\|\partial_j A_{t,n}\|$.

There exists $a > 0$ such that $\|1_{\{\sigma>n\}}\psi\|_{L^1(Y)} = O(e^{-an})$. Now $(A_{t,n}v)(y,\ell) = 1_{\{\ell=n\}} e^{it\cdot\kappa_n(y)} v(y)$ so

$$\|A_{t,n}\| \le \int_\Delta 1_{\{\ell=n\}}\, d\mu_\Delta \le \|1_{\{\sigma>n\}}\|_{L^1(Y)}.$$

Similarly, $\|\partial_j A_{t,n}\| \le \int_\Delta 1_{\{\ell=n\}}\psi\, d\mu_\Delta \le \|1_{\{\sigma>n\}}\psi\|_{L^1(Y)}$ completing the proof. ∎

For $z \in \mathbb{C}$ with $|z| \le 1$ and $t \in \mathbb{R}^d$, define

$$\widehat{B}(z,t) : L^1(\Delta) \to L^1(Y), \qquad \widehat{B}(z,t)v = \sum_{n=1}^\infty z^n B_{t,n} v$$

where

$$B_{t,n}v = 1_Y P_t^n(1_{D_n} v), \qquad D_n = \{(y, \sigma(y) - n) : y \in Y,\, \sigma(y) > n\}.$$

Proposition 5.4 *There exists $\delta > 0$ such that regarded as operators from $\mathcal{B}(\Delta)$ to $\mathcal{B}_1(Y)$,*

(a) $z \mapsto \widehat{B}(z,t)$ is analytic on $B_{1+\delta}(0)$ for all $t \in \mathbb{R}^d$;

(b) $(z,t) \mapsto (\partial_z \widehat{B})(z,t)$ is C^1 on $B_{1+\delta}(0) \times \mathbb{R}^d$.

Proof Let $\|\ \|$ denote $\|\ \|_{\mathcal{B}(\Delta) \mapsto \mathcal{B}_1(Y)}$. Again, it suffices to obtain exponential estimates for $\|B_{t,n}\|$ and $\|\partial_j B_{t,n}\|$.

We can write $B_{t,n}v = R(u_{t,n} v_n)$ where

$$u_{t,n}(y) = 1_{\{\sigma(y)>n\}} e^{it\cdot\kappa_n(y,\sigma(y)-n)}, \qquad v_n(y) = 1_{\{\sigma(y)>n\}} v(y, \sigma(y) - n).$$

Note that $u_{t,n}$ is constant on partition elements and $\|v_n\|_{\mathcal{B}_1(Y)} \le \|v\|_{\mathcal{B}}$. Also, there exists $a > 0$ such that $\|1_{\{\sigma>n\}}\psi\|_{L^1(Y)} = O(e^{-an})$.

By Proposition 5.1,

$$\|B_{t,n}\| \ll \|u_{t,n}\|_{L^1(Y)} = \|1_{\{\sigma>n\}}\|_{L^1(Y)}.$$

Similarly, $\|\partial_j B_{t,n}\| \ll \|1_{\{\sigma>n\}}\psi\|_{L^1(Y)}$ completing the proof. ∎

For $z \in \mathbb{C}$ with $|z| \le 1$ and $t \in \mathbb{R}^d$, define

$$\widehat{E}(z,t) : \mathcal{B}(\Delta) \to L^1(\Delta), \qquad \widehat{E}(z,t)v = \sum_{n=1}^{\infty} z^n E_{t,n} v$$

where $(E_{t,n}v)(y,\ell) = 1_{\{\ell>n\}}(P_t^n v)(y,\ell)$.

Proposition 5.5 *There exists $\delta > 0$ such that regarded as operators from $\mathcal{B}(\Delta)$ to $L^1(\Delta)$,*

(a) $z \mapsto \widehat{E}(z,t)$ is analytic on $B_{1+\delta}(0)$ for all $t \in \mathbb{R}^d$;

(b) $(z,t) \mapsto \widehat{E}(z,t)$ is C^0 on $B_{1+\delta}(0) \times \mathbb{R}^d$.

Proof Let $\| \ \|$ denote $\| \ \|_{\mathcal{B}(\Delta)\mapsto L^1(Y)}$. It suffices to obtain an exponential estimate for $\|E_{t,n}\|$. But $(E_{t,n}v)(y,\ell) = 1_{\{\ell>n\}} e^{it\cdot\kappa_n(y,\ell-n)} v(y,\ell-n)$, so

$$\|E_{t,n}\| \le \int_\Delta 1_{\{\ell>n\}}\, d\mu_\Delta \le \|\sigma\, 1_{\{\sigma>n\}}\|_{L^1(Y)} \ll e^{-\epsilon n}$$

as required. ∎

5.2 Further estimates

In this subsection, we obtain more refined estimates on the renewal operators from Subsection 5.1, exploiting the fact (Proposition 2.1) that $\mu_Y(\psi > n) = O(n^{-2})$.

Proposition 5.6 *There exist $C > 0$, $\delta > 0$, $b > 0$ such that*

$$\|\partial_j \partial_z \widehat{R}(z,t+h) - \partial_j \partial_z \widehat{R}(z,t)\|_{\mathcal{B}_1(Y)} \le C|h|L(h)^2 \big\{1 + |h|^{-b\log|z|} L(h)(|z|-1)\big\},$$

for all $t, h \in B_\delta(0)$, all $z \in \mathbb{C}$ with $1 \le |z| \le 1+\delta$, and all $j = 1,\dots,d$.

Proof In this argument, we take $|x| = \max_{j=1,\dots,d} |x_j|$ on $\mathbb{R}^d$ so that ψ is integer-valued. Now, $\partial_j \partial_z \widehat{R}(z,t)v = iR((\kappa_\sigma)_j e^{it\cdot\kappa_\sigma} \sigma z^{\sigma-1} v)$. By Proposition 5.1,

$$\begin{aligned}\|\partial_j \partial_z \widehat{R}(z,t+h) - \partial_j \partial_z \widehat{R}(z,t)\|_{\mathcal{B}_1(Y)} &\ll \int_Y |\kappa_\sigma||e^{ih\cdot\kappa_\sigma} - 1|\sigma|z|^\sigma\, d\mu_Y \\ &\le 2\int_Y \psi \min\{|h|\psi, 1\}\sigma|z|^\sigma\, d\mu_Y = 2\sum_{m,n=1}^{\infty} r_{m,n}\end{aligned}$$

where

$$r_{m,n} = \mu_Y\big(\psi = m, \sigma = n\big) mn \min\{|h|m, 1\}|z|^n.$$

Recall that $\mu_Y(\sigma = n) = O(e^{-an})$ for some $a > 0$. Fix $a_1 \in (0, a)$ and $\delta > 0$ so that $e^{-a}(1 + \delta) < e^{-a_1}$. Then

$$r_{m,n} \ll |h|m^2 ne^{-an}|z|^n \ll |h|m^2 e^{-a_1 n}.$$

Fixing $b > 0$ sufficiently large,

$$\sum_{n > b\log m} r_{m,n} \ll |h|m^2 e^{-a_1 b \log m}(1 - e^{-a_1})^{-1} \ll |h|m^2 m^{-a_1 b} \leq |h|m^{-2}.$$

Hence $\sum_{m=1}^{\infty} \sum_{n > b\log m} r_{m,n} \ll |h|$.

It remains to consider the terms with $n \leq b \log m$. Now

$$|z|^n = 1 + (|z|^n - 1) \leq 1 + n|z|^{n-1}(|z| - 1) \ll 1 + (\log m)m^{b\log|z|}(|z| - 1).$$

Hence

$$r_{m,n} \ll \mu_Y(\psi = m, \sigma = n)m(\log m) \min\big\{|h|m, 1\}\{1 + (\log m)m^{b\log|z|}(|z| - 1)\big\}.$$

and so

$$\sum_{n \leq b\log m} r_{m,n} \ll \mu_Y(\psi = m)m \min\big\{|h|m, 1\}\{\log m + (\log m)^2 m^{b\log|z|}(|z| - 1)\big\}.$$

Let $K = [1/|h|] \geq 1$. Then for $m \leq K$,

$$\sum_{n \leq b\log m} r_{m,n} \ll \mu_Y(\psi = m)|h|m^2\{\log K + (\log K)^2 K^{b\log|z|}(|z| - 1)\big\},$$

and so by resummation,

$$\begin{aligned} \sum_{m=1}^{K} \sum_{n \leq b\log m} r_{m,n} &\ll |h| \sum_{m=1}^{K} \mu_Y(\psi = m)m^2\{\log K + (\log K)^2 K^{b\log|z|}(|z| - 1)\big\} \\ &\ll |h|(\log K)^2 + |h|(\log K)^3 K^{b\log|z|}(|z| - 1) \\ &\ll |h|L(h)^2\big\{1 + |h|^{-b\log|z|}L(h)(|z| - 1)\big\}. \end{aligned} \tag{5.2}$$

Next,

$$\begin{aligned} \sum_{m > K} \sum_{n \leq b\log m} r_{m,n} \ll& \sum_{m > K} \mu_Y(\psi = m)m(\log m) \\ &+ (|z| - 1) \sum_{m > K} \mu_Y(\psi = m)m^{1+b\log|z|}(\log m)^2. \end{aligned}$$

