POLYNOMIAL DECAY OF CORRELATIONS FOR NONPOSITIVELY CURVED SURFACES

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Abstract. We prove polynomial decay of correlations for geodesic flows on a class of nonpositively curved surfaces where zero curvature only occurs along one closed geodesic. We also prove that various statistical limit laws, including the central limit theorem, are satisfied by this class of geodesic flows.

Dedicated to the memory of Todd Fisher

Contents

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1. Introduction

The goal of this work is to provide examples of geodesic flows on nonpositively curved manifolds with polynomial decay of correlations.

Theorem 1.1. *Let $r \in [4, \infty)$, and let S be a closed Riemannian surface of nonpositive curvature obtained by isometrically gluing two negatively curved surfaces with boundaries to the boundaries of the surface of revolution with profile $1 + |s|^r$, $|s| \leq 1$. Let $M = T^1 S$ and $a = \frac{r+2}{r-2} \in (1, 3]$. Then the geodesic flow $g_t : M \to M$ has polynomial decay of correlations with respect to the normalized Riemannian volume μ: for all $\epsilon > 0$ and all sufficiently smooth observables $\phi, \psi : M \to \mathbb{R}$, there is a constant $C(\phi, \psi)$ such that*

$$\left| \int_M \phi \cdot (\psi \circ g_t) \, d\mu - \int_M \phi \, d\mu \int_M \psi \, d\mu \right| \leq C(\phi, \psi) \frac{1}{t^{a-\epsilon}} \text{ for all } t > 0.$$

The precise meaning of "sufficiently smooth" is explained in Section 7. In addition, we state and prove statistical limit laws such as the central limit theorem (CLT) for Hölder observables $\phi : M \to \mathbb{R}$.

Figure 1 depicts the surfaces S considered in Theorem 1.1: the region between two curves α and β is a surface of revolution with profile $1 + |s|^r$, $|s| \leq 1$, thus the curve γ with $s = 0$ is a closed geodesic with zero curvature; outside this region, S has negative curvature. We call γ the *degenerate closed geodesic*.

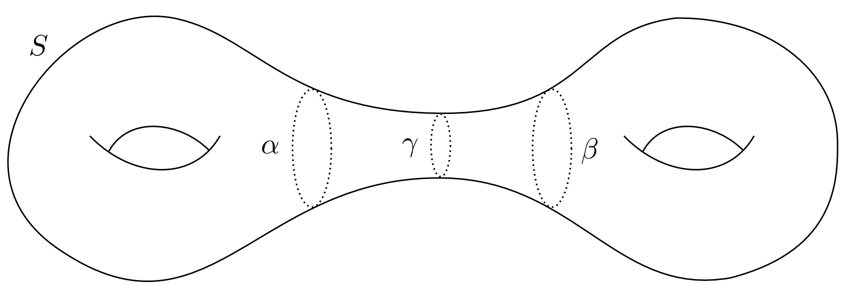

Figure 1. Surfaces with degenerate closed geodesic γ considered in Theorem 1.1.

Remark 1.2. We expect that the mixing rate in Theorem 1.1 is almost sharp. Indeed, as part of the proof of Theorem 1.1, we construct a piecewise smooth Poincaré map $g : \Sigma \to \Sigma$ with piecewise smooth first hit time that is bounded above and below (away from zero), and g has the sharp polynomial mixing rate n^{-a}.

More precisely, let μ_Σ denote the corresponding Liouville probability measure on Σ. In Section 7, we apply [You99] to show that for all Hölder observables $\phi, \psi : \Sigma \to \mathbb{R}$, there is a constant $C(\phi,\psi)$ such that for all integers $n \geq 2$,

$$\left|\int_\Sigma \phi \cdot (\psi \circ g^n)\, d\mu_\Sigma - \int_\Sigma \phi\, d\mu_\Sigma \int_\Sigma \psi\, d\mu_\Sigma\right| \leq C(\phi,\psi)\frac{(\log n)^{1+a}}{n^a}.$$

Moreover, by [BMT21], for all Hölder observables $\phi, \psi : \Sigma \to \mathbb{R}$ with nonzero mean and support bounded away from $T^1\gamma$, there is a constant $c(\phi,\psi)$ such that for all integers $n \geq 2$,

$$\left|\int_\Sigma \phi \cdot (\psi \circ g^n)\, d\mu_\Sigma - \int_\Sigma \phi\, d\mu_\Sigma \int_\Sigma \psi\, d\mu_\Sigma\right| \geq c(\phi,\psi)\frac{1}{(\log n)n^a}. \tag{1.1}$$

Obtaining rates for the geodesic flow g_t is more subtle due to the neutral flow direction. We obtain the upper bound in Theorem 1.1 by applying the recent work of [BBM19]. Since the first hit time is bounded away from zero, there is no reason to expect the flow g_t to decay faster than g; a precise statement of the form (1.1) for the flow g_t is the subject of work in progress [BMT]. (We note however that the bounds obtained in Lemmas 4.4 and 4.5 below combined with the arguments of [BMMW17b] show that g_t is certainly not exponentially mixing.)

Remark 1.3. We believe that the optimal mixing rate for g_t is t^{-a} (similarly n^{-a} for g). Indeed, we expect that the arguments in [CZ08] can be used to remove the multiplicative factor $(\log n)^{1+a}$ in Remark 1.2 and the same argument would allow us to take $\epsilon = 0$ in Theorem 1.1. However, to focus on the main ideas introduced in this paper, we do not pursue such an improvement here.

In general, the dynamical and statistical properties of geodesic flows in closed Riemannian manifolds is a fascinating topic whose origin goes back to Artin, Hadamard, Hedlund, Hopf, Klein, Poincaré, among others. Indeed, geodesic flows on manifolds with negative or nonpositive sectional curvature were the motivation of various breakthroughs in ergodic theory. One of them was given by Hopf: the nowadays called *Hopf argument* was used to prove that geodesic flows on negatively curved compact surfaces are ergodic with respect to their Liouville volume measure [Hop39]. Anosov extended Hopf's argument to prove ergodicity of geodesic flows in negative curvature to arbitrary dimensions [Ano69].

Moreover, geodesic flows in manifolds with negative sectional curvature are Bernoulli [OW73, Rat74], which is the ultimate chaotic property from a measure-theoretic point of view. After this was established, efforts were made to understand finer statistical properties such as decay of correlations. Among the developments, we mention the work of Chernov [Che98], Dolgopyat [Dol98] and Liverani [Liv04] on the exponential decay of correlations for contact Anosov flows (and in particular geodesic flows on compact manifolds with negative curvature), and the work of Burns et al. [BMMW17b, BMMW17a] on the rates of mixing of the Weil-Petersson geodesic flows on moduli spaces of Riemann surfaces. We mention that, beside its intrinsic interest, exponential mixing for geodesic (and frame) flows has applications to other fields such as the geometry of lattices [EM93], number theory [KM99], and the topology of 3–manifolds [KM12].

While geodesic flows in negative curvature are the prototypical examples of uniformly hyperbolic flows, geodesic flows in nonpositive curvature are the prototypical examples of nonuniformly hyperbolic flows, and are much harder to study. For instance, the ergodicity of the Liouville measure is still an open problem. Pesin developed a global theory for nonuniformly hyperbolic systems, nowadays called *Pesin theory*, and used it to derive the ergodicity of the Liouville measure in various contexts [Pes77a, Pes77b], see also the discussion in [Bal95, p. 5]. In particular, the geodesic flows considered in Theorem 1.1 are ergodic for the Liouville measure. There has been recent progress on the measure-theoretic properties of these flows, including the uniqueness of the measure of maximal entropy [Kni98], the uniqueness of other classes of equilibrium states [BCFT18], and the Bernoulli property [LLS16], among others. Previously, nothing was known about decay of correlations or the CLT for such flows, and this paper gives the first contribution in this direction. (Although this paper gives the first "classical" CLT, an asymptotic version of the CLT for measures converging to the measure of maximal entropy was recently obtained by [TW21].)

To prove Theorem 1.1 we use an axiomatic approach, nowadays called *Chernov axioms*, first developed by Chernov to prove exponential decay of correlations for dispersing billiard maps [Che99a]. We actually follow a simplification of this work given by Chernov and Zhang [CZ05a]. We apply these works to a uniformly hyperbolic map with singularities f, equal to the return map of a Poincaré section that does not intersect the degenerate closed geodesic. Then the method of Markarian [Mar04, CZ05a, BMT21] enables us to establish polynomial decay of correlations for h and g_t.

Remark 1.4. When establishing mixing rates for billiards, the main step is to verify complexity bounds, due to the fact that the remaining Chernov axioms had already been verified for many classes of examples in the previous twenty years starting with [BSC91], as discussed in [CZ05a, Section 4]. However, this is not the case for geodesic flows in nonpositive curvature, so the current paper aims to lay the groundwork for verifying all of the Chernov axioms for general classes of geodesic flows, in addition to treating the specific example in Theorem 1.1.

One of the Chernov axioms is that invariant manifolds have uniformly bounded curvature. This is a delicate point for the surfaces we consider. For instance, Ballmann, Brin and Burns showed that in a surface of revolution with profile $1 + s^4$ (i.e. $r = 4$ in Theorem 1.1) the invariant manifolds of the degenerate closed geodesic are not C^2, hence the curvature is not even defined [BBB87]. To avoid this, we verify that [Che99a, CZ05a] works under the weaker assumption that the invariant manifolds have uniformly bounded $C^{1+\mathrm{Lip}}$ norms, and we exploit the fact that this latter property is satisfied in the class of surfaces we consider, by a result of Gerber and Wilkinson [GW99], see Theorem 2.6.

Some of the Chernov axioms are related to hyperbolicity properties of the uniformly hyperbolic map f mentioned above. In negative curvature, these properties are usually obtained by estimating solutions of the Riccati equation. Unfortunately, the presence of zero curvature weakens such estimates, and we were not able to use them to establish the required axioms. Instead, we follow a different approach and use a system of coordinates in the unit tangent bundle of the surface of revolution, called *Clairaut coordinates*. In these coordinates, estimates for f are almost sharp.

In addition, the Poincaré section has to satisfy some geometrical and dynamical properties. One of them is the absence of triple intersections for a sufficiently large number of pre-iterates under f of a finite family of compact curves. This task would be a simple application of perturbative methods if these pre-iterates remained compact. However, f has unbounded derivative, and the pre-images of some compact curves have infinite length.

This phenomenon is related to the homoclinic points of the degenerate closed geodesic, and a thorough analysis of the dynamics of f is required to implement the perturbative methods successfully.

Remark 1.5. Chernov and Zhang considered an analogous class of dispersing billiard maps [CZ05b] for which the obstacles are convex with nonvanishing curvature except at two flat points where the obstacles have profile $\pm(1+|s|^r)$, $r>2$. They obtained the same upper bound $(\log n)^{a+1}n^{-a}$ as in Remark 1.2 and the lower bound $(\log n)^{-1}n^{-a}$ was proved in [BMT21]. The associated semiflow has the same polynomial decay of correlations [Mel07, Mel18], but the analogue of Theorem 1.1 for the billiard flow remains unproved (this is the final open question in [Mel18, Section 9]).

Contrary to [CZ05b], we require $r \geq 4$ rather than just $r>2$ because this provides the usual C^4 regularity required to apply several tools from the theory of geodesic flows described in Section 2 (from the properties of elementary ordinary differential equations like the Jacobi and Riccati equations to more recent results such as Theorem 2.6).

The paper is organized as follows. In Section 2 we review known facts about the geometry and dynamics of geodesic flows on surfaces, with special attention to the class of surfaces considered in Theorem 1.1. In particular, we state the main results of Gerber and Wilkinson [GW99] that we will use. Section 3 presents the axiomatic approach of [Che99a, CZ05a], and includes the justification that uniform bounds on the $C^{1+\mathrm{Lip}}$ norms of invariant manifolds are enough, see Remark 3.2. In Section 4 we make a systematic study of the dynamics of the geodesic flow near the degenerate closed geodesic, which is related to explosion of the derivative of f. Here we make substantial use of the Clairaut coordinates. In Section 5 we construct the Poincaré section. We also prove that the roof function of the constructed Poincaré section has polynomial tails (Lemma 5.4), and prove some hyperbolicity estimates for f, see Section 5.6. Using these results, we prove in Section 6 that f indeed satisfies the Chernov axioms. Finally, we prove Theorem 1.1 and various statistical limit laws in Section 7.

2. Surfaces with nonpositive curvature

In this section, we recall some known facts on differential geometry, most specifically on geodesic flows in nonpositively curved surfaces and surfaces of revolution. We also give a precise description of the class of surfaces we consider in this article, and describe the properties of these surfaces that will be used in the sequel.

2.1. **Geodesic flows.** Let S be a closed Riemannian surface. Let $M = T^1S$ be its unit tangent bundle, which is a closed three dimensional Riemannian manifold. There is a natural metric on M, called the *Sasaki metric*, which is the product of horizontal and vertical vectors, see e.g. [dC92, Chapter 3, Exercise 2]. We write $\|\cdot\|_{\mathrm{Sas}}$ for the norm induced by the Sasaki metric. The volume form on S induces a smooth probability measure μ on M.

Geodesic flow $\{g_t\}$: The *geodesic flow* on S is the flow $\{g_t\}_{t\in\mathbb{R}} : M \to M$ defined by $g_t(x) = \gamma_x'(t)$, where $\gamma_x : \mathbb{R} \to S$ is the unique geodesic such that $\gamma_x'(0) = x$. For simplicity, we denote the geodesic flow by g_t. The probability measure μ is invariant under g_t.

For $p \in S$, let $K(p)$ be the Riemannian curvature at p. We assume that S has *nonpositive* Riemannian curvature: $K(p) \leq 0$ for all $p \in S$. The dynamical properties of g_t are intimately related to the curvature of S.

Degenerate and Regular sets: The *degenerate set* of g_t is defined by

$$\mathrm{Deg} = \{x \in M : K(\gamma_x(t)) = 0 \text{ for all } t \in \mathbb{R}\},$$

and the *regular set* of g_t is defined by

$$\mathrm{Reg} = M \setminus \mathrm{Deg} = \{x \in M : K(\gamma_x(t)) < 0 \text{ for some } t \in \mathbb{R}\}.$$

Clearly, Deg and Reg form a partition of M, with Deg closed and Reg open.

Remark 2.1. The classical literature uses the terminology "singular set" instead of "degenerate set", but here we reserve the term "singular" for the dynamical setting of the Chernov axioms.

Theorem 2.2 (Pesin [Pes77b]). *If $\mu[\mathrm{Deg}] = 0$, then the flow (g_t, μ) is ergodic.*

See also the discussion in [Bal95, p. 5].