Now,

$$\begin{aligned}
&\sum_{m>K}\mu_Y(\psi=m)m^{1+b\log|z|}(\log m)^2\\
&\qquad=\sum_{m>K}\mu_Y(\psi\ge m)m^{1+b\log|z|}(\log m)^2-\sum_{m>K}\mu_Y(\psi>m)m^{1+b\log|z|}(\log m)^2\\
&\qquad\le\mu_Y(\psi>K)K^{1+b\log|z|}(\log K)^2\\
&\qquad\qquad+\sum_{m>K}\mu_Y(\psi\ge m)(m^{1+b\log|z|}(\log m)^2-(m-1)^{1+b\log|z|}(\log(m-1))^2\\
&\qquad\ll K^{b\log|z|-1}(\log K)^2+(1+b\log|z|)\sum_{m>K}\mu_Y(\psi\ge m)m^{b\log|z|}(\log m)^2\\
&\qquad\ll |h|^{1-b\log|z|}L(h)^2+\sum_{m>K}m^{b\log|z|-2}(\log m)^2.
\end{aligned}$$

By Karamata,

$$\sum_{m>K}m^{b\log|z|-2}(\log m)^2\ll(1-b\log|z|)^{-1}K^{b\log|z|-1}(\log K)^2\ll|h|^{1-b\log|z|}L(h)^2.$$

Hence

$$\sum_{m>K}\sum_{n\le b\log m}r_{m,n}\ll|h|L(h)\big\{1+|h|^{-b\log|z|}L(h)(|z|-1)\big\}.$$

This combined with (5.2) gives the desired estimate for $\sum_{m\ge1}\sum_{n\le b\log m}r_{m,n}$, completing the proof. ∎

Remark 5.7 Similarly,

$$\|\partial_j\widehat{R}(z,t+h)-\partial_j\widehat{R}(z,t)\|_{\mathcal{B}_1(Y)}\le C|h|L(h)\big\{1+|h|^{-b\log|z|}L(h)(|z|-1)\big\}.$$

Proposition 5.8 *There exist $C>0$, $\delta>0$, $b>0$ such that*

$$\|\partial_j\widehat{A}(z,t+h)-\partial_j\widehat{A}(z,t)\|_{\mathcal{B}_1(Y)\mapsto L^1(\Delta)}\le C|h|L(h)\big\{1+|h|^{-b\log|z|}L(h)(|z|-1)\big\},$$

for all $t,h\in B_\delta(0)$, all $z\in\mathbb{C}$ with $1\le|z|\le1+\delta$, and all $j=1,\dots,d$.

Proof We have

$$(\widehat{A}(z,t)v)(y,\ell)=\sum_{n=1}^{\infty}z^n1_{\{\ell=n\}}e^{it\cdot\kappa_n(y,0)}v(y)=z^\ell e^{it\cdot\kappa_\ell(y,0)}v(y).$$

Hence

$$\begin{aligned}
\|\partial_j\widehat{A}(z,t+h)-\partial_j\widehat{A}(z,t)\|_{\mathcal{B}_1(Y)\mapsto L^1(\Delta)}&\ll\big\||z|^\sigma\psi\min\{|h|\psi,1\}\big\|_{L^1(Y)}\\
&=\sum_{m,n=1}^{\infty}\mu_Y\big(\psi=m,\sigma=n\big)m|z|^n\min\{|h|m,1\}.
\end{aligned}$$

We now proceed as in the proof of Proposition 5.6, except that there is one less factor of n (hence one less factor of $L(h)$). ∎

Proposition 5.9 *There exist $C > 0$, $\delta > 0$, $b > 0$ such that*

$$\|\partial_j \widehat{B}(z, t+h) - \partial_j \widehat{B}(z,t)\|_{\mathcal{B}(\Delta)\mapsto\mathcal{B}_1(Y)} \le C|h|L(h)\big\{1 + |h|^{-b\log|z|}L(h)(|z|-1)\big\},$$

for all $t, h \in B_\delta(0)$, all $z \in \mathbb{C}$ with $1 \le |z| \le 1+\delta$, and all $j = 1, \dots, d$,

Proof We have

$$\begin{aligned}\|\partial_j \widehat{B}(z, t+h) - \partial_j \widehat{B}(z,t)\|_{\mathcal{B}(\Delta)\mapsto\mathcal{B}_1(Y)} &\ll \sum_{n=1}^{\infty} |z|^n \|1_{\{\sigma>n\}}\psi \min\{|h|\psi, 1\}\|_{L^1(Y)}. \\ &= \sum_{m,n=1}^{\infty} \mu_Y\big(\psi = m, \sigma > n\big) m|z|^n \min\{|h|m, 1\}.\end{aligned}$$

This is the same as in Proposition 5.8 except that $\sigma = n$ is replaced by $\sigma > n$ (which makes no difference given the exponential tails). ∎

5.3 Spectral properties for $\widehat{R}(z,t)$

In this subsection, we analyse the leading eigenvalues and spectral projections for $\widehat{R}(z,t)$. Throughout we make use of the fact that $\lambda_k^\sigma = 1$ (since σ is divisible by q and λ_k is a q'th root of unity). In particular, $\widehat{R}(\lambda_k, 0) = \widehat{R}(1,0) = R$ for $k = 0, \dots, q-1$.

Proposition 5.10 *Let $z \in \mathbb{C}$, $|z| \le 1$. Then $1 \in \operatorname{spec} \widehat{R}(z,0) : \mathcal{B}_1(Y) \to \mathcal{B}_1(Y)$ if and only if $z^q = 1$ in which case 1 is a simple eigenvalue with eigenfunction 1.*

Proof Similar arguments can be found for example in [14, Lemma 6.7] and [20, Section 5.2]. Hence we just sketch the proof.

It is easily seen that the spectral radius of $\widehat{R}(z,0)$ is no larger than $|z|$, so we can restrict to the case $|z| = 1$.

By [1, 2], the essential spectral radius of $\widehat{R}(1,0)$ is strictly less than 1. This property extends to general $|z| = 1$ as follows: In the notation of the proof of Proposition 5.1,

$$\begin{aligned}|R(z^\sigma v)(y) - R(z^\sigma v)(y')| &\le C{\textstyle\sum_a}\mu_Y(a)d_\theta(y,y')\|v\|_\infty + {\textstyle\sum_a}\xi(y'_a)\|v\|_{\mathcal{B}_1(Y)}d_\theta(y_a, y'_a) \\ &= Cd_\theta(y,y')\|v\|_\infty + \|v\|_{\mathcal{B}_1(Y)}\theta d_\theta(y,y').\end{aligned}$$

Hence, we obtain a Lasota-Yorke (or Doeblin-Fortet) inequality $\|\widehat{R}(z,0)v\|_{\mathcal{B}_1(Y)} \le (C+1)\|v\|_\infty + \theta\|v\|_{\mathcal{B}_1(Y)}$ and it follows that the essential spectral radius of $\widehat{R}(z,0)$ is at most θ.

In particular, $1 \in \operatorname{spec} \widehat{R}(z,0)$ if and only if 1 is an eigenvalue. By ergodicity, 1 is a simple eigenvalue for $\widehat{R}(1,0)$ with eigenfunction 1, hence this also holds for $\widehat{R}(z,0)$ when $z^q = 1$.

Finally, suppose that 1 is an eigenvalue for $\widehat{R}(z,0)$ for some $|z| = 1$, with eigenfunction $v \in \mathcal{B}_1(Y)$. Define $\widetilde{v} : \Delta \to \mathbb{C}$, $\widetilde{v}(y,\ell) = z^\ell v(y)$ (note that $\tilde{v} \in \mathcal{B}$). Then, $\widetilde{v}(\cdot,0) = v = \widehat{R}(z,0)v = z(P\widetilde{v})(\cdot,0)$ and, for $\ell \geq 1$, $(P\widetilde{v})(y,\ell) = \widetilde{v}(y,\ell-1) = \widetilde{v}(y,\ell)/z$. This implies that z^{-1} is an eigenvalue for the transfer operator P which, as noted in Section 2, is the case only for $z^q = 1$. ∎

By Proposition 5.2(b), for $k = 0,\dots,q-1$, the eigenvalue 1 for $\widehat{R}(\lambda_k,0)$ extends to a C^1 family of simple isolated eigenvalues $(z,t) \mapsto \tau_k(z,t)$ on $B_\delta(\lambda_k) \times B_\delta(0)$, for some $\delta > 0$, with $\tau_k(\lambda_k,0) = 1$. Let $\bar{\sigma} = \int_Y \sigma\, d\mu_Y$. Recall that $L(t) = \log(1/|t|)$.

Proposition 5.11 *Let $0 \leq k, k' \leq q-1$. There are constants $C > 0$, $\delta > 0$ such that*

(a) $|\tau_k(z,0) - 1 - \lambda_k^{-1}\bar{\sigma}(z-\lambda_k)| \leq C|z-\lambda_k|^2$ *for* $z \in B_\delta(\lambda_k)$*;*

(b) $|\tau_k(\lambda_k,t) - 1| \leq C|t|^2 L(t)$ *for* $t \in B_\delta(0)$*;*

(c) $|\tau_k(\lambda_k,t) - \tau_{k'}(\lambda_{k'},t)| \leq C|t|^2$ *for all* $t \in B_\delta(0)$*.*

Proof Let $v_k(z,t)$ denote the C^1 families of eigenfunctions corresponding to the eigenvalues $\tau_k(z,t)$, with $v_k(\lambda_k,0) = 1$. Normalise so that

$$\int_Y \widehat{R}(\lambda_k,0)v_k(z,t)\, d\mu_Y = \int_Y v_k(z,t)\, d\mu_Y = 1$$

for $(z,t) \in B_\delta(\lambda_k) \times B_\delta(0)$. Then

$$\tau_k(z,t) = \int_Y \widehat{R}(z,t)v_k(z,t)\, d\mu_Y = I_k(z,t) + J_k(z,t)$$

where

$$\begin{aligned} I_k(z,t) &= \int_Y \widehat{R}(z,t)1\, d\mu_Y = \int_Y z^\sigma e^{it\cdot\kappa_\sigma}\, d\mu_Y, \\ J_k(z,t) &= \int_Y (\widehat{R}(z,t) - \widehat{R}(\lambda_k,0))(v_k(z,t) - v_k(\lambda_k,0))\, d\mu_Y. \end{aligned}$$

Since $\widehat{R}$ and v_k are C^1, it follows that $J_k(z,0) = O(|z-\lambda_k|^2)$ and $J_k(\lambda_k,t) = O(|t|^2)$. Hence it suffices to consider the first term I_k.

For $t = 0$, using that $\lambda_k^\sigma = 1$,

$$\begin{aligned} I_k(z,0) &= \int_Y (\lambda_k + (z-\lambda_k))^\sigma\, d\mu_Y = \int_Y (1 + \lambda_k^{-1}(z-\lambda_k))^\sigma\, d\mu_Y \\ &= 1 + \lambda_k^{-1}\bar{\sigma}(z-\lambda_k) + O(|z-\lambda_k|)^2 \end{aligned}$$

yielding part (a).

Recall (see the beginning of Section 4) that $\int_\Delta \kappa\, d\mu_\Delta = 0$ and that $\int_Y \sigma\, d\mu_Y = \bar\sigma < \infty$. Hence

$$\int_Y \kappa_\sigma\, d\mu_Y = \int_Y \sum_{\ell=0}^{\sigma(y)-1} \kappa(y,\ell)\, d\mu_Y(y) = \bar\sigma \int_\Delta \kappa\, d\mu_\Delta = 0.$$

It follows that

$$I_k(\lambda_k,t) = 1 + \int_Y (e^{it\cdot\kappa_\sigma} - 1 - it\cdot\kappa_\sigma)\, d\mu_Y,$$

so

$$\begin{aligned}|I_k(\lambda_k,t)-1| &\le \int_Y |e^{it\cdot\kappa_\sigma} - 1 - it\cdot\kappa_\sigma|\, d\mu_Y \le 2\int_Y \min\{|t|^2\psi^2, |t|\psi\}\, d\mu_Y \\ &\le 2|t|^2 \sum_{0\le m\le 1/|t|} m^2\mu_Y(\psi=m) + 2|t|\sum_{m>1/|t|} m\mu_Y(\psi=m).\end{aligned}$$

Using the tail estimate $\mu_Y(\psi>n) = O(n^{-2})$ and resummation we obtain $|I_k(\lambda_k,t) - 1| \ll |t|^2 L(t)$ proving (b). Finally, $I_k(\lambda_k,t)$ is independent of k yielding part (c). ∎

It follows from Proposition 5.11(a) that $(\partial_z\tau_k)(\lambda_k,0) = \lambda_k^{-1}\bar\sigma \ne 0$. By the implicit function theorem, we can solve uniquely the equation $\tau_k(z,t) = 1$ near $(\lambda_k,0)$ to obtain a C^1 solution $z = g_k(t)$, $g_k : B_\delta(0) \to \mathbb{C}$, with $g_k(0) = \lambda_k$.

Recall that $M_b(t,h) = |h|L(h)\big\{1 + L(h)\,|t|^2L(t) + |h|^{-b|t|^2L(t)}L(h)^2\,|t|^4L(t)^2\big\}$.

Corollary 5.12 *There exist $C>0$, $\delta>0$, $b>0$ such that for all $t,h\in B_\delta(0)$, $j=1,\dots,d$, $k=0,\dots,q-1$,*

(a) $|g_k(t) - \lambda_k| \le C|t|^2L(t)$;

(b) $|\partial_j g_k(t+h) - \partial_j g_k(t)| \le CM_b(t,h)$.

Proof Write

$$\tau_k(z,t) = \tau_k(\lambda_k,t) + (z-\lambda_k)c_k(z,t). \tag{5.3}$$

It follows from Proposition 5.2(b) that $(z,t)\mapsto \partial_z\tau_k(z,t)$ is C^1. Introducing momentarily the function $\zeta(s) = \tau_k(\lambda_k + s(z-\lambda_k),t)$,

$$c_k(z,t) = (z-\lambda_k)^{-1}\int_0^1 \zeta'(s)\, ds = \int_0^1 (\partial_z\tau_k)(\lambda_k + s(z-\lambda_k),t)\, ds. \tag{5.4}$$

We deduce that $(z,t)\mapsto c_k(z,t)$ is C^1. By Proposition 5.11(a), $c_k(\lambda_k,0) = \lambda_k^{-1}\bar\sigma \ne 0$ and we can shrink δ if necessary so that $c_k(\lambda_k,t)$ is bounded away from zero for $t\in B_\delta(0)$.

Solving $\tau_k(z,t)=1$,

$$g_k(t)-\lambda_k = z-\lambda_k \sim c_k(\lambda_k,t)^{-1}(1-\tau_k(\lambda_k,t)). \tag{5.5}$$

The spectral radius of $\widehat{R}(\lambda_k,t)$ is at most 1 for all t, so $\tau_k(\lambda_k,t)\in B_1(0)$. Hence $|g_k(t)|\ge 1$ for all t. By Proposition 5.11(b),

$$|g_k(t)-\lambda_k| \sim |c_k(\lambda_k,t)|^{-1}|1-\tau_k(\lambda_k,t)| \ll |t|^2L(t),$$

proving part (a).

Implicit differentiation of $\tau_k(g_k(t),t)\equiv 1$ yields

$$\partial_j g_k(t) = -\partial_j\tau_k(g_k(t),t)/\partial_z\tau_k(g_k(t),t).$$

By smoothness of $\partial_z\tau_k$ and g, the denominator $t\mapsto \partial_z\tau_k(g_k(t),t)$ is C^1. We claim that

$$|\partial_j\tau_k(g_k(t+h),t+h)-\partial_j\tau_k(g_k(t),t)| \ll M_b(t,h) \tag{5.6}$$

from which part (b) follows.

It follows from Proposition 5.2(c) that $z\mapsto \partial_j\tau_k(z,t)$ is C^1 uniformly in t. Also, g is C^1, so $|\partial_j\tau_k(g_k(t+h),t+h)-\partial_j\tau_k(g_k(t),t+h)|\ll |h|$. By (5.3),

$$\begin{aligned}|\partial_j\tau_k(z,t+h)-&\partial_j\tau_k(z,t)| \\ &\le |\partial_j\tau_k(\lambda_k,t+h)-\partial_j\tau_k(\lambda_k,t)| + |z-\lambda_k|\,|\partial_j c_k(z,t+h)-\partial_j c_k(z,t)|.\end{aligned}$$

By Remark 5.7, $|\partial_j\tau_k(\lambda_k,t+h)-\partial_j\tau_k(\lambda_k,t)|\ll |h|L(h)$. By (5.4) and Proposition 5.6,

$$|\partial_j c_k(z,t+h)-\partial_j c_k(z,t)| \ll |h|L(h)^2\big\{1+|h|^{-b\log|z|}L(h)|z-\lambda_k|\big\}.$$

Hence

$$|\partial_j\tau_k(z,t+h)-\partial_j\tau_k(z,t)| \ll |h|L(h)+|h|L(h)^2|z-\lambda_k| + |h|^{1-b\log|z|}L(h)^3|z-\lambda_k|^2,$$

for $|z|\ge 1$. But $|g_k(t)|\ge 1$, so by part (a),

$$|\partial_j\tau_k(g_k(t),t+h)-\partial_j\tau_k(g_k(t),t)| \ll M_b(t,h)$$

completing the proof of the claim. ∎

Let $\pi_k(z,t):\mathcal{B}_1(Y)\to\mathcal{B}_1(Y)$ denote the spectral projection corresponding to $\tau_k(z,t)$.

Lemma 5.13 *There exists $\delta>0$ such that*

$$(1-\tau_k(z,t))^{-1}\pi_k(z,t) = (g_k(t)-z)^{-1}\tilde{\pi}_k(t) + H_k(z,t) \tag{5.7}$$

where $\tilde{\pi}_k(t)$, $H_k(z,t):\mathcal{B}_1(Y)\to\mathcal{B}_1(Y)$ are families of bounded operators satisfying *(a)* $\tilde{\pi}_k$ *is* C^1 *on* $B_\delta(0)$*;*

(b) H_k *is* C^0 *on* $B_\delta(\lambda_k) \times B_\delta(0)$*;*

(c) $z \mapsto H_k(z,t)$ *is analytic on* $B_\delta(\lambda_k)$ *for* $t \in B_\delta(0)$*.*

Moreover, there are constants $C > 0$*,* $b > 0$ *such that* $|\partial_j \tilde{\pi}_k(t+h) - \partial_j \tilde{\pi}_k(t)| \le CM_b(t,h)$ *for* $t, h \in B_\delta(0)$*,* $j = 1\dots,d$*.*

Proof Fix j and k. Throughout this proof, we use the following abbreviations (for $r \ge 0$):

(a) "C^r uniformly in z" means C^r on $B_\delta(0)$ uniformly in $z \in B_\delta(\lambda_k)$;
(b) "jointly C^r" means C^r on $B_\delta(\lambda_k) \times B_\delta(0)$;
(c) "analytic" means analytic on $B_\delta(\lambda_k)$ for all $t \in B_\delta(0)$.