The dynamical properties of g_t are usually studied via Jacobi fields.

Jacobi field: A vector field $J : t \mapsto J(t) \in T_{\gamma(t)}S$ along a geodesic γ is called a *Jacobi field* if it satisfies the *Jacobi equation*

$$J''(t) + K(\gamma(t))J(t) = 0.$$

If $J(t), J'(t)$ are perpendicular to $\gamma'(t)$ for some (and hence all) t, then J is called a *perpendicular Jacobi field*.

For every $x \in M$ there is an isomorphism

$$T_xM \leftrightarrow \{(J(0), J'(0)) : J \text{ is a Jacobi field along } \gamma_x \text{ with } J'(0) \perp \gamma_x'(0)\}.$$

Under this identification, the Sasaki metric is equal to $\|(J, J')\|^2_{\mathrm{Sas}} = \|J\|^2 + \|J'\|^2$. Additionally, we have $dg_t(J(0), J'(0)) = (J(t), J'(t))$, and this is one of the reasons why Jacobi fields provide dynamical information of g_t. Under our curvature assumptions, we can characterize stable and unstable subspaces.

Stable and unstable Jacobi fields: A Jacobi field J is called *stable* if $\|J(t)\|$ is uniformly bounded for all $t \geq 0$, and *unstable* if $\|J(t)\|$ is uniformly bounded for all $t \leq 0$.

Stable and unstable subspaces: The *stable subspace* of $x \in M$ is

$$\widehat{E}^s_x = \{(J(0), J'(0)) : J \text{ is a stable perpendicular Jacobi field along } \gamma_x\},$$

and the *unstable subspace* of $x \in M$ is

$$\widehat{E}^u_x = \{(J(0), J'(0)) : J \text{ is an unstable perpendicular Jacobi field along } \gamma_x\}.$$

Remark 2.3. We reserve the notation $E^{s/u}_x$ for the stronger notion of stable/unstable subspace in the sense of hyperbolic dynamics, as described in Section 5.

Let Z_x denote the one-dimensional subspace of T_xM tangent to the geodesic flow. The following are known facts of these subspaces, see e.g. [Ebe01].

Lemma 2.4. *The families $\{\widehat{E}^s_x\}, \{\widehat{E}^u_x\}$ satisfy the following properties:*

(1) Invariance: *$\{\widehat{E}^s_x\}, \{\widehat{E}^u_x\}$ are dg_t–invariant for all $t \in \mathbb{R}$.*
(2) Dimension: *$\widehat{E}^s_x, \widehat{E}^u_x$ have dimension one and are orthogonal to Z_x.*
(3) Continuity: *The maps $x \mapsto \widehat{E}^s_x, \widehat{E}^u_x$ are continuous.*
(4) Relation with Deg, Reg: *$\widehat{E}^s_x = \widehat{E}^u_x$ if and only if $x \in \mathrm{Deg}$, hence $\widehat{E}^s_x \oplus Z_x \oplus \widehat{E}^u_x = T_xM$ if and only if $x \in \mathrm{Reg}$.*

Next, we consider the invariant manifolds of g_t. We first define the invariant manifolds for the geodesic flow $\widetilde{g}_t$ on the universal cover $\widetilde{S}$ of S. For that, we consider Busemann functions and horospheres. Our discussion follows [Ebe01, Section IV.A]. For each $v \in T^1\widetilde{S}$, let $\widetilde{\gamma}_v$ be the unique geodesic of $\widetilde{S}$ with $\widetilde{\gamma}_v'(0) = v$. Given $t \in \mathbb{R}$, define the function $B_{v,t} : \widetilde{S} \to \mathbb{R}$ by $B_{v,t}(x) = d(x, \widetilde{\gamma}_v(t)) - t$. (Here, d is the unique metric on $\widetilde{S}$ making the covering map a local isometry.) The *Busemann function* of v is the limit function $B_v : \widetilde{S} \to \mathbb{R}$ defined by $B_v = \lim\limits_{t\to+\infty} B_{v,t}$. Since S has nonpositive curvature, it follows from Eberlein that each B_v is C^2 [Ebe01, Section IV.A], see also [HIH77, Prop. 3.1].

Horospheres: The *stable horosphere* at $v \in T^1\widetilde{S}$ is the set $H^s(v) \subset \widetilde{S}$ defined as $H^s(v) = B_v^{-1}(0)$. The *unstable horosphere* at $v \in T^1\widetilde{S}$ is the set $H^u(v) \subset \widetilde{S}$ defined by $H^u(v) = (B_{-v})^{-1}(0)$.

Each $H^{s/u}(v)$ is a C^2 curve of $\widetilde{S}$, and $v \mapsto H^{s/u}(v)$ is continuous, see e.g. [Bal95, p. 25]. These curves define invariant foliations for $\widetilde{g}_t$.

Invariant manifolds for $\widetilde{g}_t$: The *stable manifold* for $\widetilde{g}_t$ at $v \in T^1\widetilde{S}$ is the graph over $H^s(v)$ defined by

$$\widetilde{W}^s_v = \left\{ w \in T^1\widetilde{S} : \begin{array}{l} w \text{ is perpendicular to and has basepoint at} \\ H^s(v), \text{ pointing in the same direction of } v \end{array} \right\}.$$

The *unstable manifold* for $\widetilde{g}_t$ at $v \in T^1\widetilde{S}$ is defined analogously.

Since $H^{s/u}(v)$ is C^2, its normal subbundle is C^1, i.e. each leaf $\widetilde{W}^{s/u}_v$ is C^1.

Invariant manifolds for g_t: The stable/unstable manifolds of g_t at $x \in M$ are the projections to M of the stable/unstable manifolds of $\widetilde{g}_t$ at some (every) $v \in T^1\widetilde{S}$ that projects to x. We denote them by $\widehat{W}^{s/u}_x$.

By the above discussion the curves $\widehat{W}^{s/u}_x$ are C^1 for all $x \in M$. Under additional conditions, Gerber & Wilkinson proved a stronger regularity [GW99] and also a property about the tangent distributions $\widehat{E}^{s/u}$, see Theorem 2.6 below.

Next, we discuss the link between Jacobi fields and horospheres. Fix $x \in M$, and let $J_-(t)$ be a stable perpendicular Jacobi field along γ_x. If $E(t)$ is a unitary parallel vector field orthogonal to γ_x, then $J_-(t) = j_-(t)E(t)$ where $j_-(t) = \|J_-(t)\|$ satisfies the *scalar Jacobi equation*

$$j''(t) + K(\gamma_x(t))j(t) = 0.$$

The logarithmic derivative $u_{x,-}(t) = \frac{j_-'(t)}{j_-(t)} = [\log j_-(t)]'$ satisfies the *Riccati equation*

$$u'(t) + u(t)^2 + K(\gamma_x(t)) = 0.$$

Define $u_- : M \to \mathbb{R}$ by $u_-(x) = u_{x,-}(0)$. It is known that $u_-(x)$ is the geodesic curvature of the curve $H^s(v)$ at the basepoint of v for some (every) $v \in T^1\widetilde{S}$ that projects to x. Similarly, for each $x \in M$ define a function $u_{x,+}$; then $u_+ : M \to \mathbb{R}$ defined by $u_+(x) = u_{x,+}(0)$ is the geodesic curvature of $H^u(v)$ at the basepoint of v for some (every) $v \in T^1\widetilde{S}$ that projects to x. The following properties of $u_\pm$ will be essential to us.

Proposition 2.5. *The functions $u_\pm$ are continuous, $u_- \leq 0 \leq u_+$ and $u_\pm(x) = 0$ if and only if $x \in \mathrm{Deg}$.*

The continuity of $u_\pm$ follows from the regularity of the Busemann functions mentioned above. The functions $u_\pm$ also provide the growth rate of the derivative of g_t, as follows. Fix $x \in M$. For an unstable perpendicular Jacobi field $J_+(t)$ along γ_x, we have

$$\begin{aligned} j_+(t) &= j_+(0)\exp\left[\int_0^t u_+(g_s x)ds\right] \\ j_+'(t) &= j_+(0)\exp\left[\int_0^t u_+(g_s x)ds\right] u_+(g_t x) \end{aligned}$$

and so

$$\begin{aligned} \frac{\|dg_t(J_+(0), J_+'(0))\|_{\mathrm{Sas}}}{\|(J_+(0), J_+'(0))\|_{\mathrm{Sas}}} &= \frac{\|(J_+(t), J_+'(t))\|_{\mathrm{Sas}}}{\|(J_+(0), J_+'(0))\|_{\mathrm{Sas}}} \\ &= \sqrt{\frac{1+u_+(g_t x)^2}{1+u_+(x)^2}}\exp\left[\int_0^t u_+(g_s x)ds\right]. \end{aligned}$$

By Proposition 2.5, if the orbit segment $g_{[0,t]}x$ is far from Deg, then $\widehat{E}^u_x$ is indeed an expanding direction. A similar calculation holds for stable perpendicular Jacobi fields. We actually work with a variant of the Sasaki metric, as in [KH95, §17.6]. Let $\delta > 0$.

δ–Sasaki metric: The *δ–Sasaki metric* is the metric $\|\cdot\|_{\delta-\mathrm{Sas}}$ satisfying the equality

$$\|(J, J')\|^2_{\delta-\mathrm{Sas}} = \|J\|^2 + \delta\|J'\|^2$$

for all Jacobi field J.

The Sasaki metric is the 1–Sasaki metric. In our calculations, we will fix a δ–Sasaki metric for δ small enough and denote it simply by $\|\cdot\|$. Hence

$$\frac{\|dg_t(J_+(0), J_+'(0))\|}{\|(J_+(0), J_+'(0))\|} = \sqrt{\frac{1+\delta u_+(g_t x)^2}{1+\delta u_+(x)^2}}\exp\left[\int_0^t u_+(g_s x)ds\right],$$

thus by the continuity of u_+ we get that

$$C_\delta^{-1}\exp\left[\int_0^t u_+(g_s x)ds\right] \leq \frac{\|dg_t(J_+(0), J_+'(0))\|}{\|(J_+(0), J_+'(0))\|} \leq C_\delta \exp\left[\int_0^t u_+(g_s x)ds\right] \tag{2.1}$$

where $\lim_{\delta\to 0} C_\delta = 1$. Similar considerations apply to J_-.

2.2. **Surfaces of revolution.** Let $I \subset \mathbb{R}$ be a compact interval, and let $\xi : I \to \mathbb{R}$ be a positive C^4 function. The *surface of revolution* defined by ξ around the x axis is the surface S with global chart $\Xi : I \times [0, 2\pi] \to \mathbb{R}^3$ given by $\Xi(s, \theta) = (s, \xi(s)\cos\theta, \xi(s)\sin\theta)$. We collect some known facts about these surfaces, see [dC76].

Curvature: The curvature at $p = \Xi(s, \theta)$ is equal to

$$K(p) = -\frac{\xi''(s)}{\xi(s)[1 + (\xi'(s))^2]^2}.$$

See [dC76, Example 4, p. 161].

Geodesics on surfaces of revolution have a simple description. They satisfy the so-called *Clairaut relation*, which allows to reduce the second order ordinary differential equation (ODE) defining the geodesic to a first order ODE. Let $\gamma(t) = \Xi(s(t), \theta(t))$ be a geodesic, and let $\psi(t) \in \mathbb{S}^1$ be the angle that the circle $s = s(t)$, more precisely its image under Ξ, makes with γ at $\gamma(t)$, see Figure 2.

Clairaut relation: The value

$$c = \xi(s(t)) \cos\psi(t) = \xi(s(t))^2\theta'(t) \tag{2.2}$$

is constant along γ. We call c the *Clairaut constant* of γ and of all of its tangent vectors.

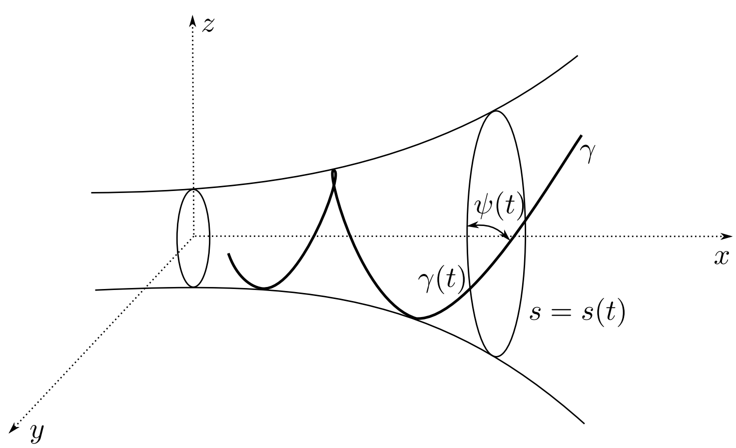

Figure 2. Clairaut relation: $c = \xi(s(t)) \cos\psi(t) = \xi(s(t))^2\theta'(t)$ is constant along γ.

Equation of geodesics: If $\gamma(t) = \Xi(s(t), \theta(t))$ is a geodesic with Clairaut constant c, then $s(t)$ satisfies

$$\left[1 + \xi'(s)^2\right](s')^2 + \frac{c^2}{\xi(s)^2} = 1. \tag{2.3}$$

See [dC76, Example 5, p. 255] for the proof.

For a fixed $s_0 \in \mathbb{R}$, the curve $\Xi(s_0, \theta)$ is called a *meridian*. The meridian $\Xi(s_0, \theta)$ is a geodesic if and only if $\xi'(s_0) = 0$ (see [dC76, p. 256]).

Observe that S is diffeomorphic to $I\times\mathbb{S}^1$ and M is diffeomorphic to $S\times\mathbb{S}^1\cong I\times\mathbb{S}^1\times\mathbb{S}^1$, where $(p,\psi)\in S\times\mathbb{S}^1$ is identified to the unit tangent vector with basepoint p that makes an angle ψ with the meridian passing though p. We can use this identification to define another metric on M.

Clairaut coordinates and Clairaut metric: The *Clairaut coordinates* on M are $(s,\theta,\psi)\in I\times\mathbb{S}^1\times\mathbb{S}^1$, and the *Clairaut metric* on M is the Riemannian metric $\|\cdot\|_{\rm C}$ on M given by the canonical product on $I\times\mathbb{S}^1\times\mathbb{S}^1$.

The Clairaut metric induces a distance, which we call the *Clairaut distance* and denote by $d_{\rm C}$. Above, the canonical metrics are the induced metrics of $I\subset\mathbb{R}$ and $\mathbb{S}^1\subset\mathbb{R}^2$. Since I is compact, the metrics $\|\cdot\|_{\delta-{\rm Sas}}$ and $\|\cdot\|_{\rm C}$ are equivalent.