Step 1 Write

$$\pi_k(z,t) = \pi_k(g_k(t),t) + (g_k(t) - z)\widetilde{H}(z,t).$$

It follows from Proposition 5.2(a) that π_k is analytic and hence that $\widetilde{H}$ is analytic.

Next, $\partial_j(\pi_k(g_k(t),t)) = G_1(t) + G_2(t)$ where

$$G_1(t) = (\partial_z \pi_k)(g_k(t),t) \cdot \partial_j g_k(t), \qquad G_2(t) = (\partial_j \pi_k)(g_k(t),t).$$

It follows from Proposition 5.2(b) that $\partial_z \pi_k$ is jointly C^1. Also, g_k is C^1. Hence, by Corollary 5.12(b),

$$|G_1(t+h) - G_1(t)| \ll |h| + |\partial_j g_k(t+h) - \partial_j g_k(t)| \ll M_b(t,h).$$

Next, we note that G_2, with π_k changed to τ_k, was estimated in (5.6), and the identical argument shows that $|G_2(t+h) - G_2(t)| \ll M_b(t,h)$. Hence

$$|\partial_j(\pi_k(g_k(t+h),t+h)) - \partial_j(\pi_k(g_k(t),t))| \ll M_b(t,h).$$

Writing $\widetilde{H}(z,t) = \int_0^1 (\partial_z \pi_k)((1-s)z + sg_k(t),t)\,ds$, we obtain that $\widetilde{H}$ is jointly C^0.

Step 2 Write

$$1 - \tau_k(z,t) = \tau_k(g_k(t),t) - \tau_k(z,t) = (g_k(t) - z)\beta(z,t).$$

Again, it follows from Proposition 5.2(a) that τ_k is analytic and hence that β is analytic. Also, it follows from Proposition 5.2(b) that $\partial_z^2 \tau_k$ is jointly C^1. Writing $\beta(z,t) = \int_0^1 \partial_z \tau_k((1-s)z + sg_k(t),t)\,ds$, we obtain that $\partial_z \beta$ is jointly C^1. By Proposition 5.6,

$$|\partial_j \beta(g_k(t+h),t+h) - \partial_j \beta(g_k(t),t)| \ll M_b(t,h).$$

By Proposition 5.11(a), $|\beta(\lambda_k,0)| = \bar\sigma > 0$ and we can shrink δ if necessary so that β is bounded away from zero on $B_\delta(\lambda_k)\times B_\delta(0)$. Let $\tilde\beta(z,t)=\beta(z,t)^{-1}$. Then, we can write

$$(1-\tau_k(z,t))^{-1} = (g_k(t)-z)^{-1}\big\{\tilde\beta(g_k(t),t)+(g_k(t)-z)q(z,t)\big\},$$

where q is analytic and jointly C^0 and

$$|\partial_j\tilde\beta(g_k(t+h),t+h)-\partial_j\tilde\beta(g_k(t),t)| \ll M_b(t,h).$$

Step 3 Combining Steps 1 and 2, we obtain (5.7) with

$$\begin{aligned}\tilde\pi_k(t) &= \tilde\beta(g_k(t),t)\pi_k(g_k(t),t),\\ H_k(z,t) &= q(z,t)\pi_k(g_k(t),t)+\tilde\beta(g_k(t),t)\widetilde H(z,t)+(g_k(t)-z)q(z,t)\widetilde H(z,t).\end{aligned}$$

The desired regularity properties of $\tilde\pi_k$ and H_k follow immediately from the regularity properties established in Steps 1 and 2. ∎

Corollary 5.14 *There exists $\delta>0$ such that*

$$(I-\widehat R(z,t))^{-1} = \sum_{k=0}^{q-1}(g_k(t)-z)^{-1}\tilde\pi_k(t)+\widehat H(z,t), \quad (z,t)\in B_{1+\delta}(0)\times B_\delta(0),$$

where $\tilde\pi_k$ is as in Lemma 5.13 and $\widehat H(z,t):\mathcal{B}_1(Y)\to\mathcal{B}_1(Y)$ is a family of bounded operators satisfying

- *(a) $\widehat H$ is C^0 on $B_{1+\delta}(0)\times B_\delta(0)$;*
- *(b) $z\mapsto\widehat H(z,t)$ is analytic on $B_{1+\delta}(0)$ for $t\in B_\delta(0)$.*

Proof Let $t\in B_\delta(0)$. For $z\in B_\delta(\lambda_k)$, the spectrum of $\widehat R(z,t)$ is bounded uniformly away from 1 except for the simple eigenvalue $\tau_k(z,t)$ near 1. Hence

$$(I-\widehat R)^{-1} = (1-\tau_k)^{-1}\pi_k+\widehat H_{k,0} \quad \text{on } B_\delta(\lambda_k)\times B_\delta(0),$$

where $z\mapsto\widehat H_{k,0}(z,t)$ is analytic on $B_\delta(\lambda_k)$ for $t\in B_\delta(0)$ and $\widehat H_{k,0}$ is C^0 on $B_\delta(\lambda_k)\times B_\delta(0)$. Applying Lemma 5.13 and relabelling,

$$(I-\widehat R(z,t))^{-1} = (g_k(t)-z)^{-1}\tilde\pi_k(t)+\widehat H_k(z,t).$$

In addition, $z\mapsto(I-\widehat R(z,t))^{-1}$ is analytic on $B_{1+\delta}(0)\setminus\bigcup_k B_\delta(\lambda_k)$ and $(z,t)\mapsto(I-\widehat R(z,t))^{-1}$ is C^0 on $(B_{1+\delta}(0)\setminus\bigcup_k B_\delta(\lambda_k))\times B_\delta(0)$. Hence, we obtain the desired result on $B_{1+\delta}(0)\times B_\delta(0)$. ∎

5.4 Completion of the proof of Lemmas 4.1 and 4.2

Define for $t \in \mathbb{R}^d$, $n \geq 1$,

$$T_{t,n} : L^1(Y) \to L^1(Y), \qquad T_{t,n}v = 1_Y P_t^n(1_Y v).$$

For $z \in \mathbb{C}$, $t \in \mathbb{R}^d$, define $\widehat{P}(z,t) = \sum_{n=0}^{\infty} z^n P_t^n$, $\widehat{T}(z,t) = \sum_{n=0}^{\infty} z^n T_t^n$. By [20], we have the renewal equation $\widehat{T} = (I - \widehat{R})^{-1}$. Also, by [15], $\widehat{P} = \widehat{A}\widehat{T}\widehat{B} + \widehat{E}$.

Throughout, we work on the domain $B_{1+\delta}(0) \times B_\delta(0) \subset \mathbb{C} \times \mathbb{R}^d$. Applying the renewal equation, Corollary 5.14 becomes

$$\widehat{T}(z,t) = \sum_{k=0}^{q-1} (g_k(t) - z)^{-1} \tilde{\pi}_k(t) + \widehat{H}(z,t), \tag{5.8}$$

where $\tilde{\pi}_k$, $\widehat{H} : \mathcal{B}_1(Y) \to \mathcal{B}_1(Y)$ are families of bounded operators satisfying: $\tilde{\pi}_k$ is C^1; $\widehat{H}$ is C^0; $z \mapsto \widehat{H}(z,t)$ is analytic for all t. Moreover, $|\partial_j \tilde{\pi}_k(t+h) - \partial_j \tilde{\pi}_k(t)| \ll M_b(t,h)$.