The Clairaut relation (2.2) leads us to the following definition.

Clairaut function: The *Clairaut function* is the function $c:M\to\mathbb{R}$ defined by $c(s,\theta,\psi)=\xi(s)\cos\psi$.

2.3. **Surfaces with degenerate closed geodesic.** We now define a class of surfaces that exhibit two special features: the only region of zero curvature is a closed geodesic, and on a neighborhood of this geodesic the surface is a particular surface of revolution.

Surface with degenerate closed geodesic: A surface of nonpositive curvature S is a *surface with degenerate closed geodesic* γ if there are $r\in[4,\infty)$ and $\varepsilon_0>0$ such that:

- S is C^r with everywhere negative curvature except at a closed geodesic γ.
- There are two closed curves α,β defining a set $\mathcal{N}\subset S$ that contains γ such that $\mathcal{N}$ is a surface of revolution with $\xi(s)=1+|s|^r$ for $|s|\leq\varepsilon_0$. Moreover, $\alpha=\Xi(-\varepsilon_0,\theta)$, $\gamma=\Xi(0,\theta)$, $\beta=\Xi(\varepsilon_0,\theta)$ are meridians. We call $\mathcal{N}$ the *neck*, see Figure 3.

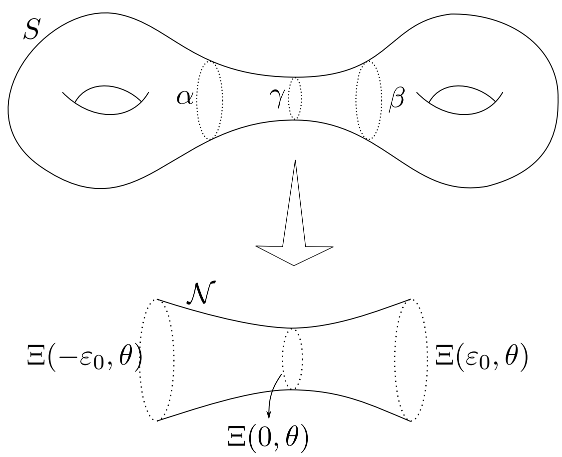

Figure 3. An example of a surface S with degenerate closed geodesic.

Such surfaces indeed exist, and can be obtained by interpolating the neck with a hyperbolic surface ($K\equiv-1$) with one cusp on each side. Since near the cusp a hyperbolic surface is a surface of revolution, it is enough to interpolate its profile function with the function

ξ, in a way that the resulting function is strictly convex for $s \neq 0$. This can be made as in [Don88, Appendix A.2], using a partition of unity.

In the sequel, we fix a surface S with degenerate closed geodesic γ. Following the notation of Section 2.1, let $M = T^1S$, $g_t : M \to M$ the geodesic flow on S, and μ the smooth probability measure on M induced by the Riemannian metric. Recall that the invariant manifolds $\widehat{W}^{s/u}_x$ are C^1 for all $x \in M$. Gerber & Wilkinson proved a stronger regularity [GW99] and also a property about the tangent distributions $\widehat{E}^{s/u}$.

Theorem 2.6 (Gerber & Wilkinson [GW99])**.** *Let S be a surface with degenerate closed geodesic. Then the curves $\widehat{W}^{s/u}_x$ are uniformly $C^{1+\mathrm{Lip}}$, and $x \mapsto \widehat{E}^{s/u}_x$ is Hölder continuous.*

In other words, the $C^{1+\mathrm{Lip}}$ norms of all $\widehat{W}^{s/u}_x$, $x \in M$, are bounded by a uniform constant, and the tangent direction $\widehat{E}^{s/u}_x$ is Hölder continuous as a function of $x \in M$. Actually, Gerber & Wilkinson established Theorem 2.6 in a context that does not cover surfaces with degenerate closed geodesic with non-integer r, but their proof can be easily adapted to prove the above theorem. In the Appendix, we show how to make such changes.

In the Clairaut coordinates, let $\gamma_0 = \{0\} \times \mathbb{S}^1 \times \{0\}$ and $\gamma_\pi = \{0\} \times \mathbb{S}^1 \times \{\pi\}$. We have $\mathrm{Deg} = \gamma_0 \cup \gamma_\pi$ and so Theorem 2.2 implies that the flow g_t is ergodic. Actually, g_t is Bernoulli; see [Pes77a] and [BP07, Thm. 12.2.13] for the classical proofs, and [LLS16] for a proof using symbolic dynamics.

Next, we use the Clairaut function to distinguish some vectors of M that will play a key role in the next sections. The only meridian that is a geodesic is $\gamma = \Xi(0, \theta)$. In M, this corresponds to the two geodesics γ_0 and γ_π. The Clairaut constants are $c = 1$ and $c = -1$ respectively. Let $x = (s, \theta, \psi) \in M$ with $s \neq 0$ such that $g_{[0,\varepsilon]}(x) \subset [-|s|, |s|] \times \mathbb{S}^1 \times \mathbb{S}^1$ for some $\varepsilon > 0$, i.e. the geodesic starting at x points towards γ.

Asymptotic, bouncing, crossing vectors and geodesics: A vector $x \in M$ as above is called:

- *Asymptotic* if $c(x) = \pm 1$: the geodesic path $g_{[0,\infty)}(x)$ is asymptotic to γ.
- *Bouncing* if $|c(x)| > 1$: there is $t > 0$ such that $\psi(t) = 0$ or π, i.e. the geodesic path $g_{[0,t]}(x)$ spirals towards γ, $g_t(x)$ is tangent to a meridian and after that the geodesic path spirals away from γ. In such cases, the geodesic does not reach γ.
- *Crossing* if $|c(x)| < 1$: there is $t > 0$ such that $s(t) = 0$, i.e. the geodesic path $g_{[0,t]}(x)$ spirals towards γ, $g_t(x)$ crosses γ and after that the geodesic path spirals away from γ.

The corresponding geodesic with initial condition x is called *asymptotic, bouncing, crossing* respectively.

See Figure 4. The statements above are easily verified using the Clairaut relation (2.2). For instance, if $c(x) > 1$ then $s(t)$ never vanishes during the neck transition and $g_t(x)$ is bounded away from γ. It follows that $\psi(t_0) = 0$ for some $t_0 > 0$, and $s(t_0)$ is uniquely determined by the equation $\xi(s(t_0)) = c$. In particular, the value of t_0 is unique and we obtain bouncing as claimed.

We end this section making a comment on the number of closed geodesics. Letting $P(T)$ denote the number of closed orbits of length $\leq T$, Knieper [Kni83] proved that

$$\lim_{T \to \infty} \frac{1}{T} \log P(T) = h_{\mathrm{top}}(g_1). \tag{2.4}$$

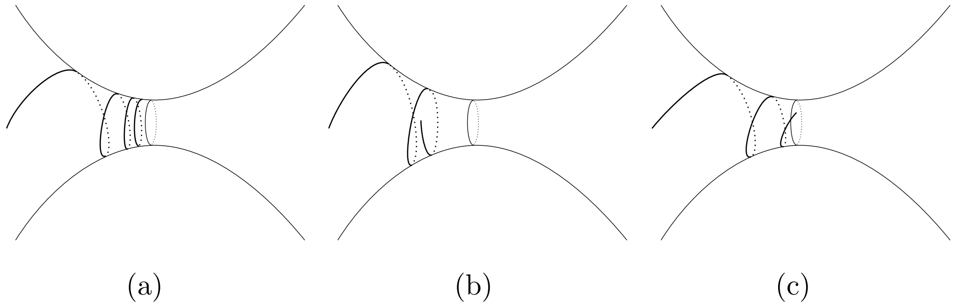

FIGURE 4. (a) Asymptotic vector. (b) Bouncing vector. (c) Crossing vector.

3. Chernov axioms for exponential mixing

Young introduced a powerful scheme [You98], nowadays called *Young towers*, that implies exponential mixing for a vast class of dynamical systems, especially finite horizon dispersing billiards. Based on some previous work [Che99b] and on Young's novel method, Chernov introduced a set of axioms that implies exponential mixing, and applied it to numerous further classes of planar dispersing billiards [Che99a]. These axioms are nowadays called *Chernov axioms*. Using the ideas of Young and Chernov, many authors proved exponential and polynomial decay of correlations for other classes of billiards [You99, Mar04, CZ05a, CZ05b, BT08].

In this paper, we only require the existence of a Young tower together with its consequences; hence we omit the precise definition of Young tower and instead refer the reader to [You98].

We pay special attention to [CZ05a], where the presentation of Chernov axioms is more suitable to our context, as we now explain. Firstly, they focus on two-dimensional maps. Secondly, they give a simpler criterion on the axiom that is usually hardest to prove, commonly called *growth of unstable manifolds*. The simpler criterion assumes, additionally to the low dimension of the phase space, four facts:

- Alignment: $\mathscr{S}^+$ is tangent to stable cones and $\mathscr{S}^-$ is tangent to unstable cones. Here, $\mathscr{S}^\pm$ are the singular sets, see Section 3.1 below.
- Structure of the singular set: control on the rate of accumulation of singularity curves.
- Growth bound: control on the inverses of least expansions of smooth pieces of unstable manifolds.
- Complexity bound: control on the growth rate of self-intersections of *primary* singularities.

These conditions are stated in axioms (A3) and (A8) below. Hence, in this work the Chernov axioms consist of eight conditions for an abstract smooth hyperbolic map with singularities to have exponential decay of correlations. Except for axioms (A3) and (A8), our presentation is based on [BT08, Appendix A], with one crucial difference in axiom (A5): while the Chernov axioms require the invariant manifolds to be C^2 curves with uniformly bounded curvature, we only require them to be $C^{1+\mathrm{Lip}}$ with uniformly bounded $C^{1+\mathrm{Lip}}$ norm.

3.1. **Chernov axioms.** Here, we state the axioms (A1) to (A8).

(A1) Dynamical system. We consider X_0 an open subset of a C^2 Riemannian surface $\widehat{X}$ such that its closure X is compact. We let $\mathscr{S}^+, \mathscr{S}^-$ be closed subsets of X, and let $f : X_0 \backslash \mathscr{S}^+ \to X_0 \backslash \mathscr{S}^-$ be a C^2 diffeomorphism. We call $\mathscr{S}^+$ the *singular set* of f, and $\mathscr{S}^-$ the *singular set* of f^{-1}. (Derivatives are allowed to blow up at the boundary of X_0 and at the singular sets.)

For $n \geq 1$, define

$$\mathscr{S}_n = \mathscr{S}^+ \cup f^{-1}(\mathscr{S}^+) \cup \cdots \cup f^{-n+1}(\mathscr{S}^+)$$
$$\mathscr{S}_{-n} = \mathscr{S}^- \cup f(\mathscr{S}^-) \cup \cdots \cup f^{n-1}(\mathscr{S}^-).$$

Then $\mathscr{S}_n$ is the singular set of f^n, and $\mathscr{S}_{-n}$ is the singular set of f^{-n}. Observe that $\mathscr{S}_1 = \mathscr{S}^+$ and $\mathscr{S}_{-1} = \mathscr{S}^-$.

(A2) Uniform hyperbolicity. There are two families of cones $\{C^u_x\}, \{C^s_x\}$ in the tangent planes $T_x\widehat{X}$, $x \in X$, called unstable and stable cones, and a constant $\Lambda > 1$ such that:

(A2.1) *Continuity*: $\{C^u_x\}, \{C^s_x\}$ are continuous on X.
(A2.2) *Full hyperbolicity*: The axes of $\{C^u_x\}, \{C^s_x\}$ are one-dimensional.
(A2.3) *Transversality:* $\min_{x \in X} \angle(C^u_x, C^s_x) > 0$.
(A2.4) *Invariance*: $Df(C^u_x) \subset C^u_{fx}$ and $Df(C^s_x) \supset C^s_{fx}$ whenever Df exists.
(A2.5) *Uniform hyperbolicity*: For all $v^u \in C^u_x$ and $v^s \in C^s_x$,

$$\|Df(v^u)\| \geq \Lambda \|v^u\| \quad \text{and} \quad \|Df^{-1}(v^s)\| \geq \Lambda \|v^s\|.$$

Let $x \in X_0$. For $x \notin \bigcup_{n \geq 0} \mathscr{S}_{-n}$ let $E^u_x = \bigcap_{n \geq 0} Df^n(C^u_{f^{-n}x})$, and for $x \notin \bigcup_{n \geq 0} \mathscr{S}_n$ let $E^s_x = \bigcap_{n \geq 0} Df^{-n}(C^s_{f^n x})$. These are called the *unstable* and *stable* subspaces, respectively. Axiom (A2) implies that every E^u_x, E^s_x is one-dimensional. Moreover, if $x \notin \bigcup_{n \in \mathbb{Z}} \mathscr{S}_n$ then $E^u_x \oplus E^s_x = T_x\widehat{X}$, with E^u_x being spanned by vectors with positive Lyapunov exponents and E^s_x spanned by vectors with negative Lyapunov exponents.

For the remaining axioms, we need to introduce some terminology. Let ρ, m be respectively the Riemannian metric and Lebesgue measure on $\widehat{X}$. Given a curve $W \subset \widehat{X}$, let ρ_W, m_W be respectively the Riemannian metric and Lebesgue measure on W induced by ρ, m.

Local unstable manifold (LUM): A *local unstable manifold* (LUM) is a curve $W \subset X$ such that:

(i) f^{-n} is well-defined and smooth on W for all $n \geq 0$.
(ii) $\rho(f^{-n}x, f^{-n}y) \to 0$ exponentially quickly as $n \to \infty$ for all $x, y \in W$.

We usually write W^u_x to represent a LUM containing x. The tangent space of W^u_x at x is E^u_x. Similarly, we define the notion of local stable manifold (LSM) and write W^s_x to represent a LSM containing x.

Now let W_1, W_2 be sufficiently small and close enough LUM's such that small LSM's intersect each of W_1, W_2 at most once. let $W_1' = \{x \in W_1 : W^s_x \cap W_2 \neq \emptyset\}$, and let $H : W_1' \to W_2$ be the *holonomy map* obtained by sliding along local stable manifolds, i.e.

$H(x)$ is the unique intersection between W^s_x and W_2. Also, let $\Lambda(x) = |\det(Df \restriction_{E^u_x})|$ be the Jacobian of f in the direction of E^u_x, which is the factor of expansion on W^u_x at x.[1]

(A3) Alignment. The angle between $\mathscr{S}^+$ and LUM's is bounded away from zero; the angle between $\mathscr{S}^-$ and LSM's is bounded away from zero.