The same argument as in Step 1 of Lemma 5.13 (using Propositions 5.4 and 5.3 instead of Proposition 5.2) shows that

$$\widehat{A}(z,t) = \widetilde{A}_k(t) + (g_k(t) - z)\widehat{H}_{k,1}(z,t), \qquad \widetilde{A}_k(t) = \widehat{A}(g_k(t), t), \tag{5.9}$$

$$\widehat{B}(z,t) = \widetilde{B}_k(t) + (g_k(t) - z)\widehat{H}_{k,2}(z,t), \qquad \widetilde{B}_k(t) = \widehat{B}(g_k(t), t), \tag{5.10}$$

where $\widetilde{A}_k$, $\widehat{H}_{k,1} : \mathcal{B}_1(Y) \to L^1(\Delta)$ and $\widetilde{B}_k$, $\widehat{H}_{k,2} : \mathcal{B}(\Delta) \to \mathcal{B}_1(Y)$ are families of bounded operators satisfying: $\widetilde{A}_k$, $\widetilde{B}_k$ are C^1; $\widehat{H}_{k,r}$ is C^0; $z \mapsto \widehat{H}_{k,r}(z,t)$ is analytic for all t; for $r = 1, 2$. Moreover, by Propositions 5.8 and 5.9,

$$\|\partial_j \widetilde{A}_k(t+h) - \partial_j \widetilde{A}_k(t)\|_{\mathcal{B}(Y_1) \mapsto L^1(\Delta)} \ll M_b(t,h),$$
$$\|\partial_j \widetilde{B}_k(t+h) - \partial_j \widetilde{B}_k(t)\|_{\mathcal{B}(\Delta) \mapsto \mathcal{B}_1(Y)} \ll M_b(t,h).$$

Combining (5.8), (5.9) and (5.10) together with Proposition 5.5,

$$\widehat{P}(z,t) = \widehat{A}(z,t)\widehat{T}(z,t)\widehat{B}(z,t) + \widehat{E}(z,t) = \sum_{k=0}^{q-1} \Big((g_k(t) - z)^{-1} \tilde{\pi}_{k,1}(t) + \widehat{H}_{k,3}(z,t) \Big),$$

where $\tilde{\pi}_{k,1}$, $\widehat{H}_{k,3} : \mathcal{B}(\Delta) \to L^1(\Delta)$ are families of bounded operators satisfying: $\tilde{\pi}_{k,1}$ is C^1; $\widehat{H}_{k,3}$ is C^0; $z \mapsto \widehat{H}_{k,3}(z,t)$ is analytic for all t; and $|\partial_j \tilde{\pi}_{k,1}(t+h) - \partial_j \tilde{\pi}_{k,1}(t)| \ll M_b(t,h)$.

Let $\| \ \|$ denote $\| \ \|_{\mathcal{B}(\Delta) \mapsto L^1(\Delta)}$. An immediate consequence of the regularity properties of $\widehat{H}_{k,3}$ is that there exists $\gamma \in (0,1)$ such that the Taylor coefficients of $\widehat{H}_{k,3}$ satisfy $\|(H_{k,3})_{t,n}\| \ll \gamma^n$. Hence,

$$\Big\| P_t^n - \sum_{k=0}^{q-1} g_k(t)^{-(n+1)} \tilde{\pi}_{k,1}(t) \Big\| \ll \gamma^n.$$

By (2.1) and (2.2), $\|P_t^n - \sum_{k=0}^{q-1} \lambda_{k,t}^n \Pi_{k,t}\| \ll \gamma^n$ for some $\gamma \in (0,1)$. Altogether, we have shown that there exist $\gamma \in (0,1)$, $C > 0$ such that

$$\Big\| \sum_{k=0}^{q-1} \Big(\lambda_{k,t}^n \Pi_{k,t} - g_k(t)^{-(n+1)} \tilde{\pi}_{k,1}(t) \Big) \Big\| \le C\gamma^n \quad \text{for all } t \in B_\delta(0),\ n \ge 1.$$

Since $|\lambda_{k,0}| = |g_k(0)| = 1$, we can shrink $\delta > 0$ so that $|\lambda_{k,t}| > \gamma$ and $|g_k(t)^{-1}| > \gamma$. It follows that $\{\lambda_{k,t}\}_k = \{g_k(t)^{-1}\}_k$ and $\{\Pi_{k,t}\}_k = \{g_k(t)^{-1}\tilde{\pi}_{k,1}(t)\}_k$. The desired regularity properties of $\lambda_{k,t}$ and $\Pi_{k,t} : \mathcal{B}(\Delta) \mapsto L^1(\Delta)$ now follow from those for g_k and $\tilde{\pi}_{k,1}$, completing the proof of Lemma 4.1.

Proof of Lemma 4.2 After relabelling (by a permutation in k), we can suppose that $\lambda_{k,t} = g_k(t)^{-1}$ with $g_k(0) = \lambda_k^{-1}$ as before and $\lambda_{k,0} = \lambda_k^{-1}$. (In particular, $\lambda_{0,t}$ is unchanged, but λ_k becomes λ_k^{-1}.) By (2.3),

$$g_0(t) - 1 = \lambda_{0,t}^{-1}(1 - \lambda_{0,t}) \sim \Sigma t \cdot t\, L(t).$$

By (5.5) with $k = 0$,

$$1 - \tau_0(1,t) \sim c_0(1,t)(g_0(t) - 1) \sim \bar{\sigma}\, \Sigma t \cdot t\, L(t).$$

Hence by Proposition 5.11(c),

$$1 - \tau_k(\lambda_k^{-1}, t) \sim \bar{\sigma}\, \Sigma t \cdot t\, L(t)$$

for all $k = 0, \ldots, q-1$. Applying (5.5) once more,

$$g_k(t) - \lambda_k^{-1} \sim c_k(\lambda_k^{-1}, t)^{-1}(1 - \tau_k(\lambda_k^{-1}, t)) \sim \lambda_k^{-1}\, \Sigma t \cdot t\, L(t).$$

Finally,

$$\lambda_k - \lambda_{k,t} = \lambda_k - g_k(t)^{-1} = \lambda_k g_k(t)^{-1}(g_k(t) - \lambda_k^{-1}) \sim \lambda_k\, \Sigma t \cdot t\, L(t)$$

completing the proof. ∎

6 Proof of the main result

In this section, we complete the proof of Theorem 1.1. We continue to work on the one-sided tower Δ.

Fix δ as in Section 4. Let $r : \mathbb{R}^d \to \mathbb{C}$ be C^2 with $\operatorname{supp} r \subset B_\delta(0)$ and define $A_{n,N} = \int_{\mathbb{R}^d} e^{-it\cdot N} r(t) P_t^n\, dt$. By (2.1),

$$A_{n,N} = \sum_{k=0}^{q-1} \int_{B_\delta(0)} e^{-it\cdot N} r(t) \lambda_{k,t}^n \Pi_{k,t}\, dt + \int_{B_\delta(0)} e^{-it\cdot N} r(t) Q_t^n\, dt.$$

Following [18], the main step in the proof of Theorem 1.1 is to estimate $\|A_{n,N}\|$. Throughout this section, $\|\ \|$ denotes $\|\ \|_{\mathcal{B}\mapsto L^1}$.

The next result suffices in the range $|N| \le a_n$.

Corollary 6.1 *There exists $C > 0$ such that $\|A_{n,N}\| \le C a_n^{-d}$ for all $n \ge 1$, $N \in \mathbb{Z}^d$.*

Proof By (2.2) and Corollary 4.3, $\|A_{n,N}\| \ll \sum_{k=0}^{q-1} \int_{B_\delta(0)} |\lambda_{k,t}|^n\, dt + \gamma^n \ll a_n^{-d}$. ∎

Recall from the proof of Corollary 4.3 that there is a constant $c > 0$ such that $\log|\lambda_{k,t}| \le -c|t|^2 L(t)$. Let $b > 0$ be as in Lemma 4.1 and define $\epsilon_1 = c/(2b)$. We now focus on the range

$$a_n \le |N| \le e^{\epsilon_1 n}.$$

Choose j so that $|N_j| = \max\{|N_1|, \dots, |N_d|\}$ and set $h = \pi N_j^{-1} e_j$ (where $e_j \in \mathbb{R}^d$ is the j'th canonical unit vector).

Proposition 6.2 *There exist $C > 0$, $\delta > 0$ such that*

$$\int_{B_{2\delta}(0)} \big|\partial_j(\lambda^n)_{k,t} - \partial_j(\lambda^n)_{k,t-h}\big|\, dt \le C \frac{n}{a_n^d} \frac{\log|N|}{|N|}$$

for all $n \ge 1$, $|N| > \pi/\delta$ with $a_n \le |N| \le e^{\epsilon_1 n}$, $k = 0 \dots, q-1$.

Proof In this proof we abbreviate $B_a(0)$ to B_a and suppress dt. Set $s = t - h$ and relabel so that $\int_{B_{2\delta}} M_b(t,h)|\lambda_{k,s}|^n \le \int_{B_{2\delta}} M_b(t,h)|\lambda_{k,t}|^n$. Then $\int_{B_{2\delta}} |\partial_j(\lambda^n)_{k,t} - \partial_j(\lambda^n)_{k,s}| \le J + K$ where

$$J = n\int_{B_{2\delta}} |\lambda_{k,t}^{n-1} - \lambda_{k,s}^{n-1}||\partial_j \lambda_{k,t}|, \qquad K = n\int_{B_{2\delta}} |\lambda_{k,s}|^{n-1}|\partial_j\lambda_{k,t} - \partial_j\lambda_{k,s}|.$$

By Lemma 4.1,

$$K \ll n\int_{B_{2\delta}} M_b(t,h)|\lambda_{k,t}|^n \ll \frac{n\log|N|}{|N|}\int_{B_{2\delta}} |\lambda_{k,t}|^n + \frac{n(\log|N|)^2}{|N|}\int_{B_{2\delta}} |\lambda_{k,t}|^n |t|^2 L(t)$$
$$+ \frac{n(\log|N|)^3}{|N|}\int_{B_{2\delta}} |\lambda_{k,t}|^n |N|^{b|t|^2 L(t)} |t|^4 L(t)^2. \qquad (6.1)$$

Since

$$|N|^{b|t|^2L(t)} = e^{b(\log|N|)|t|^2L(t)} \le e^{b\epsilon_1 n|t|^2L(t)} = e^{\frac{1}{2}cn|t|^2L(t)} \le |\lambda_{k,t}|^{-n/2},$$

it follows from Corollary 4.3 that

$$\int_{B_{2\delta}} |\lambda_{k,t}|^n |N|^{b|t|^2L(t)}|t|^4L(t)^2 \le \int_{B_{2\delta}} |\lambda_{k,t}|^{n/2}|t|^4 L(t)^2 \ll \frac{(\log n)^2}{a_n^{d+4}}.$$

The other integrals in (6.1) are also estimated using Corollary 4.3 and we obtain

$$K \ll \frac{n}{a_n^d}\frac{\log|N|}{|N|}\Big\{1 + \frac{\log|N|\log n}{a_n^2} + \frac{(\log|N|)^2(\log n)^2}{a_n^4}\Big\} \ll \frac{n}{a_n^d}\frac{\log|N|}{|N|}.$$

(Here, we used that $\log|N| \ll n = a_n^2/\log n$.)