(A4) SRB measure. The map f preserves an ergodic volume measure μ such that a.e. $x \in X_0$ has a LUM W^u_x and the conditional measure on W^u_x induced by μ is absolutely continuous with respect to $m_{W^u_x}$. Furthermore, f^n is ergodic for all $n \geq 1$.

(A5) Uniformly bounded $C^{1+\mathrm{Lip}}$ norms. The leaves $W^{u/s}_x$ are uniformly $C^{1+\mathrm{Lip}}$.

In other words, there exists a universal constant $K > 0$ such that if $W^{u/s}_x$ is an LUM or LSM then the graph representing $W^{u/s}_x$ locally at x has $C^{1+\mathrm{Lip}}$ norm bounded by K. Axiom (A5) is weaker than those required in the literature, and is discussed further at the end of this section, see Remark 3.2.

(A6) Uniform distortion bounds. There is a function $\psi : [0, \infty) \to [0, \infty)$ with $\lim_{x\to 0} \psi(x) = 0$ for which the following holds: if W is a LUM, then for all x, y belonging to the same connected component V of $W \cap \mathscr{S}_{n-1}$,

$$\log \left[\prod_{i=0}^{n-1} \frac{\Lambda(f^i x)}{\Lambda(f^i y)} \right] \leq \psi(\rho_{f^n(V)}(f^n x, f^n y)).$$

(A7) Uniform absolute continuity. There is a constant $C > 0$ with the following property: if W_1, W_2 are two sufficiently small and close enough LUM's, then the holonomy map $H : W_1' \to W_2$ is absolutely continuous with respect to m_{W_1}, m_{W_2} and

$$\frac{1}{C} \leq \frac{m_{W_2}(H[W_1'])}{m_{W_1}(W_1')} \leq C.$$

Now we proceed to the crucial axiom, which states that the expansion of the system prevails over the fragmentation caused by the singularities. Let us recall once more that condition (A8) below follows [CZ05a]. Indeed, we require the practical scheme described in [CZ05a, §6], for the following reasons:

- The singular set $\mathscr{S}_1 = \mathscr{S}^+$ is usually decomposed into two components $\mathscr{S}_{\mathrm{P}}$ and $\mathscr{S}_{\mathrm{S}}$. The set $\mathscr{S}_{\mathrm{P}}$ is made of intrinsic singularities, which we call *primary*, and $\mathscr{S}_{\mathrm{S}}$ is made of artificial ones, which we call *secondary*. The set $\mathscr{S}_{\mathrm{S}}$ is artificially added to guarantee bounded distortion, near places where the derivative explodes.
- Since the expansion factor Λ might be close to 1, we often need to consider an iterate f^n so that Λ^n is large enough.

Hence we assume that $\mathscr{S}_1 = \mathscr{S}^+ = \mathscr{S}_{\mathrm{P}} \cup \mathscr{S}_{\mathrm{S}}$, and we let $\mathscr{S}_n = \mathscr{S}_{\mathrm{P},n} \cup \mathscr{S}_{\mathrm{S},n}$ be the corresponding decomposition of $\mathscr{S}_n$ for $n \geq 1$, where

$$\mathscr{S}_{\mathrm{P},n} = \mathscr{S}_{\mathrm{P}} \cup f^{-1}(\mathscr{S}_{\mathrm{P}}) \cup \cdots \cup f^{-n+1}(\mathscr{S}_{\mathrm{P}})$$
$$\mathscr{S}_{\mathrm{S},n} = \mathscr{S}_{\mathrm{S}} \cup f^{-1}(\mathscr{S}_{\mathrm{S}}) \cup \cdots \cup f^{-n+1}(\mathscr{S}_{\mathrm{S}}).$$

[1] For simplicity, the original work of Chernov also required an assumption called *nonbranching of unstable manifolds*, see [Che99a, p. 516]. As already remarked in [Che99a], this assumption can be dropped [vdB01].

Since secondary singularities are accompanied by strong hyperbolicity, the philosophy that "expansion prevails over fragmentation" is guaranteed if the expansion caused by $\mathscr{S}_{\mathrm{S}}$ prevails over the fragmentation caused by $\mathscr{S}_{\mathrm{P},n}$. We only need to check this for some $n \geq 1$ for which Λ^n is large enough. The precise conditions are the following.

(A8) Growth of unstable manifolds.

(A8.1) *Structure of the singularity set*: There are constants $C, d > 0$ such that if W is a LUM, then $W \cap \mathscr{S}_1$ is at most countable. Furthermore, $W \cap \mathscr{S}_1$ has at most one accumulation point x_∞, and if $\{x_n\}$ is the monotonic sequence in $W \cap \mathscr{S}_1$ coverging to x_∞ then

$$\rho(x_n, x_\infty) \leq \frac{C}{n^d} \quad \text{for all } n \geq 1.$$

(A8.2) *Growth bound (assumption on secondary singularities)*:

$$\theta_0 := \liminf_{\delta \to 0} \sup_{|W| < \delta} \sum_n \tfrac{1}{\Lambda_n} < 1,$$

where the supremum is taken over all LUM W with $W \cap \mathscr{S}_{\mathrm{P}} = \emptyset$, the connected components of $W \backslash \mathscr{S}_{\mathrm{S}}$ are $\{W_n\}$ and $\Lambda_n = \min\{\Lambda(x) : x \in W_n\}$.

(A8.3) *Complexity bound (assumption on primary singularities)*: Let

$$K_{\mathrm{P,n}} := \lim_{\delta \to 0} \sup_{|W| < \delta} K_{\mathrm{P},n}(W)$$

where the supremum is taken over all LUM W and $K_{\mathrm{P},n}(W)$ is the number of connected components of $W \backslash \mathscr{S}_{\mathrm{P},n}$; then $K_{\mathrm{P},n} < \min\{\theta_0^{-1}, \Lambda\}^n$ for some $n \geq 1$.

We can now state the result of Chernov & Zhang that will be important to us.

Theorem 3.1 (Chernov & Zhang [CZ05a]). *If f satisfies* (A1)–(A8)*, then f is modelled by a Young tower with exponential tails.*

In particular, f enjoys exponential decay of correlations by [You98].

Remark 3.2. Previous work requires invariant manifolds to have uniformly bounded sectional curvature in axiom (A5). This condition is solely used to uniformly approximate pieces of invariant manifolds by hyperplanes. More specifically, [Che99a, Estimate (4.1)] states the existence of $C > 0$ such that if W is a δ–LUM then $Z[W, W, 0] \leq C Z[H, H, 0]$, where H is the projection of W to $T_x M$ for some (every) $x \in W$. The function $Z[W, W, 0]$ characterizes, in some sense, the "size" of W. The curvature assumption is used to show that the "size" of W is of the same order of its projection to hyperplanes. The weaker assumption of uniformly bounded $C^{1+\mathrm{Lip}}$ norm ensures that the jacobians of the projection map and of its inverse have uniformly bounded Lipschitz constants, and this is enough to imply the above estimate. In particular, Theorem 3.1 remains valid under (A1)–(A8) with this slightly weakened version of (A5).

4. Dynamics of transitions in the neck $\mathcal{N}$

We initiate the study of the geodesic flow on a surface with degenerate closed geodesic as defined in Section 2.3 with $r \geq 4$ and $\varepsilon_0 > 0$ fixed. Recall that we are taking $\delta > 0$ small enough and considering the δ–Sasaki metric $\|\cdot\| = \|\cdot\|_{\delta-\mathrm{Sas}}$. By Proposition 2.5 and equation (2.1), the loss of uniform hyperbolicity of g_t occurs when a geodesic approaches Deg, i.e. when it spends a large amount of time spiraling close to the degenerate closed geodesic γ. Hence in this section we focus on giving a detailed description of the dynamics of the transitions in the neck $\mathcal{N}$.

In Subsection 4.1 we construct a two-dimensional section Ω, the *transition section*, and an associated *transition map* f_0. In Subsection 4.2 we obtain an explicit formula for f_0. It is this representation that allows us to avoid using the Riccati equation and its (not so precise) estimates and, instead, to perform more accurate calculations. In Subsection 4.3 we estimate transition times and derivatives of f_0.

In the rest of the paper, we use the following notation:

- $a(u) \ll b(u)$ or $a(u) = O(b(u))$ if there is a constant C such that $a(u) \leq Cb(u)$ for all u small enough.
- $a(u) \approx b(u)$ if $a(u) \ll b(u)$ and $b(u) \ll a(u)$.
- $a(u) \sim b(u)$ if $\lim\limits_{u \to 0} \frac{a(u)}{b(u)} = 1$.

The calculations in this section use the Clairaut coordinates $x = (s, \theta, \psi)$ on $T^1 A \cong [-\varepsilon_0, \varepsilon_0] \times \mathbb{S}^1 \times \mathbb{S}^1$.

4.1. **Transition section Ω and map f_0.** The transition section Ω we construct allows a very simple description of the transitions of geodesics in the neck. As detailed below, we define $\Omega = \Omega_+ \cup \Omega_- \cup \Omega_0$ where:

- $\Omega_+ \subset \{\pm\varepsilon_0\} \times \mathbb{S}^1 \times \mathbb{S}^1$ is a neighborhood of four families of geodesics entering the neck and asymptotic to γ;
- $\Omega_- \subset \{\pm\varepsilon_0\} \times \mathbb{S}^1 \times \mathbb{S}^1$ is a neighborhood of four families of geodesics exiting the neck and asymptotic to γ in backwards time;
- Each of Ω_+ and Ω_- is a disjoint union of four annular regions (diffeomorphic to $\mathbb{S}^1 \times (-1, 1)$);
- Ω_0 is defined in a neighborhood of γ and is a disjoint union of two open disks.

To construct $\Omega_\pm$, we use the Clairaut function c from Section 2.2. Recall that $\xi(s) = 1 + |s|^r$. Hence there is a unique $\psi_0 \in (0, \frac{\pi}{2})$ such that $\xi(\varepsilon_0) \cos\psi_0 = \xi(-\varepsilon_0)\cos\psi_0 = 1$. Recalling the notation of asymptotic vector introduced in Section 2.3, the vectors $(\pm\varepsilon_0, \theta, \pm\psi_0)$, $\theta \in \mathbb{S}^1$, constitute four families of asymptotic vectors with Clairaut constant $c = 1$ and asymptotic to γ. Similarly, $(\pm\varepsilon_0, \theta, \pm(\pi - \psi_0))$, $\theta \in \mathbb{S}^1$, constitute four families of asymptotic vectors with $c = -1$ and asymptotic to γ. Of these, $(-\varepsilon_0, \theta, \psi_0)$, $(\varepsilon_0, \theta, -\psi_0)$, $(-\varepsilon_0, \theta, \pi - \psi_0)$, $(\varepsilon, \theta, -(\pi - \psi_0))$ correspond to the four families of trajectories that enter $\mathcal{N}$ and are asymptotic to γ as $t \to \infty$. The remaining four families of trajectories exit $\mathcal{N}$ and are asymptotic to γ as $t \to -\infty$.

Focusing momentarily on $x = (-\varepsilon_0, \theta, \psi_0)$, we define

$$\Omega_1 = \{x = (-\varepsilon_0, \theta, \psi) \in T^1 A : |c(x) - 1| < \chi\}.$$

Shrinking $\chi = \chi(\varepsilon_0, r) > 0$, we can ensure that $\Omega_1 = \{-\varepsilon_0\} \times \mathbb{S}^1 \times I$ where I is an open interval containing ψ_0 with $\overline{I} \subset (0, \frac{\pi}{2})$. Treating the other three families of entering asymptotic trajectories similarly, we obtain Ω_+ as the union of four sets isomorphic to Ω_1.

Similarly, the set Ω_-, isomorphic to Ω_+, is obtained by considering the four families of exiting asymptotic trajectories.

Finally, let

$$\Omega_0 = (-\chi, \chi) \times \{0\} \times ((-\chi, \chi) \cup (\pi - \chi, \pi + \chi)) \subset T^1 A.$$

TRANSITION SECTION Ω: Define $\Omega = \Omega_- \cup \Omega_0 \cup \Omega_+$.

We now prove that Ω is transverse to the flow direction.

- ◦ The tangent space at every point of Ω_0 is $\mathbb{R} \times \{0\} \times \mathbb{R}$. Since the flow directions at $(0,0,0)$ and $(0,\pi,0)$ are spanned by $(0,1,0)$ and $(0,-1,0)$ respectively, transversality holds at $(0,0,0)$ and $(0,\pi,0)$. Since $\chi > 0$ is small enough, transversality holds at every point of Ω_0.
- ◦ The tangent space at every $x = (\pm\varepsilon_0, \theta, \psi) \in \Omega_+$ is $\{0\} \times \mathbb{R} \times \mathbb{R}$. If $(s(t), \theta(t), \psi(t))$ is the geodesic defined by x, then the flow direction at x is $(s'(0), \theta'(0), \psi'(0))$. Since $\psi(0) \neq 0$, we have $\xi(s(0)) > c(x)$ and so the equation of geodesics (2.3) implies that $s'(0) \neq 0$. Again for $\chi > 0$ small enough, Ω_+ is transverse to the flow direction at x.
- ◦ Analogously, Ω_- is transverse to the flow direction.

The transition section Ω captures all trajectories that approach Deg. To better understand them, consider the partition of $\Omega_+ = \Omega_+^= \cup \Omega_+^> \cup \Omega_+^<$ induced by c:

$$\begin{aligned} \Omega_+^= &= \{x \in \Omega_+ : |c(x)| = 1\}, \text{ (asymptotic)}, \\ \Omega_+^> &= \{x \in \Omega_+ : |c(x)| > 1\}, \text{ (bouncing)}, \\ \Omega_+^< &= \{x \in \Omega_+ : |c(x)| < 1\}, \text{ (crossing)}. \end{aligned}$$

Similarly, we define $\Omega_- = \Omega_-^= \cup \Omega_-^> \cup \Omega_-^<$.

We have the following transitions of segments of trajectories that enter $\mathcal{N}$ and approach Deg.

- ◦ Starting at asymptotic vectors: $\Omega_+^= \to \Omega_0 \to \Omega_0 \to \cdots$, and these trajectories get trapped in Ω_0.
- ◦ Starting at bouncing vectors: $\Omega_+^> \to \Omega_0 \to \cdots \to \Omega_0 \to \Omega_-^>$.
- ◦ Starting at crossing vectors: $\Omega_+^< \to \Omega_0 \to \cdots \to \Omega_0 \to \Omega_-^<$.

We have thus understood the transition in the neck of every $x \in \Omega_+$. Next, we introduce the map that performs the transitions from Ω_+ to Ω_-.

TRANSITION MAP f_0: Let $f_0 : \Omega_+ \setminus \Omega_+^= \to \Omega_- \setminus \Omega_-^=$ be the map induced by the flow, i.e. $f_0(x) = g_t(x)$ where $t > 0$ is least such that $g_t(x) \in \Omega_-$.