Next, $|\lambda_{k,t}^{n-1}-\lambda_{k,s}^{n-1}| \le (n-1)(|\lambda_{k,t}|^{n-2}+|\lambda_{k,s}|^{n-2})|\lambda_{k,t}-\lambda_{k,s}|$ so by the mean value theorem,

$$|\lambda_{k,t}^{n-1}-\lambda_{k,s}^{n-1}| \ll \frac{n}{|N|}(|\lambda_{k,t}|^n+|\lambda_{k,s}|^n)|\partial_j\lambda_{k,u}|$$

for some u between t and s. Accordingly,

$$J \ll \frac{n^2}{|N|}\int_{B_{2\delta}}(|\lambda_{k,t}|^n+|\lambda_{k,s}|^n)|\partial_j\lambda_{k,u}||\partial_j\lambda_{k,t}|.$$

By Lemma 4.1,

$$|\partial_j\lambda_{k,t}| = |\partial_j\lambda_{k,t}-\partial_j\lambda_{k,0}| \ll M_b(0,t) = |t|L(t), \qquad |\partial_j\lambda_{k,u}| \ll |u|\log(1/|u|). \quad (6.2)$$

Now $|u| \le |t|+|h|$, so

$$\begin{aligned}|u|\log(1/|u|) &\le (|t|+|h|)\log(1/(|t|+|h|))\\ &= |t|\log(1/(|t|+|h|))+|h|\log(1/(|t|+|h|))\\ &\le |t|\log(1/|t|)+|h|\log(1/|h|) \ll |t|\log(1/|t|)+\tfrac{\log|N|}{|N|} = |t|L(t)+\tfrac{\log|N|}{|N|}.\end{aligned}$$

In this way, it follows from (6.2) that $|\partial_j\lambda_{k,t}| \ll |t|L(t)$, $|\partial_j\lambda_{k,u}| \ll |t|L(t)+\frac{\log|N|}{|N|}$. Similarly, $|\partial_j\lambda_{k,t}| \ll |s|L(s)+\frac{\log|N|}{|N|}$, $|\partial_j\lambda_{k,u}| \ll |s|L(s)+\frac{\log|N|}{|N|}$. By Corollary 4.3,

$$\begin{aligned}\int_{B_{2\delta}}|\lambda_{k,s}|^n|\partial_j\lambda_{k,t}||\partial_j\lambda_{k,u}| &\ll \int_{B_{2\delta}}|\lambda_{k,s}|^n|s|^2L(s)^2+\frac{\log|N|}{|N|}\int_{B_{2\delta}}|\lambda_{k,s}|^n|s|L(s)\\ &\qquad +\frac{(\log|N|)^2}{|N|^2}\int_{B_{2\delta}}|\lambda_{k,s}|^n\\ &\ll \int_{B_{3\delta}}|\lambda_{k,t}|^n|t|^2L(t)^2+\frac{\log|N|}{|N|}\int_{B_{3\delta}}|\lambda_{k,t}|^n|t|L(t)+\frac{(\log|N|)^2}{|N|^2}\int_{B_{3\delta}}|\lambda_{k,t}|^n\\ &\ll \frac{1}{a_n^d}\Big(\frac{\log^2 n}{a_n^2}+\frac{\log|N|}{|N|}\frac{\log n}{a_n}+\frac{(\log N)^2}{|N|^2}\Big)\\ &= \frac{\log|N|}{na_n^d}\Big(\frac{n\log^2 n}{a_n^2\log|N|}+\frac{n\log n}{a_n|N|}+\frac{n\log N}{|N|^2}\Big)\\ &\ll \frac{\log|N|}{na_n^d}\Big(\frac{\log n}{\log|N|}+\frac{a_n}{|N|}+\frac{a_n^2}{\log a_n}\frac{\log N}{|N|^2}\Big) \ll \frac{\log|N|}{na_n^d}.\end{aligned}$$

A simpler calculation shows that $\int_{B_{2\delta}}|\lambda_{k,t}|^n|\partial_j\lambda_{k,t}||\partial_j\lambda_{k,u}| \ll \dfrac{\log|N|}{na_n^d}$. Hence $J \ll \dfrac{n}{a_n^d}\dfrac{\log|N|}{|N|}$. This completes the proof. ∎

Lemma 6.3 *There exists $C > 0$ such that*

$$\Big\| \int_{B_\delta(0)} e^{-it\cdot N} r(t)\lambda_{k,t}^n \Pi_{k,t}\, dt \Big\| \le C \frac{n}{a_n^d} \frac{\log |N|}{|N|^2}$$

for all $n \ge 1$, $|N| > \pi/\delta$ with $a_n \le |N| \le e^{\epsilon_1 n}$, $k = 0, \ldots, q-1$.

Proof Again, we abbreviate $B_\delta(0)$ to B_δ and suppress dt. Let $I = \int_{B_\delta} e^{-it\cdot N} r(t)\lambda_{k,t}^n \Pi_{k,t}$. Integrating by parts,

$$I = \frac{1}{iN_j} \int_{B_\delta} e^{-it\cdot N} \partial_j r(t)\lambda_{k,t}^n \Pi_{k,t} + \frac{1}{iN_j} \int_{B_\delta} e^{-it\cdot N} r(t)\partial_j(\lambda^n \Pi)_{k,t} = I_1 + I_2 + I_3$$

where

$$I_1 = -\frac{1}{N_j^2} \int_{B_\delta} e^{-it\cdot N} \partial_j^2 r(t)\lambda_{k,t}^n \Pi_{k,t}, \quad I_2 = -\frac{1}{N_j^2} \int_{B_\delta} e^{-it\cdot N} \partial_j r(t)\partial_j(\lambda^n \Pi)_{k,t},$$
$$I_3 = \frac{1}{iN_j} \int_{B_\delta} e^{-it\cdot N} r(t)\partial_j(\lambda^n \Pi)_{k,t}.$$

Recall that r is C^2 and that $t \mapsto \lambda_{k,t}$, $t \mapsto \Pi_{k,t}$ are C^1 by Lemma 4.1. Hence by Corollary 4.3,

$$\|I_1\| \ll \frac{1}{|N|^2} \int_{B_\delta} |\lambda_{k,t}|^n \ll \frac{1}{a_n^d} \frac{1}{|N|^2}, \qquad \|I_2\| \ll \frac{n}{|N|^2} \int_{B_\delta} |\lambda_{k,t}|^n \ll \frac{n}{a_n^d} \frac{1}{|N|^2}.$$

To estimate I_3, we use a modulus of continuity argument (see for instance [16, Chapter 1]). Set $s = t - h$ where $h = \pi N_j^{-1} e_j$ and notice that $I_3 = -\frac{1}{iN_j} \int_{B_{2\delta}} e^{-it\cdot N} r(s)\partial_j(\lambda^n \Pi)_{k,s}$. Hence

$$I_3 = \frac{1}{2iN_j} \int_{B_{2\delta}} e^{-it\cdot N} (r(t)\partial_j(\lambda^n \Pi)_{k,t} - r(s)\partial_j(\lambda^n \Pi)_{k,s}).$$

Setting $I_4 = \frac{1}{|N|} \int_{B_{2\delta}} |r(s)| \|\partial_j(\lambda^n \Pi)_{k,t} - \partial_j(\lambda^n \Pi)_{k,s}\|$, we obtain

$$\begin{aligned} \|I_3\| &\ll \frac{1}{|N|} \int_{B_{2\delta}} |r(t) - r(s)| \|\partial_j(\lambda^n \Pi)_{k,t}\| + I_4 \\ &\ll \frac{n}{|N|^2} \int_{B_{2\delta}} |\lambda_{k,t}|^n + I_4 \ll \frac{n}{a_n^d} \frac{1}{|N|^2} + I_4. \end{aligned}$$