4.2. **Formula for the transition map f_0.** In this subsection, we obtain an explicit formula for f_0.

Denote geodesics in the neck by $\mathbf{x} = \mathbf{x}(t) = (s(t), \theta(t), \psi(t))$. As we have seen, they are characterized by the equations

$$c = \xi(s) \cos \psi = \xi(s)^2 \theta', \tag{4.1}$$

$$[1 + \xi'(s)^2](s')^2 + \frac{c^2}{\xi(s)^2} = 1, \tag{4.2}$$

where $c = c(\mathbf{x})$ is the Clairaut constant of $\mathbf{x}$. We parametrize bouncing geodesics $\mathbf{x}$ taking $s'(0) = 0$ and $\psi(0) = 0$ or π, i.e. $\mathbf{x}$ bounces back exactly at time $t = 0$. Similarly, we parametrize crossing geodesics $\mathbf{x}$ taking $s(0)=0$, i.e. $\mathbf{x}$ crosses γ exactly at time $t=0$. The next result describes the symmetry properties of bouncing/crossing geodesics.

Lemma 4.1. *The following are true.*

(1) *A bouncing geodesic* $(s(t),\theta(t),\psi(t))$ *with* $s'(0)=0$ *and* $\psi(0)=0$ *satisfies*
$$(s(-t),\theta(-t),\psi(-t))=(s(t),-\theta(t)+2\theta(0),-\psi(t))$$
for all t.

(2) *A crossing geodesic* $(s(t),\theta(t),\psi(t))$ *with* $s(0)=0$ *satisfies*
$$(s(-t),\theta(-t),\psi(-t))=(-s(t),-\theta(t)+2\theta(0),\psi(t))$$
for all t.

Proof. (1) Define
$$\overline{\mathbf{x}}=(\overline{s}(t),\overline{\theta}(t),\overline{\psi}(t))=(s(-t),-\theta(-t)+2\theta(0),-\psi(-t)).$$
We show that $\overline{\mathbf{x}}$ is a geodesic with the same initial conditions of $\mathbf{x}$. Start by observing that $\overline{\mathbf{x}}(0)=(s(0),\theta(0),0)=\mathbf{x}(0)$. Since $\overline{s}'(t)=-s'(-t)$, $\overline{\theta}'(t)=\theta'(-t)$ and $\overline{\psi}'(t)=\psi'(-t)$,
$$c(\overline{\mathbf{x}})=\xi(\overline{s}(t))^2\overline{\theta}'(t)=\xi(s(-t))^2\theta'(-t)=c(\mathbf{x}),$$
$$[1+\xi'(\overline{s}(t))^2](\overline{s}'(t))^2+\tfrac{c(\overline{\mathbf{x}})^2}{\xi(\overline{s}(t))^2}=[1+\xi'(s(-t))^2](s'(-t))^2+\tfrac{c(\mathbf{x})^2}{\xi(s(-t))^2}=1.$$
Hence $\overline{\mathbf{x}}=\mathbf{x}$, which proves part (1).

(2) Similarly, define
$$\overline{\mathbf{x}}=(\overline{s}(t),\overline{\theta}(t),\overline{\psi}(t))=(-s(-t),-\theta(-t)+2\theta(0),\psi(-t)).$$
Then $\overline{\mathbf{x}}(0)=(0,\theta(0),\psi(0))=\mathbf{x}(0)$ and, since $\overline{s}'(t)=s'(-t)$, $\overline{\theta}'(t)=\theta'(-t)$ and $\overline{\psi}'(t)=-\psi'(-t)$,
$$c(\overline{\mathbf{x}})=\xi(\overline{s}(t))^2\overline{\theta}'(t)=\xi(s(-t))^2\theta'(-t)=c(\mathbf{x}),$$
$$[1+\xi'(\overline{s}(t))^2](\overline{s}'(t))^2+\tfrac{c(\overline{\mathbf{x}})^2}{\xi(\overline{s}(t))^2}=[1+\xi'(s(-t))^2](s'(-t))^2+\tfrac{c(\mathbf{x})^2}{\xi(s(-t))^2}=1.$$
We obtain again that $\overline{\mathbf{x}}=\mathbf{x}$, which proves part (2). □

Let $\mathbf{x}=(s(t),\theta(t),\psi(t))$ be a bouncing/crossing geodesic parametrized as above.

TRANSITION TIME Υ_0: Define $\Upsilon_0(\mathbf{x})=\min\{t>0:|s(t)|=\varepsilon_0\}$.

Since ξ is an even function, we can assume that $\mathbf{x}$ enters the neck $\mathcal{N}$ at $\{s=-\varepsilon_0\}$. Lemma 4.1 implies the identities
$$x=g_{-\Upsilon_0(\mathbf{x})}(s(0),\theta(0),\psi(0))\in\Omega_+,\qquad f_0(x)=g_{\Upsilon_0(\mathbf{x})}(s(0),\theta(0),\psi(0))\in\Omega_-.$$
In particular, the transition time of $\mathbf{x}$ from Ω_+ to Ω_- is actually equal to $2\Upsilon_0(\mathbf{x})$ (but it is convenient to call Υ_0 the transition time). Moreover:

- If $\mathbf{x}$ is bouncing then $s\restriction_{[-\Upsilon_0(\mathbf{x}),0]}$ is strictly increasing and $s\restriction_{[0,\Upsilon_0(\mathbf{x})]}$ is strictly decreasing. We have $s(-\Upsilon_0(\mathbf{x}))=-\varepsilon_0=s(\Upsilon_0(\mathbf{x}))$.
- If $\mathbf{x}$ is crossing then $s\restriction_{[-\Upsilon_0(\mathbf{x}),\Upsilon_0(\mathbf{x})]}$ is strictly increasing. We have $s(-\Upsilon_0(\mathbf{x}))=-\varepsilon_0$, $s(\Upsilon_0(\mathbf{x}))=\varepsilon_0$.

The next proposition gives the remaining ingredient to obtain the explicit formula for f_0.

Proposition 4.2. *Let $\mathbf{x}$ be a bouncing/crossing geodesic. Then*

$$|\theta(\Upsilon_0(\mathbf{x})) - \theta(-\Upsilon_0(\mathbf{x}))| = 2\int_{|s(0)|}^{\varepsilon_0} \frac{|c(\mathbf{x})|}{\xi(s)} \left[\frac{1+\xi'(s)^2}{\xi(s)^2 - c(\mathbf{x})^2}\right]^{\frac{1}{2}} ds.$$

Proof. Equations (4.1) and (4.2) can be rewritten as

$$\theta' = c(\mathbf{x})\xi(s)^{-2} \quad \text{and} \quad |s'| = \xi(s)^{-1}\left[\frac{\xi(s)^2 - c(\mathbf{x})^2}{1+\xi'(s)^2}\right]^{\frac{1}{2}}.$$

Dividing one equation by the other yields

$$\left|\frac{d\theta}{ds}\right| = \frac{|c(\mathbf{x})|}{\xi(s)} \left[\frac{1+\xi'(s)^2}{\xi(s)^2 - c(\mathbf{x})^2}\right]^{\frac{1}{2}}.$$

Since $s\restriction_{[0,\Upsilon_0(\mathbf{x})]}$ is monotone,

$$|\theta(\Upsilon_0(\mathbf{x})) - \theta(0)| = \int_{|s(0)|}^{\varepsilon_0} \frac{|c(\mathbf{x})|}{\xi(s)} \left[\frac{1+\xi'(s)^2}{\xi(s)^2 - c(\mathbf{x})^2}\right]^{\frac{1}{2}} ds.$$

By Lemma 4.1, we have that $\theta(\Upsilon_0(\mathbf{x})) - \theta(-\Upsilon_0(\mathbf{x})) = 2[\theta(\Upsilon_0(\mathbf{x})) - \theta(0)]$ and the proof is complete. $\square$

By the rotational symmetry, the integral in Proposition 4.2 only depends on ψ. Hence we define

$$\zeta(\psi) = 2\int_{|s(0)|}^{\varepsilon_0} \frac{|c(\mathbf{x})|}{\xi(s)} \left[\frac{1+\xi'(s)^2}{\xi(s)^2 - c(\mathbf{x})^2}\right]^{\frac{1}{2}} ds.$$

With this notation,

$$f_0(-\varepsilon_0, \theta, \psi) = \begin{cases} (-\varepsilon_0, \theta \pm \zeta(\psi), -\psi) & \text{for bouncing vectors,} \\ (\varepsilon_0, \theta \pm \zeta(\psi), \psi) & \text{for crossing vectors.} \end{cases}$$

4.3. **Transition times and derivatives of f_0.** Before proceeding further, we derive two elementary integral estimates.

Proposition 4.3.

(1) *Let $r, \alpha, \varepsilon > 0$ with $\alpha r > 1$. Then*

$$\int_0^{\varepsilon} (s^r + b)^{-\alpha}\, ds \sim C_1 b^{-\alpha + \frac{1}{r}} \quad \text{as } b \to 0^+,$$

where $C_1 = \int_0^\infty (x^r+1)^{-\alpha}\, dx$.

(2) *Let $q, r, \alpha, \varepsilon > 0$ and $\beta \geq 0$ with $\alpha r - \beta q > 1$. Then*

$$\int_b^{\varepsilon} (s^r - b^r)^{-\alpha}(s^q - b^q)^{\beta}\, ds \sim C_2 b^{\beta q - \alpha r + 1} \quad \text{as } b \to 0^+,$$

and

$$\int_b^{\varepsilon + b} (s^r - b^r)^{-\alpha}(s^q - b^q)^{\beta}\, ds \sim C_2 b^{\beta q - \alpha r + 1} \quad \text{as } b \to 0^+,$$

where $C_2 = \int_1^\infty (x^r - 1)^{-\alpha}(x^q-1)^{\beta}\, dx$.

Proof. Note that $\alpha r > 1$ implies that C_1 is finite. By direct computation and the change of variables $x = b^{-\frac{1}{r}} s$,

$$\int_0^\varepsilon (s^r + b)^{-\alpha}\, ds = b^{-\alpha} \int_0^\varepsilon (b^{-1}s^r + 1)^{-\alpha}\, ds = b^{-\alpha+\frac{1}{r}} \int_0^{\varepsilon b^{-\frac{1}{r}}} (x^r + 1)^{-\alpha}\, dx$$

and part (1) follows.

Proceeding similarly to the proof of part (1),

$$\begin{aligned}\int_b^\varepsilon (s^r - b^r)^{-\alpha}(s^q - b^q)^\beta ds &= b^{\beta q - \alpha r} \int_b^\varepsilon \left[\left(\tfrac{s}{b}\right)^r - 1\right]^{-\alpha} \left[\left(\tfrac{s}{b}\right)^q - 1\right]^\beta ds \\ &= b^{\beta q - \alpha r + 1} \int_1^{\varepsilon b^{-1}} (x^r - 1)^{-\alpha}(x^q - 1)^\beta dx\end{aligned}$$

and the first estimate in part (2) follows. The argument for the second estimate is identical. □

To better analyse f_0 near the set of asymptotic vectors, we introduce a partition of Ω_+, as follows.

Homogeneity bands on Ω_+: For each $n \geq 1$, the *homogeneity band* with index n is $\mathscr{C}_n = \mathscr{C}_n^> \cup \mathscr{C}_n^<$ where

$$\begin{aligned}\mathscr{C}_n^> &= \left\{x \in \Omega_+ : 1 + \tfrac{1}{(n+1)^2} < |c(x)| < 1 + \tfrac{1}{n^2}\right\} \\ \mathscr{C}_n^< &= \left\{x \in \Omega_+ : 1 - \tfrac{1}{n^2} < |c(x)| < 1 - \tfrac{1}{(n+1)^2}\right\}.\end{aligned}$$

The next result estimates Υ_0 in the homogeneity bands.

Lemma 4.4. *Let $\mathbf{x}$ be a geodesic with entry vector in $\mathscr{C}_n$. Then $\Upsilon_0(\mathbf{x}) \approx n^{\frac{r-2}{r}}$.*

Proof. We continue to assume without loss that $s(-\Upsilon_0(\mathbf{x})) = -\varepsilon_0$. Also, we suppose without loss that the Clairaut constant $c = c(\mathbf{x})$ is positive. By assumption, $\frac{1}{(n+1)^2} < |c - 1| < \frac{1}{n^2}$. By (4.2),

$$(s')^2 = (1 + (\xi')^2)^{-1}\xi^{-2}(\xi + c)(\xi - c) \approx \xi(s) - c. \tag{4.3}$$

We have two cases:

- If $\mathbf{x}$ is bouncing, then $c = 1 + |s(0)|^r$ where $s(0) \sim -n^{-2/r}$. By (4.3), $(s')^2 \approx |s|^r - |s(0)|^r$ and so $(|s|^r - |s(0)|^r)^{-\frac{1}{2}} s' \approx 1$ on the interval $[-\Upsilon_0(\mathbf{x}), 0]$. Applying Proposition 4.3(2) with $\alpha = \frac{1}{2}$, $\beta = 0$ and $b = |s(0)|$, we conclude that

$$\begin{aligned}&\Upsilon_0(\mathbf{x}) \approx -\int_{-\varepsilon_0}^{s(0)} (|s|^r - |s(0)|^r)^{-\frac{1}{2}} ds = \int_{|s(0)|}^{\varepsilon_0} (s^r - |s(0)|^r)^{-\frac{1}{2}} ds \\ &\approx |s(0)|^{-\frac{r}{2}+1} \sim n^{\frac{r-2}{r}}.\end{aligned}$$

∘ If $\mathbf{x}$ is crossing, then $c = \cos\psi(0) \sim 1 - n^{-2}$. By (4.3), $(s')^2 \approx |s|^r + n^{-2}$ and so $(|s|^r + n^{-2})^{-\frac{1}{2}} s' \approx 1$ on the interval $[-\Upsilon_0(\mathbf{x}), \Upsilon_0(\mathbf{x})]$. Applying Proposition 4.3(1) with $\alpha = \frac{1}{2}$ and $b = n^{-2}$, we conclude that

$$\Upsilon_0(\mathbf{x}) \approx \int_0^{\varepsilon_0} (s^r + n^{-2})^{-\frac{1}{2}} ds \approx \left(n^{-2}\right)^{-\frac{1}{2}+\frac{1}{r}} = n^{\frac{r-2}{r}}.$$

This concludes the proof of the lemma. □

Now we estimate, in terms of c, how the ψ–coordinate varies under f_0. Without loss, we restrict to positive values of c.

Lemma 4.5. *Suppose that $\mathbf{x}, \overline{\mathbf{x}}$ are both bouncing or both crossing geodesics. Then*

$$|\psi(\Upsilon_0(\mathbf{x})) - \overline{\psi}(\Upsilon_0(\overline{\mathbf{x}}))| \approx |c(\mathbf{x}) - c(\overline{\mathbf{x}})|.$$

In particular, if the entry vectors of $\mathbf{x}, \overline{\mathbf{x}}$ are both in the same connected component of $\mathscr{C}_n^<$ or of $\mathscr{C}_n^>$ then

$$|\psi(\Upsilon_0(\mathbf{x})) - \overline{\psi}(\Upsilon_0(\overline{\mathbf{x}}))| = 2(\varepsilon_0^{2r} + 2\varepsilon_0^r)^{-1/2} n^{-3} + O(n^{-4}).$$

Proof. Let $a = (1+\varepsilon_0^r)^{-1}$. By (4.1), $(1+\varepsilon_0^r)\cos\psi(\Upsilon_0(\mathbf{x})) = c(\mathbf{x})$ and so $\cos\psi(\Upsilon_0(\mathbf{x})) = ac(\mathbf{x})$. Similarly, $\cos\overline{\psi}(\Upsilon_0(\overline{\mathbf{x}})) = ac(\overline{\mathbf{x}})$. By the mean value theorem,

$$\begin{aligned} a|c(\mathbf{x}) - c(\overline{\mathbf{x}})| &= |\cos\psi(\Upsilon_0(\mathbf{x})) - \cos\overline{\psi}(\Upsilon_0(\overline{\mathbf{x}}))| \\ &= |\sin\psi^*| \cdot |\psi(\Upsilon_0(\mathbf{x})) - \overline{\psi}(\Upsilon_0(\overline{\mathbf{x}}))| \end{aligned}$$

for some ψ^* between $\psi(\Upsilon_0(\mathbf{x}))$ and $\overline{\psi}(\Upsilon_0(\overline{\mathbf{x}}))$. Since $\cos\psi(\Upsilon_0(\mathbf{x})), \cos\overline{\psi}(\Upsilon_0(\overline{\mathbf{x}})) \sim a$, we have $|\sin\psi^*| \sim (1-a^2)^{1/2}$ and so

$$|\psi(\Upsilon_0(\mathbf{x})) - \overline{\psi}(\Upsilon_0(\overline{\mathbf{x}}))| = a|\sin\psi^*|^{-1}|c(\mathbf{x}) - c(\overline{\mathbf{x}})| \sim a(1-a^2)^{-1/2}|c(\mathbf{x}) - c(\overline{\mathbf{x}})|.$$

Now suppose that the entry vectors of $\mathbf{x}, \overline{\mathbf{x}}$ are in the same connected component of $\mathscr{C}_n^>$. Assuming without loss of generality that $c(\mathbf{x}), c(\overline{\mathbf{x}}) > 0$, then $1 + (n+1)^{-2} < c(\mathbf{x}), c(\overline{\mathbf{x}}) \leq 1 + n^{-2}$. It follows that $|c(\mathbf{x}) - c(\overline{\mathbf{x}})| = 2n^{-3} + O(n^{-4})$. Also, $\sin\psi^* = (1-a^2)^{-1/2} + O(n^{-2})$. Hence

$$|\psi(\Upsilon_0(\mathbf{x})) - \overline{\psi}(\Upsilon_0(\overline{\mathbf{x}}))| = 2a(1-a^2)^{-1/2} n^{-3} + O(n^{-4}).$$

An analogous calculation holds if the entry vectors are in $\mathscr{C}_n^<$. □

To conclude this section, we obtain estimates for the derivatives of ζ.

Lemma 4.6. *The following are true.*

(1) *If $(-\varepsilon_0, \theta, \psi) \in \mathscr{C}_n^<$ then $\zeta'(\psi) \approx -n^{3-\frac{2}{r}}$ and $\zeta''(\psi) \approx n^{5-\frac{2}{r}}$.*
(2) *If $(-\varepsilon_0, \theta, \psi) \in \mathscr{C}_n^>$ then $\zeta'(\psi) \approx n^{3-\frac{2}{r}}$ and $|\zeta''(\psi)| \ll n^{5-\frac{2}{r}}$.*

Note that the first three estimates in Lemma 4.6 give upper and lower bounds, while the fourth estimate gives only an upper bound.

Proof. (1) Let $a = 1+\varepsilon_0^r$ (this notation is different from Lemma 4.5). We have $c(\mathbf{x}) = a\cos\psi$, hence $\cos\psi \sim a^{-1} \approx 1$ and $\sin\psi \sim (1-a^{-2})^{\frac{1}{2}} \approx 1$. Also, $s(0) = 0$, so

$$\zeta(\psi) = 2a\cos\psi \int_0^{\varepsilon_0} A(s) \left[\xi(s)^2 - a^2\cos^2\psi\right]^{-\frac{1}{2}} ds$$

where $A(s)=\frac{[1+\xi'(s)^2]^{\frac{1}{2}}}{\xi(s)}\approx 1$. By direct calculation,

$$\begin{aligned}\zeta'(\psi)&=-2a\sin\psi\int_0^{\varepsilon_0}A(s)\xi(s)^2\left[\xi(s)^2-a^2\cos^2\psi\right]^{-\frac{3}{2}}ds\\&\approx-\int_0^{\varepsilon_0}[\xi(s)-c(\mathbf{x})]^{-\frac{3}{2}}ds\end{aligned}$$

and

$$\begin{aligned}\zeta''(\psi)&=2a\cos\psi\int_0^{\varepsilon_0}A(s)\xi(s)^2\left[3a^2\sin^2\psi+a^2\cos^2\psi-\xi(s)^2\right]\left[\xi(s)^2-a^2\cos^2\psi\right]^{-\frac{5}{2}}ds\\&\approx\int_0^{\varepsilon_0}[\xi(s)-c(\mathbf{x})]^{-\frac{5}{2}}ds.\end{aligned}$$

Noting that $\xi(s)-c(\mathbf{x})\approx s^r+n^{-2}$, Proposition 4.3(1) applied with:

- $\alpha=\frac{3}{2}$ and $b=n^{-2}$ gives that $\zeta'(\psi)\approx-(n^{-2})^{-\frac{3}{2}+\frac{1}{r}}=-n^{3-\frac{2}{r}}$.
- $\alpha=\frac{5}{2}$ and $b=n^{-2}$ gives that $\zeta''(\psi)\approx(n^{-2})^{-\frac{5}{2}+\frac{1}{r}}=n^{5-\frac{2}{r}}$.

This proves part (1).

(2) This part is more difficult, for two reasons: the interval of integration in ζ is not fixed, and the denominator $\xi(s)^2-c(\mathbf{x})^2$ is harder to control since both $\xi(s),c(\mathbf{x})>1$. Since $s'(0)=0$, we have $c(\mathbf{x})=a\cos\psi=1+|s(0)|^r$ with, as before, $\cos\psi\approx 1$ and $\sin\psi\approx 1$. Introduce the variable $y=|s(0)|$. Then $c(\mathbf{x})-1=y^r$ and so $y\sim n^{-\frac{2}{r}}$. Write $\zeta(\psi)=2I(y)$, where

$$\begin{aligned}I(y)&=\int_y^{\varepsilon_0}\frac{(1+y^r)}{\xi(s)}\left[\frac{1+\xi'(s)^2}{\xi(s)^2-(1+y^r)^2}\right]^{\frac{1}{2}}ds\\&=\int_0^{\varepsilon_0-y}\frac{(1+y^r)}{\xi(s+y)}\left[\frac{1+\xi'(s+y)^2}{\xi(s+y)^2-(1+y^r)^2}\right]^{\frac{1}{2}}ds\\&=\int_0^{\varepsilon_0-y}A(s,y)\left[(s+y)^r-y^r\right]^{-\frac{1}{2}}ds.\end{aligned}$$

Here, $A(s,y)=\frac{(1+y^r)}{\xi(s+y)}\left[\frac{1+\xi'(s+y)^2}{\xi(s+y)+(1+y^r)}\right]^{\frac{1}{2}}$ is C^2 with $A(s,y)\approx 1$.

Next, write $I(y)=I_1(y)+I_2(y)$ where

$$\begin{aligned}I_1(y)&=\int_0^{\varepsilon_0}A(s,y)\left[(s+y)^r-y^r\right]^{-\frac{1}{2}}ds,\\I_2(y)&=-\int_{\varepsilon_0-y}^{\varepsilon_0}A(s,y)\left[(s+y)^r-y^r\right]^{-\frac{1}{2}}ds.\end{aligned}$$

Now, I_2 is C^2 for y small and in particular I_2' and I_2'' are bounded. Therefore, it remains to estimate I_1' and I_1''. We have $I_1' = Q_1 + Q_2$ where

$$Q_1(y) = \int_0^{\varepsilon_0} \partial_y A(s,y)\,[(s+y)^r - y^r]^{-\frac{1}{2}}\,ds,$$
$$Q_2(y) = -\tfrac{r}{2}\int_0^{\varepsilon_0} A(s,y)\,[(s+y)^r - y^r]^{-\frac{3}{2}}\left[(s+y)^{r-1} - y^{r-1}\right]ds.$$

Using that A is C^2 and applying Proposition 4.3(2) with $\alpha = \frac{1}{2}$, $\beta = 0$, $b = y$ gives that

$$|Q_1(y)| \ll \int_0^{\varepsilon_0} [(s+y)^r - y^r]^{-\frac{1}{2}}\,ds = \int_y^{\varepsilon_0+y} [s^r - y^r]^{-\frac{1}{2}}\,ds \approx y^{-\frac{r}{2}+1}.$$

Now, since $A(s,y) \approx 1$, applying Proposition 4.3(2) with $\alpha = \frac{3}{2}$, $\beta = 1$, $q = r-1$, $b = y$ implies that

$$\begin{aligned} Q_2(y) &\approx -\int_0^{\varepsilon_0} [(s+y)^r - y^r]^{-\frac{3}{2}}\left[(s+y)^{r-1} - y^{r-1}\right]ds \\ &= -\int_y^{\varepsilon_0+y} [s^r - y^r]^{-\frac{3}{2}}\left[s^{r-1} - y^{r-1}\right]ds \approx -y^{-\frac{r}{2}}. \end{aligned}$$

Hence, $I'(y) \approx I_1'(y) \approx -y^{-\frac{r}{2}}$. Next we transform back to the variable ψ. Differentiating $1+y^r = a\cos\psi$ with respect to ψ, we get that $ry^{r-1}\frac{dy}{d\psi} = -a\sin\psi \approx -1$ and so $\frac{dy}{d\psi} \approx -y^{1-r}$, which implies that

$$\zeta'(\psi) = 2I'(y)\tfrac{dy}{d\psi} \approx (-y^{-\frac{r}{2}})(-y^{1-r}) = y^{1-\frac{3r}{2}} \sim (n^{-\frac{2}{r}})^{1-\frac{3r}{2}} \sim n^{3-\frac{2}{r}}.$$

This is the desired estimate for ζ'.

Similarly, we can write $I_1'' = Q_3 + Q_4 + Q_5 + Q_6$ with

$$\begin{aligned} |Q_3(y)| &\ll \int_y^{\varepsilon_0+y} [s^r - y^r]^{-\frac{1}{2}}\,ds \approx y^{-\frac{r}{2}+1}, \\ |Q_4(y)| &\ll \int_y^{\varepsilon_0+y} [s^r - y^r]^{-\frac{3}{2}}\left[s^{r-1} - y^{r-1}\right]ds \approx y^{-\frac{r}{2}}, \\ |Q_5(y)| &\approx \int_y^{\varepsilon_0+y} [s^r - y^r]^{-\frac{3}{2}}\left[s^{r-2} - y^{r-2}\right]ds \approx y^{-\frac{r}{2}-1}, \\ |Q_6(x)| &\approx \int_y^{\varepsilon_0+y} [s^r - y^r]^{-\frac{5}{2}}\left[s^{r-1} - y^{r-1}\right]^2 ds \approx y^{-\frac{r}{2}-1}. \end{aligned}$$

Here, the estimates of Q_3, Q_4 are the same as those of Q_1, Q_2, the estimate of Q_5 follows from Proposition 4.3(2) with $\alpha = \frac{3}{2}$, $\beta = 1$, $q = r-2$, $b = y$ and the estimate of Q_6 follows from Proposition 4.3(2) with $\alpha = \frac{5}{2}$, $\beta = 2$, $q = r-1$, $b = y$. Hence $|I''(y)| \ll y^{-\frac{r}{2}-1}$. Differentiating $ry^{r-1}\frac{dy}{d\psi} = -a\sin\psi$ with respect to ψ and recalling that $\frac{dy}{d\psi} \approx -y^{1-r}$, we get that

$$-1 \approx -a\cos\psi = r(r-1)y^{r-2}\big(\tfrac{dy}{d\psi}\big)^2 + ry^{r-1}\tfrac{d^2y}{d\psi^2} \approx r(r-1)y^{-r} + ry^{r-1}\tfrac{d^2y}{d\psi^2}.$$

Since y^{-r} is large, both terms on the right-hand side have the same order, hence $\frac{d^2y}{d\psi^2} \approx -y^{-2r+1}$. We thus conclude that

$$|\zeta''(\psi)| = \left|2I''(y)\left(\tfrac{dy}{d\psi}\right)^2 + 2I'(y)\tfrac{d^2y}{d\psi^2}\right| \ll y^{-\frac{r}{2}-1}y^{2(1-r)} + y^{-\frac{r}{2}}y^{-2r+1}$$
$$\approx y^{-\frac{5r}{2}+1} \sim (n^{-\frac{2}{r}})^{-\frac{5r}{2}+1} = n^{5-\frac{2}{r}}$$

which is the required estimate for ζ''. □

5. Poincaré section and first return map f

In this section, we construct a suitable Poincaré first return map $f : \Sigma_0 \to \Sigma_0$ with unbounded Poincaré return time $\tau : \Sigma_0 \to (0, \infty]$ such that $\inf \tau > 0$. Keeping in mind the Chernov axioms (Section 3), we require that the two-dimensional cross-section Σ_0 satisfies:

- Σ_0 is the disjoint union of finitely many codimension one submanifolds of M each of which is almost orthogonal to the flow direction and hence, by Lemma 2.4(2), almost parallel to $\widehat{E}^s \oplus \widehat{E}^u$. In particular, the flow projections of the one-dimensional stable/unstable directions $\widehat{E}^{s,u}$ of g_t define stable/unstable directions $E^{s,u}$ for f. Moreover, the hyperbolicity of f along $E^{s,u}$ is almost the same as that of g_t along $\widehat{E}^{s,u}$, as given by equation (2.1).
- The boundary of Σ_0 is transverse to E^s and E^u. This condition ensures (A3).
- There are no triple intersections for a certain family of curves and iterates under f. This family of curves is the union of boundaries of Σ_0 and finitely many curves of asymptotic vectors, and makes up the primary singular set $\mathscr{S}_{\mathrm{P}}$. This condition is used in the verification of (A8.3).