Now,

$$\begin{aligned} I_4 &\ll \frac{1}{|N|}\int_{B_{2\delta}} \|\partial_j(\lambda^n)_{k,t}\Pi_{k,t} - \partial_j(\lambda^n)_{k,s}\Pi_{k,s}\| + \frac{1}{|N|}\int_{B_{2\delta}} \|\lambda^n_{k,t}\partial_j\Pi_{k,t} - \lambda^n_{k,s}\partial_j\Pi_{k,s}\| \\ &\ll \frac{1}{|N|}\int_{B_{2\delta}} |\partial_j(\lambda^n)_{k,t} - \partial_j(\lambda^n)_{k,s}|\|\Pi_{k,t}\| + \frac{1}{|N|}\int_{B_{2\delta}} |\partial_j(\lambda^n)_{k,s}|\|\Pi_{k,t} - \Pi_{k,s}\| \\ &\qquad\qquad + \frac{1}{|N|}\int_{B_{2\delta}} |\lambda^n_{k,t} - \lambda^n_{k,s}|\|\partial_j\Pi_{k,t}\| + \frac{1}{|N|}\int_{B_{2\delta}} |\lambda_{k,s}|^n\|\partial_j\Pi_{k,t} - \partial_j\Pi_{k,s}\| \\ &\ll \frac{1}{|N|}\int_{B_{2\delta}} |\partial_j(\lambda^n)_{k,t} - \partial_j(\lambda^n)_{k,s}| + \frac{n}{|N|^2}\int_{B_{2\delta}} |\lambda_{k,s}|^n \\ &\quad + \frac{n}{|N|^2}\int_{B_{2\delta}} |\lambda_{k,t}|^n + \frac{1}{|N|}\int_{B_{2\delta}} |\lambda_{k,s}|^n\|\partial_j\Pi_{k,t} - \partial_j\Pi_{k,s}\|. \end{aligned}$$

To complete the proof, we show that $I_4 \ll \frac{n}{a_n^d}\frac{\log|N|}{|N|^2}$. The first integral on the right-hand side was estimated in Proposition 6.2 while the second and third are dominated by $\frac{n}{a_n^d}\frac{1}{|N|^2}$. The same calculation that was used for the integral K in Proposition 6.2 shows that $\int_{B_{2\delta}} |\lambda_{k,s}|^n\|\partial_j\Pi_{k,t} - \partial_j\Pi_{k,s}\| \ll \frac{1}{a_n^d}\frac{\log|N|}{|N|}$. The desired estimate for I_4 follows. ∎

Corollary 6.4 *There exist $\epsilon_1 > 0$, $C > 0$ such that*

$$\|A_{n,N}\| \le C\frac{n}{a_n^d}\frac{\log|N|}{1+|N|^2} \quad \textit{for all } n \ge 1,\ N \in \mathbb{Z}^d \textit{ with } |N| \le e^{\epsilon_1 n}.$$

Proof By Corollary 6.1, $\|A_{n,N}\| \ll a_n^{-d}$. Hence for $n \gg (1+|N|^2)/\log|N|$,

$$\|A_{n,N}\| \ll a_n^{-d} \ll \frac{n}{a_n^d}\frac{\log|N|}{1+|N|^2}.$$

Hence we can reduce to the case $n \le \frac{1}{2}(1+|N|^2)/\log|N|$. Then we can suppose without loss that $|N| > \pi/\delta$. In this way, we reduce to proving $\|A_{n,N}\| \ll \frac{n}{a_n^d}\frac{\log|N|}{|N|^2}$ under the constraints

$$|N| > \pi/\delta, \qquad |N| \le e^{\epsilon_1 n}, \qquad n \le \tfrac{1}{4}|N|^2/\log|N|.$$

Since $a_n^2/\log a_n \sim 2n$, the last constraint can be weakened to $a_n^2/\log a_n \le |N|^2/\log|N|$, equivalently $a_n \le |N|$.

By (2.2), there exists $\gamma \in (0,1)$ such that $\|\int_{B_\delta(0)} e^{-it\cdot N} r(t)Q_t^n\,dt\| \ll \gamma^n$. Together with Lemma 6.3, this implies that

$$\|A_{n,N}\| \ll \frac{n}{a_n^d}\frac{\log|N|}{|N|^2} + \frac{1}{|N|^2}\gamma^n|N|^2.$$

Shrinking ϵ_1 if necessary, $\gamma^n|N|^2 \le \gamma^n e^{2\epsilon_1 n} \le \gamma^{n/2} \ll \frac{n}{a_n^d}$, and so $\|A_{n,N}\| \ll \frac{n}{a_n^d}\frac{\log|N|}{|N|^2}$. ∎

Proof of Theorem 1.1 By Lemma 3.1, it remains to consider the range $\log|N| \le \epsilon_1 n$. By [18, Lemma 3.9], there exists an even C^2 function $r : \mathbb{R}^d \to \mathbb{R}$ supported in $B_\delta(0)$ such that

$$1_{\{\kappa_n=N\}} \le \int_{\mathbb{R}^d} e^{-it\cdot N} r(t) e^{it\cdot\kappa_n}\, dt$$

for $n \ge 1$, $N \in \mathbb{Z}^d$. Hence

$$P^n 1_{\{\kappa_n=N\}} \le \int_{\mathbb{R}^d} e^{-it\cdot N} r(t) P^n e^{it\cdot\kappa_n}\, dt = A_{n,N}\, 1_\Delta.$$

It follows that $\mu_\Delta(\kappa_n = N) = \int_\Delta P^n 1_{\{\kappa_n=N\}}\, d\mu_\Delta \ll \|A_{n,N}\|$. By Corollary 6.4, we obtain the desired estimate for $\log|N| \le \epsilon_1 n$. ∎

7 LLD for nonuniformly hyperbolic systems modelled by Young towers

In this section, we state and prove an abstract version of Theorem 1.1 for systems modelled by a Young tower with exponential towers for a general class of observables κ. The observables take values in $\mathbb{Z}^d$ where there is no restriction on the value of $d \ge 1$.

Let (T, M, μ) be a general nonuniformly hyperbolic map modelled by a two-sided Young tower Δ with exponential tails (as in Section 2). Let $\kappa : M \to \mathbb{Z}^d$ be an integrable observable with $\int_M \kappa\, d\mu = 0$ and $\int_M |\kappa|^2\, d\mu = \infty$. Define the lifted observable $\hat\kappa = \kappa \circ \pi : \Delta \to \mathbb{Z}^d$. We require that $\hat\kappa$ is constant on $\bar\pi^{-1}(a \times \{\ell\})$ for each $a \in \alpha$, $\ell \in \{0, \dots, \sigma(a) - 1\}$. Then $\hat\kappa$ projects to an observable $\bar\kappa : \bar\Delta \to \mathbb{Z}^d$ constant on the partition elements $a \times \{\ell\}$ of the one-sided tower $\bar\Delta$.

Define P_t, $\lambda_{k,t}$ and so on as in Section 2. Properties (2.1) and (2.2) remain valid. Our further assumptions in the abstract setting are that there exist continuous slowly varying[4] functions ℓ_1, $\ell_2 : [0, \infty) \to (0, \infty)$ and a positive-definite matrix $\Sigma \in \mathbb{R}^{d\times d}$ such that

$$\mu(|\kappa| > x) \le x^{-2} \ell_1(x) \quad \text{for all } x > 1. \tag{7.1}$$

$$1 - \lambda_{0,t} \sim \Sigma t \cdot t\, \ell_2(1/|t|) \quad \text{as } t \to 0. \tag{7.2}$$

Define the slowly varying function $\tilde\ell_1(x) = \int_1^{1+x} u^{-1} \ell_1(u/\log u)\, du$.[5] We require that there is a constant $C > 0$ such that

$$(\log x)^2 \tilde\ell_1(x) \le C \ell_2(x) \quad \text{for all } x > 1. \tag{7.3}$$

[4]So $\lim_{t\to\infty} \ell_1(\lambda t)/\ell_1(t) = 1$ for all $\lambda > 0$, and similarly for ℓ_2.

[5]To optimise the results, we should take $\tilde\ell_1(x) = \int_1^{1+x} u^{-1} (\log u)^2 \ell_1(u/\log u)\, du$. Then $\widetilde{L}(t) = \tilde\ell_1(1/|t|)$ below but the formula for $\widetilde{M}_b$ is much more complicated.

Choose a_n so that

$$na_n^{-2}\ell_2(a_n) \sim 1.$$

Theorem 7.1 (LLD in abstract setting) *Let $d \geq 1$. There exist $C > 0$ and a slowly varying function ℓ_3 (depending on ℓ_1, ℓ_2 and d) such that*

$$\mu(\kappa_n = N) \leq C\frac{n}{a_n^d}\frac{\ell_3(|N|)}{1+|N|^2} \quad \textit{for all } n \geq 1,\ N \in \mathbb{Z}^d.$$

Remark 7.2 The slowly varying function ℓ_3 can be determined by modifying the proof of Theorem 1.1. Some of the steps are indicated below.

In the case of billiards, assumptions (7.1) and (7.2) hold with $\ell_1 \equiv 1$ and $\ell_2(x) = \log x$. We note that even with these ℓ_1, ℓ_2 and $d \leq 2$, obtaining $\ell_3(x) = \log x$ in Theorem 1.1 requires extra structure for billiards beyond the abstract setting of Theorem 7.1. This extra structure was used in Proposition 2.1 and Lemma 3.1. Similarly, assumption (7.3) is not required in the billiard setting due to the extra structure.

Remark 7.3 (a) In the simpler situation of Gibbs-Markov maps studied in [18], the underlying assumption is that $\mu(|\kappa| > x) \sim x^{-2}\ell_1(x)$ and that κ lies in the nonstandard domain of a nondegenerate multivariate normal distribution. A consequence is that $1 - \lambda_{0,t} \sim \Sigma t \cdot t\, \ell_2(1/|t|)$ with $\ell_2(x) = 1 + \int_1^{1+x} \ell_1(u)/u\, du$. Moreover, $\ell_3 = \ell_2$.