The construction of Σ_0 is rather technical, and is done as follows. Recall the transition section Ω constructed in Section 4.1. Using Ω, we construct a "security" section $\widetilde{\Sigma}$ that is almost parallel to $\widehat{E}^s \oplus \widehat{E}^u$. Then we choose $\widehat{\Sigma} \subset \widetilde{\Sigma}$ so that its boundary is transverse to the invariant directions and the "no triple intersections" requirement stated above holds. Ideally, we would take $\Sigma_0 = \widehat{\Sigma}$, but unfortunately this is not enough to guarantee axiom (A2), since the hyperbolicity rate of the Poincaré return map (and its induced maps) depends on two effects that compete one against the other:

- The angle between the Poincaré section and $\widehat{E}^{s,u}$: the smaller the angle is, the closer are the hyperbolicity rates of the flow and of the Poincaré return map.
- The Poincaré return time of the section: the smaller the return time is, the weaker is the hyperbolicity of the flow (and hence of the Poincaré return map).

If the connected components of $\widehat{\Sigma}$ are small to guarantee a small angle, then the Poincaré return time is close to zero. To bypass this difficulty, we divide each connected component of $\widehat{\Sigma}$ into small pieces, each of them still with boundary transverse to the invariant directions, and displace each of them by a small amount in the flow direction so that the new angle with $\widehat{E}^{s,u}$ is as close to zero as we wish and no triple intersection appears. Letting Σ_0 be the union of the displaced pieces, its Poincaré return time is close to that of $\widehat{\Sigma}$. In other words, this division procedure decreases the angle whilst almost preserving the Poincaré return time. This allows us to prove (A2) in Section 6.

5.1. **Construction of $\widetilde{\Sigma}$.** Since $\Omega_\pm$ might not be almost parallel to $\widehat{E}^{s,u}$, we take instead a finite union of small discs that are almost parallel to $\widehat{E}^{s,u}$ and whose union of flow boxes contains $\Omega_\pm$. Choosing the discs small enough, the transitions in the neck from $\widetilde{\Sigma}$ to itself coincide with the transitions from Ω_+ to Ω_- up to small flow displacements at the beginning and end. Then we complete $\widetilde{\Sigma}$ by adding a disjoint union of finitely many small discs such that their flow boxes do not intersect the trajectories in the neck that are close to the asymptotic ones. In other words, we complete the section so that the asymptotic trajectories and nearby ones remain the same.

We begin introducing some notation. For each $x \in M$, let $\exp_x : T_xM \to M$ denote the exponential map of M at x. For $U \subset M$ and $I \subset \mathbb{R}$, we let $g_IU = \bigcup_{t\in I} g_t(U)$. Recall that Z_x is the one-dimensional subspace of T_xM tangent to the geodesic flow.

su–DISC AND FLOW BOX: The *su–disc* at $x \in M$ with radius $\lambda > 0$ is the surface

$$D_\lambda(x) = \{\exp_x(v) : v \perp Z_x \text{ and } \|v\| \le \lambda\}.$$

The *flow box* at $x \in M$ with radius $\lambda > 0$ is $g_{[-\lambda,\lambda]}D_\lambda(x)$.

Fix $\chi = \chi(r, \varepsilon_0) > 0$ small. We take $\lambda < \chi$ so that $D_\lambda(x)$ is an immersed surface and $T_yD_\lambda(x)$ is almost orthogonal to the flow and hence almost parallel to $\widehat{E}^s_y \oplus \widehat{E}^u_y$ for every $y \in D_\lambda(x)$. Now we proceed to construct $\widetilde{\Sigma}$.

STEP 1 (CONSTRUCTION OF $\widetilde{\Sigma}$ NEAR $\Omega_\pm$): Choose points $x_1, \dots, x_m \in M$ and $0 < a < b < \chi$ such that $\{D_b(x_i)\}_{1\le i\le m}$ are pairwise disjoint and

$$\Omega_+ \cup \Omega_- \subset \bigcup_{1\le i\le m} g_{[-a,a]}D_a(x_i) \subset \bigcup_{1\le i\le m} g_{[-b,b]}D_b(x_i). \tag{5.1}$$

In other words, Step 1 "approximates" $\Omega_\pm$ by finitely many su–discs. This can be done e.g. by taking a sufficiently fine net of points in $\Omega_- \cup \Omega_+$ and then displacing each of them a small amount in the flow direction so that their su–discs are all disjoint; see a similar argument in [LS19, Lemma 2.7].

STEP 2 (SECURITY NEIGHBORHOOD $\widetilde{N}$): Choose a compact neighborhood $N_+ \subset \Omega_+$ such that $\Omega_+^{\mp} \subset \mathrm{int}(N_+)$. For $x \in N_+$, let $t(x) = \inf\{t > 0 : g_t(x) \in \Omega_-\} \in (0, +\infty]$, and let $\widetilde{N}$ be the closure of $\{g_t(x) : x \in N_+ \text{ and } 0 \le t \le t(x)\}$.

Since $\Omega_+^{\mp}$ is the disjoint union of four closed curves, N_+ is the disjoint union of four compact sets. The set $\widetilde{N}$ is a compact neighborhood of all asymptotic vectors with $|s| \le \varepsilon_0$.

STEP 3 (CONSTRUCTION OF $\widetilde{\Sigma}$ FAR FROM $\Omega_\pm$): Choose $x_{m+1}, \dots, x_{m+n} \in M$ such that:

- $\{D_b(x_i)\}_{m+1\le i\le m+n}$ are pairwise disjoint su–discs, each of them disjoint from $\widetilde{N}$ and disjoint from $\{D_b(x_i)\}_{1\le i\le m}$;
- the disjoint union $\widetilde{\Sigma}(r) \uplus \Omega_0$ is a global Poincaré section (i.e. a cross-section with bounded first return time) for all $\lambda \in [a, b]$, where $\widetilde{\Sigma}(\lambda) = \bigcup_{1\le i\le m+n} D_\lambda(x_i)$.

Again, Step 3 can be carried out similarly to [LS19, Lemma 2.7]. Note that all flow trajectories that start in N_+ make the transition in the neck without visiting any $D_b(x_i)$, $m + 1 \le i \le m + n$.

The security section $\widetilde{\Sigma}$: Define $\widetilde{\Sigma}=\widetilde{\Sigma}(b)$.

Thus $\widetilde{\Sigma}\uplus\Omega_0$ is the largest global Poincaré section constructed in Steps 1–3. Before continuing, let us introduce some further notation. For $\lambda\in[a,b]$, let $\widetilde{h}_\lambda:\widetilde{\Sigma}(\lambda)\uplus\Omega_0\to\widetilde{\Sigma}(\lambda)\uplus\Omega_0$ be the corresponding Poincaré return map. Since the return times of $\widetilde{h}_a,\widetilde{h}_b$ are bounded away from zero and infinity, we have $\widetilde{h}_a=\widetilde{h}_b^N$ where $N:\widetilde{\Sigma}(a)\uplus\Omega_0\to\{1,2,\ldots,N_0\}$ is bounded. The same holds for every $\lambda\in[a,b]$, namely $\widetilde{h}_\lambda=\widetilde{h}_b^{N_\lambda}$ where $N_\lambda:\widetilde{\Sigma}(\lambda)\uplus\Omega_0\to\{1,2,\ldots,N_0\}$ is bounded (the bound N_0 is the same).

For $\lambda\in[a,b]$, define the following objects:

- $\tau_\lambda:\widetilde{\Sigma}(\lambda)\to\mathbb{N}\cup\{\infty\}$ such that $\widetilde{h}_\lambda^{\tau_\lambda(x)}(x)$ is the first return of x to $\widetilde{\Sigma}(\lambda)$.
- $\mathscr{S}^+(\lambda)=\partial\widetilde{\Sigma}(\lambda)\cup\{\tau_\lambda=\infty\}$.
- $f_\lambda:\widetilde{\Sigma}(\lambda)\backslash\mathscr{S}^+(\lambda)\to\widetilde{\Sigma}(\lambda)$ the Poincaré return map.[2]

Note that $\tau_\lambda(x)=\infty$ only for asymptotic vectors, and that the flow time function of f_λ is unbounded exactly when approaching asymptotic vectors. Yet, f_a and f_b differ by a bounded number of iterates, as we now explain. Write $\tau=\tau_a(x)$, and let $0\le i,j<\tau$ such that $\widetilde{h}_b(x),\ldots,\widetilde{h}_b^i(x)\in\widetilde{\Sigma}(b)\setminus\widetilde{\Sigma}(a)$, $\widetilde{h}_b^{i+1}(x),\ldots,\widetilde{h}_b^{\tau-j-1}(x)\in\Omega_0$ and $\widetilde{h}_b^{\tau-j}(x),\ldots,\widetilde{h}_b^{\tau-1}(x)\in\widetilde{\Sigma}(b)\setminus\widetilde{\Sigma}(a)$, i.e. i is the last iterate before entering Ω_0 and $\tau-j-1$ is the last iterate before leaving Ω_0. Clearly $f_a(x)=f_b^{i+j+1}(x)$. Observing that $\widetilde{h}_a(x)=\widetilde{h}_b^{i+1}(x)$, it follows that $i+1\le N_a$. Similarly, $j+1\le N_a$. Letting $\ell_0:=2N_0-1$, we conclude that $i+j+1\le\ell_0$, hence $f_a=f_b^\ell$ for some $\ell:\widetilde{\Sigma}(a)\backslash\mathscr{S}^+(a)\to\{1,\ldots,\ell_0\}$.

Finally, observe that for any $\widetilde{\Sigma}(a)\subset X\subset\widetilde{\Sigma}(b)$ we can similarly define $\tau_X,\mathscr{S}^+(X)$, f_X, and that $f_X=f_b^{\ell_X}$ for some $\ell_X:X\backslash\mathscr{S}^+(X)\to\{1,\ldots,\ell_0\}$ (the bound ℓ_0 is the same). Hence, controlling pre-iterates of f_X up to order n_0 say follows from controlling pre-iterates of f_b up to order $\ell_0 n_0$. In summary, we just need to analyze a bounded number of iterates of a single map. In the next subsection, we consider a multi-parameter family of such sections X and show that for some choice of parameters the section X satisfies the required properties.

Remark 5.1. We can also apply Step 3 above to extend $\Omega_+\cup\Omega_-$ to a Poincaré section $\widetilde{\Omega}$. The construction is simpler, since we only require transversality with the flow direction (and not necessarily almost perpendicularity). Hence the transition map f_0 can be extended to $f_0:\widetilde{\Omega}\backslash\mathscr{S}^+(\widetilde{\Omega})\to\widetilde{\Omega}$, where $\mathscr{S}^+(\widetilde{\Omega})$ is the union of the boundary of $\widetilde{\Omega}$ and the points that never return to $\widetilde{\Omega}$ under the flow. The transition time function Υ_0 can be extended accordingly. Similarly to the maps f_λ, the map f_0 is a Poincaré return map of g that captures all flow trajectories not asymptotic to γ. Hence f_λ and f_0 are conjugate, with transition time bounded from above.

5.2. **Construction of $\widehat{\Sigma}$.** Fix an integer n_0. Let $A\subset\widetilde{\Sigma}$ be a connected curve. The next lemma is essential to the construction of $\widehat{\Sigma}$, and shows how to perturb A to avoid triple intersections of pre-iterates under f_b up to order $\ell_0 n_0$. Since $\widetilde{\Sigma}$ is perpendicular to the flow, Lemma 2.4(2) implies that the stable/unstable subspaces $\widehat{E}^{s/u}$ for the flow project

[2] In general, the maps f_λ need not be related to the map f_0 introduced in Section 4.1.

to directions $E^{s/u}$ in $\widetilde{\Sigma}$. (For the moment, this is purely notational and no dynamical properties are claimed for $E^{s/u}$.)

Lemma 5.2. *Given a connected curve $A \subset \widetilde{\Sigma}$ transverse to E^s, E^u and $\varepsilon > 0$, there is a one-parameter family $\{A(t)\}_{|t|\leq 1}$ of disjoint curves, each of them ε–close to A in the C^1–norm, such that*

$$A(t) \cap f_b^{-i}[A(t)] \cap f_b^{-j}[A(t)] = \emptyset$$
$$A(t) \cap f_b^{-i}[A(t)] \cap f_b^{-j}[C] = \emptyset$$
$$C \cap f_b^{-i}[A(t)] \cap f_b^{-j}[A(t)] = \emptyset$$

for all $0 < i < j \leq \ell_0 n_0$ and $|t| \leq 1$.

To prove Lemma 5.2, we provide a combinatorial description of the trajectories of A, according to the transitions in the neck $\mathcal{N}$ that spend a long time. This combinatorial description (decomposition) and some notation (long backward transition and parameter $\eta > 0$) will be only used in this section.

Projection map to $\Omega_\pm$: The *projection map* to $\Omega_\pm$ is the map $\mathfrak{p} : g_{[-\chi,\chi]}(\Omega_+ \cup \Omega_-) \to \Omega_+ \cup \Omega_-$ defined by $\mathfrak{p}(g^t(x)) = x$ for $(x, t) \in (\Omega_+ \cup \Omega_-) \times [-\chi, \chi]$.

This map allows to localize our analysis inside $\Omega_\pm$. Recall the transition map f_0 studied in Section 4.2. Observe that if $B \subset \Omega_-$ is a curve intersecting $\Omega_-^=$, then $f_0^{-1}(B) \subset \Omega_+$ is the union of two disjoint curves of infinite length, each of them accumulating at $\Omega_+^=$, see Figure 5. The proof of this fact is easy. By symmetry, it is enough to prove the analogous

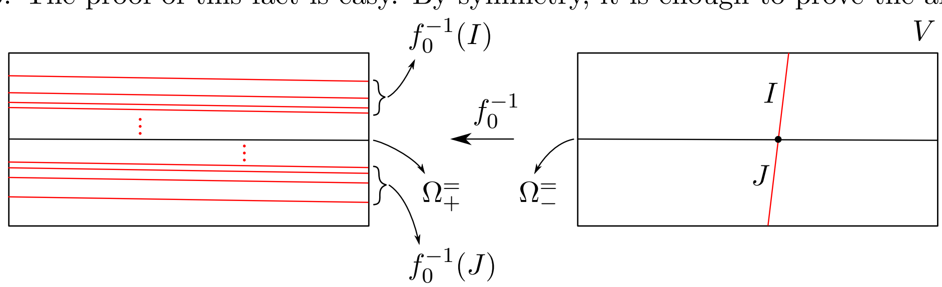

Figure 5. The pre-iterates of a curve intersecting $\Omega_-^=$ equals two curves of infinite length accumulating at $\Omega_+^=$.

result for the forward iterate of f_0, and we know for instance on Ω_1 that $\zeta(\psi) \to \infty$ as $\psi \to \psi_0^-$ and as $\psi \to \psi_0^+$ (and similarly on the other three parts of Ω_+ with ψ_0 replaced by $-\psi_0$ or $\pm(\pi - \psi_0)$ as appropriate).