(b) As in [18], the proof of Theorem 7.1 does not rely on aperiodicity assumptions and hence the result applies in situations where the local limit theorem fails.

(c) More generally, one could consider situations where the underlying limit laws are α-stable laws, $\alpha \in (0, 2)$ (rather than normal distributions with nonstandard normalisation). We already mentioned that the study of such stable LLD started with [9] and [6] in the i.i.d. case for $d = 1$, extended to $d \geq 2$ [7]. The Gibbs-Markov case was studied in [18] for $\alpha \in (0, 1) \cup (1, 2]$ and general $d \geq 1$. We expect that Theorem 7.1, in the abstract setting where M is modelled by a Young tower with exponential tails, extends to the cases $\alpha \in (0, 1) \cup (1, 2)$ with minor (and obvious) modifications. However, for purposes of readability we do not pursue this extension here.

In the remainder of this section, we sketch the proof of Theorem 7.1. Again, the range $n \ll \log |N|$ is handled at the level of $T : M \to M$ and $\kappa : M \to \mathbb{Z}^d$. Lemma 3.1 is replaced by

Lemma 7.4 *Let $d \geq 1$, $\omega > 0$, $\epsilon > 0$. There exists $C > 0$ such that*

$$\mu(\kappa_n = N) \leq C\frac{n}{a_n^d}\frac{\ell_1(|N|)(\log |N|)^{\frac{d}{2}+1+\epsilon}}{|N|^2}$$

for all $n \geq 1$, $N \in \mathbb{Z}^d$ with $n \leq \omega \log |N|$.

Proof Define $\tilde{\kappa} = \tilde{\kappa}(N) = \min\{|\kappa|, |N|\}$ and $M_n = \max_{0\le j\le n-1} |\kappa| \circ T^j$. We use $|x| = \max_{j=1,\dots,d} |x_j|$ so that $|\tilde{\kappa}|$ is integer-valued.

Now,

$$\mu(|\kappa_n| \ge |N|) \le \mu(|\kappa_n| \ge |N|,\, M_n \le |N|) + \mu(M_n > |N|).$$

Note that

$$\mu(M_n > |N|) \le \sum_{j=0}^{n-1} \mu(|\kappa| \circ T^j > |N|) = n\mu(|\kappa| > |N|) \ll n|N|^{-2}\ell_1(|N|).$$

Next, for any $r > 2$,

$$\mu(|\kappa_n| \ge |N|,\, M_n \le |N|) \le \mu(\tilde{\kappa}_n \ge |N|) \le \|\tilde{\kappa}_n\|_r^r/|N|^r \le n^r\|\tilde{\kappa}\|_r^r/|N|^r.$$

By resummation and (7.1),

$$\|\tilde{\kappa}\|_r^r = \sum_{j=1}^{\infty} j^r\mu(|\tilde{\kappa}| = j) \le \sum_{j=1}^{|N|} j^r\mu(|\kappa| = j) \ll \sum_{j=1}^{|N|} j^{r-1}\mu(|\kappa| > j) \ll \sum_{j=1}^{|N|} j^{r-3}\ell_1(j),$$

so by Karamata, $\|\tilde{\kappa}\|_r^r \ll |N|^{r-2}\ell_1(|N|)$. Hence

$$\mu(\kappa_n = N) \le \mu(|\kappa_n| \ge |N|) \le n^r|N|^{-2}\ell_1(|N|) = na_n^{-d}|N|^{-2}\ell_1(|N|)n^{r-1}a_n^d.$$

Since a_n is regularly varying of index $\frac{1}{2}$, and $r > 2$ is arbitrary, it follows that $n^{r-1}a_n^d \ll n^{\frac{d}{2}+1+\epsilon} \ll (\log|N|)^{\frac{d}{2}+1+\epsilon}$. ∎

The remainder of the proof of Theorem 7.1 is carried out on the one-sided tower. As in Section 2, we define $\psi(y) = \sum_{\ell=0}^{\sigma(y)-1} |\kappa(y,\ell)|$. The analogue of Proposition 2.1 is:

Proposition 7.5 *There exist C, $n_0 > 1$ such that*

$$\mu_Y(\psi > n) \le Cn^{-2}(\log n)^2\ell_1(n/\log n) \quad \textit{for all } n \ge n_0.$$

In particular, $\psi \in L^r(Y)$ for all $r < 2$.

Proof A standard argument (see for example [5, Proposition A.1]) shows that

$$\mu_Y(\psi > n) \le \mu_Y(\sigma > k) + \bar{\sigma}\mu(|\kappa| > n/k),$$

for $k, n > 1$. In particular, there exists $a > 0$ such that

$$\mu_Y(\psi > n) \ll e^{-ak} + n^{-2}k^2\ell_1(n/k).$$

Taking $k = q\log n$ for any $q > 2/a$ and using that ℓ_1 is slowly varying,

$$\mu_Y(\psi > n) \ll n^{-2}(\log n)^2\ell_1(n/\log n).$$

Let $\epsilon \in (0, 2-r)$. Since ℓ_1 is slowly varying, $\ell_1(n/\log n) \ll (n/\log n)^{\epsilon/2} \ll n^{\epsilon/2}$. Hence $\mu_Y(\psi > n) \ll n^{-(2-\epsilon)}$ and it follows that $\psi \in L^r$. ∎

Define

$$\widetilde{M}_b(t,h) = |h|\widetilde{L}(h)\big\{1 + L(h)|t|^2\widetilde{L}(t) + |h|^{-b|t|^2\widetilde{L}(t)}L(h)^2|t|^4\widetilde{L}(t)^2\big\},$$

where $L(t) = \log(1/|t|)$ and $\widetilde{L}(t) = L(t)^2\tilde{\ell}_1(1/|t|)$.

Lemma 7.6 *The conclusions of Lemma 4.1 and 4.2 hold with M_b and $L(t)$ replaced by $\widetilde{M}_b$ and $\ell_2(1/|t|)$ respectively.*

Proof The modifications are elementary, but heavy on notation, so we only sketch the details.

Since $\psi \in L^r$ for all $r < 2$, the arguments in Section 5.1 are unchanged. The changes in the proof of Proposition 5.6 are as follows. By resummation, Proposition 7.5 and the definition of $\tilde{\ell}_1$,

$$\sum_{m=1}^{K}\mu_Y(\psi = m)m^2 \ll \sum_{m=1}^{K}\mu_Y(\psi \ge m)m \ll \sum_{m=1}^{K}m^{-1}(\log m)^2\ell_1(m/\log m)$$
$$\ll (\log K)^2\sum_{m=1}^{K}m^{-1}\ell_1(m/\log m) \ll (\log K)^2\tilde{\ell}_1(K).$$

Using this in (5.2), we obtain

$$\sum_{m=1}^{K}\sum_{n\le b\log m} r_{m,n} \ll |h|L(h)^3\tilde{\ell}_1(1/|h|)\big\{1 + (|z|-1)|h|^{-b\log|z|}L(h)\big\}.$$

Similarly,

$$\sum_{m>K}\sum_{n\le b\log m} r_{m,n} \ll |h|L(h)^3\ell_1(|h|^{-1}L(h)^{-1})\big\{1 + (|z|-1)|h|^{-b\log|z|}L(h)\big\}.$$

Hence the estimate corresponding to Proposition 5.6 is

$$\|\partial_j\partial_z\widehat{R}(z,t+h) - \partial_j\partial_z\widehat{R}(z,t)\|_{\mathcal{B}_1(Y)} \ll |h|L(h)^3\tilde{\ell}_1(1/|h|)\big\{1 + (|z|-1)|h|^{-b\log|z|}L(h)\big\}.$$

The corresponding estimates for $\partial_j\widehat{R}$, $\partial_j\widehat{A}$ and $\partial_j\widehat{B}$ are the same but with one less factor of $L(h)$.

Parts (a) and (c) of Proposition 5.11 are unchanged. Part (b) goes through with L replaced by $\widetilde{L}$. Hence, Corollary 5.12 becomes that

$$|g_k(t) - \bar{\lambda}_k| \ll |t|^2\widetilde{L}(t), \qquad |\partial_j g_k(t+h) - \partial_j g_k(t)| \ll \widetilde{M}_b(t,h).$$

The result follows. ∎

Corollary 7.7 *Let $\beta \ge 0$, $r \in \mathbb{R}$, $k = 0, \dots, q-1$. There exist $C > 0$, $\delta > 0$ such that*

$$\int_{B_{2\delta}(0)} |t|^\beta \widetilde{L}(t)^r |\lambda_{k,t}|^n \, dt \le C \frac{(\widetilde{L}(1/a_n))^r}{a_n^{d+\beta}} \quad \text{for all } n \ge 1.$$

Proof Following the proof of Corollary 4.3, we obtain $|\lambda_{k,t}| \le \exp\{-b|t|^2\ell_2(t)\}$. Now use that a_n is defined using ℓ_2 instead of L. ∎

Proof of Theorem 7.1 The arguments are identical to those in Section 6 up to slowly varying factors. Various simplifications no longer hold as the slowly varying functions ℓ_1, ℓ_2, $\tilde{\ell}_1$ and log are less well related, so the exact formulas are rather complicated and hence are omitted. ∎

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