The transition of Figure 5 is the only source of unboundedness when considering pre-iterates of f_b, since otherwise the return time of the flow is uniformly bounded. To better analyze this phenomenon, fix $\eta > 0$ small and define

$$U = \{x \in \Omega_- : d(x, \Omega_-^=) < \eta\}, \qquad V = \{x \in \Omega_- : d(x, \Omega_-^=) < 2\eta\}.$$

Long backward transition (LBT): The point $x \in \widetilde{\Sigma}$ has a *long backward transition (LBT)* at time i if $f_b^{-i}x \in g_{[-\chi,\chi]}V$ and $f_b^{-(i+1)}x \in g_{[-\chi,\chi]}\Omega_+$. When this happens, we say that the LBT occurs at the point $f_b^{-i}x$.

Decomposition of A: Given a curve $A \subset \widetilde{\Sigma}$ with finite length and finitely many connected components, write $A = A_0 \uplus \cdots \uplus A_{\ell_0 n_0}$ where

$$A_i = \{x \in A : x \text{ makes the first LBT at time } i\},\ 0 \le i < \ell_0 n_0$$
$$A_{\ell_0 n_0} = A \setminus (A_0 \uplus \cdots \uplus A_{\ell_0 n_0 - 1}).$$

Note that $A_{\ell_0 n_0}$ is the set of points with first LBT with time at least $\ell_0 n_0$, and includes points with no LBT. Each A_i, $i < \ell_0 n_0$, is the disjoint union of finitely many open pieces of A, and $A_{\ell_0 n_0}$ is the disjoint union of finitely many pieces of A.

Consider $C = g_{[-\chi,\chi]}\Omega_+^= \cap \widetilde{\Sigma}$, which is the disjoint union of finitely many pieces of curves asymptotic to γ under the map $\widetilde{h}_b$. Decomposing $C = C_0 \uplus C_1 \uplus \cdots \uplus C_{\ell_0 n_0 - 1}$ as above, the first LBT's associated to C occur at the set $\widetilde{C} = C_0 \uplus f_b^{-1}(C_1) \uplus \cdots \uplus f_b^{-\ell_0 n_0 + 1}(C_{\ell_0 n_0 - 1})$, equal to the disjoint union of finitely many curves asymptotic to γ under the map $\widetilde{h}_b$. Then

$$\mathfrak{p}[\widetilde{C}] = \mathfrak{p}[C_0] \uplus \mathfrak{p}[f_b^{-1}(C_1)] \uplus \cdots \uplus \mathfrak{p}[f_b^{-\ell_0 n_0 + 1}(C_{\ell_0 n_0 - 1})]$$

is the disjoint union of finitely many curves asymptotic to γ under the flow. Let $H = \mathfrak{p}[\widetilde{C}] \cap \Omega_-^=$, which is a finite set equal to all homoclinic intersections associated to first LBT's. We claim that all other LBT's accumulate in H. The proof is by induction on the number of LBT's. Assume that $I \subset C_i$ has the second LBT at time $j > i$. The set $\mathfrak{p}[f_b^{-(i+1)}(C_i)]$ accumulates at $\Omega_+^=$, and $f_b^{-j}(C_i)$ is obtained from $f_b^{-(i+1)}(C_i)$ by a uniformly bounded flow time, hence $\mathfrak{p}[f_b^{-j}(C_i)]$ accumulates on H. The claim follows.

For $\eta > 0$ small enough, all first LBT's of C are associated to a point of H, i.e. the piece $\mathfrak{p}[f_b^{-i}(C_i)]$ not only intersects V but indeed crosses $\Omega_-^=$. See Figure 6 to understand the dynamics of accumulations around $\Omega_-^=$. The red/green intervals in the left figure are pieces of $\mathfrak{p}[C_i]/\mathfrak{p}[C_k]$, and the vertical red/green curves in the right figure are $\mathfrak{p}[f_b^{-i}(C_i)]/\mathfrak{p}[f_b^{-k}(C_k)]$, equal to the pre-iterates right before the first LBT's. In the figure, they define four homoclinic points. We also depict an interval $\mathfrak{p}(I)$ that makes two LBT's. The first LBT generates the two curves of infinite length in the left figure, both accumulating at $\Omega_-^=$. Finally, the vertical blue curves in the right figure are $\mathfrak{p}[f_b^{-j}(I)]$, equal to the pre-iterates right before the second LBT of I. Observe that they accumulate on each of the four homoclinic points.

Now we are able to prove Lemma 5.2.

Proof of Lemma 5.2. To obtain a one-parameter family, it is enough to perturb A so that the intersection conditions of the statement hold robustly. We have

$$\{f_b^{-k}(A) : 0 \le k \le \ell_0 n_0\} = \underbrace{\left\{f_b^{-k}(A_i) : 0 \le k \le i\right\}}_{\mathscr{F}_A} \cup \underbrace{\left\{f_b^{-k}(A_i) : i < k \le \ell_0 n_0\right\}}_{\mathscr{G}_A}$$
$$\{f_b^{-k}(C) : 0 \le k \le \ell_0 n_0\} = \underbrace{\left\{f_b^{-k}(C_i) : 0 \le k \le i\right\}}_{\mathscr{F}_C} \cup \underbrace{\left\{f_b^{-k}(C_i) : i < k \le \ell_0 n_0\right\}}_{\mathscr{G}_C}.$$

Let $\mathscr{F} = \mathscr{F}_A \cup \mathscr{F}_C$ and $\mathscr{G} = \mathscr{G}_A \cup \mathscr{G}_C$. Since C is asymptotic to γ under the flow, $\mathscr{F}_C \cup \mathscr{G}_C$ is fixed (does not depend on A) and has no double intersections. Observe that $\mathscr{F}$ is a finite

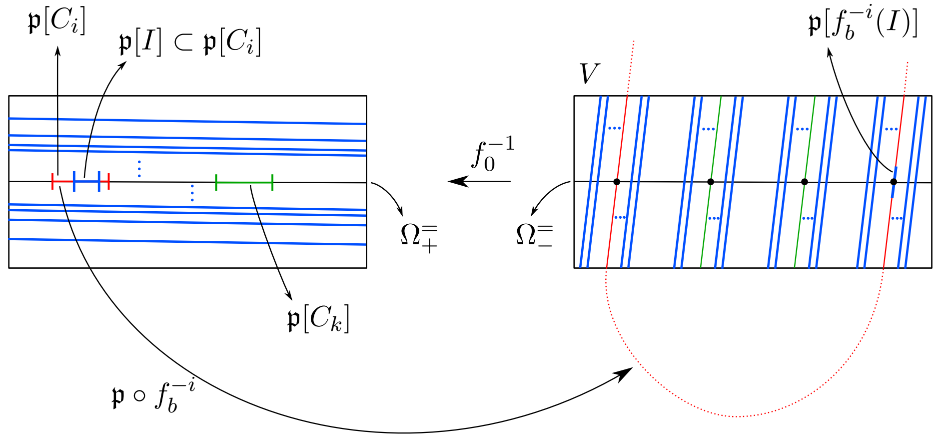

FIGURE 6. Dynamics near homoclinic points.

family of bounded curves, obtained from pieces of A and C by uniformly bounded flow time displacements. Let T_0 be a bound on such time.

We start by controlling all possible triple intersections of A and pre-iterates of A, C. There are three possible types of such intersections:

Type 1: $A \cap F_1 \cap F_2$, where $F_1, F_2 \in \mathscr{F}$.
Type 2: $A \cap F \cap G$, where $(F, G) \in \mathscr{F} \times \mathscr{G}$.
Type 3: $A \cap G_1 \cap G_2$, where $G_1, G_2 \in \mathscr{G}$.

For each type, we perform finitely many C^1 perturbations on A to prevent triple intersections robustly.

Type 1: We can assume that $F_1 \in \mathscr{F}_A$. If $F_2 \in \mathscr{F}_A$, then the intersection $A \cap F_1 \cap F_2$ is associated to flow displacements of time $\leq T_0$. By (2.4), the flow has finitely many closed orbits of length $\leq T_0$. Hence we can perform an arbitrarily small C^1 perturbation of A to destroy these intersections robustly. If $F_2 \in \mathscr{F}_C$, the same applies to guarantee that $A \cap F_1$ does not intersect F_2 robustly.

Type 2: The set $A \cap \mathscr{F} = \bigcup_{F \in \mathscr{F}} \{A \cap F\}$ is finite, and the set $A \cap \mathscr{G} = \bigcup_{G \in \mathscr{G}} \{A \cap G\}$ is countable with a finite set of accumulation points, coming from pre-iterates of $\Omega_+^=$. A C^1 perturbation of order $O(\varepsilon)$ of A changes $A \cap \mathscr{F}$ by $O(\varepsilon)$ inside A, and $A \cap \mathscr{G}$ around its accumulation points by $o(\varepsilon)$ inside A, hence we can destroy all such triple intersections robustly.

Type 3: We divide this type into three subtypes.

- Type 3.1: $A \cap f_b^{-k}(A_i) \cap f_b^{-m}(A_j)$, with $k > i$ and $m > j$.
- Type 3.2: $A \cap f_b^{-k}(A_i) \cap f_b^{-m}(C_j)$, with $k > i$, $m > j$ and $k - i \leq m - j$.
- Type 3.3: $A \cap f_b^{-k}(C_i) \cap f_b^{-m}(A_j)$, with $k > i$, $m > j$ and $k - i \leq m - j$.

The idea is to push this intersection to V. The hardest case is Type 3.1, where all three sets are simultaneously perturbed. Let us start with it. Assuming that $k - i \leq m - j$, iterate the intersection $k - i$ times (the intersection belongs to $f_b^{-m}(A_j)$ and hence can actually be iterated m times), so that $f_b^{-i}(A_i) \cap f_b^{-(m-k+i)}(A_j) \neq \emptyset$. We show that a small C^1

perturbation of A_i makes this intersection empty inside U. This is enough for us, since intersections outside U are associated to uniformly bounded flow times, which can be treated as in Type 1. Actually, we show how to guarantee that $\mathfrak{p}[f_b^{-i}(A_i)] \cap \mathfrak{p}[f_b^{-(m-k+i)}(A_j)] \cap U = \emptyset$. The argument is similar to Type 2. Since the set $\Omega_-^= \cap \mathfrak{p}[f_b^{-i}(A_i)]$ is finite and all accumulation points of $\Omega_-^= \cap \mathfrak{p}[f_b^{-(m-k+i)}(A_j)]$ are contained in H, a C^1 perturbation of order $O(\varepsilon)$ of the piece $B_i \subset A_i$ such that $\mathfrak{p}[f_b^{-i}(B_i)] = \mathfrak{p}[f_b^{-i}(A_i)] \cap U$ changes $\mathfrak{p}[f_b^{-i}(A_i)]$ by $O(\varepsilon)$, and $\Omega_-^= \cap \mathfrak{p}[f_b^{-(m-k+i)}(A_j)]$ around its accumulation points by $o(\varepsilon)$. Therefore a small perturbation guarantees that $\mathfrak{p}[f_b^{-i}(A_i)] \cap \mathfrak{p}[f_b^{-(m-k+i)}(A_j)] \cap U = \emptyset$.

Types 3.2 and 3.3 are simpler, since C is not perturbed. In Type 3.2, we perform the same argument described above to have $\mathfrak{p}[f_b^{-i}(A_i)] \cap \mathfrak{p}[f_b^{-(m-k+i)}(C_j)] \cap U = \emptyset$. Again, $\Omega_-^= \cap \mathfrak{p}[f_b^{-i}(A_i)]$ is finite and all accumulation points of $\Omega_-^= \cap \mathfrak{p}[f_b^{-(m-k+i)}(C_j)]$ are contained in H. A C^1 perturbation of order $O(\varepsilon)$ of the piece $B_i \subset A_i$ such that $\mathfrak{p}[f_b^{-i}(B_i)] = \mathfrak{p}[f_b^{-i}(A_i)] \cap U$ changes $\mathfrak{p}[f_b^{-i}(A_i)]$ by $O(\varepsilon)$, while $\Omega_-^= \cap \mathfrak{p}[f_b^{-(m-k+i)}(C_j)]$ remains fixed, so we can make $\mathfrak{p}[f_b^{-i}(A_i)] \cap \mathfrak{p}[f_b^{-(m-k+i)}(C_j)] \cap U = \emptyset$.

In Type 3.3, once again $\Omega_-^= \cap \mathfrak{p}[f_b^{-i}(C_i)]$ is finite (and contained in H) and all accumulation points of $\Omega_-^= \cap \mathfrak{p}[f_b^{-(m-k+i)}(A_j)]$ are contained in H. A C^1 perturbation of order $O(\varepsilon)$ of the piece $B_j \subset A_j$ such that $\mathfrak{p}[f_b^{-(m-k+i)}(B_j)] = \mathfrak{p}[f_b^{-(m-k+i)}(A_j)] \cap U$ changes $\mathfrak{p}[f_b^{-(m-k+i)}(A_j)]$, while $\Omega_-^= \cap \mathfrak{p}[f_b^{-i}(C_i)]$ remains fixed, so we can make $\mathfrak{p}[f_b^{-i}(C_i)] \cap \mathfrak{p}[f_b^{-(m-k+i)}(A_j)] \cap U = \emptyset$.

To finish the lemma, we deal with triple intersections of C and pre-iterates of A. There are also three types of such intersections:

Type 1': $C \cap F_1 \cap F_2$, where $F_1, F_2 \in \mathscr{F}_A$.
Type 2': $C \cap F \cap G$, where $(F, G) \in \mathscr{F}_A \times \mathscr{G}_A$.
Type 3': $C \cap G_1 \cap G_2$, where $G_1, G_2 \in \mathscr{G}_A$.

The analysis of these types is simpler than the previous ones, since C is fixed.

Type 1': Proceed as in Type 1.

Type 2': Proceed as in Type 2.

Type 3': Proceed as in Type 3.1, guaranteeing that, in its notation, $\mathfrak{p}[f_b^{-i}(A_i)] \cap \mathfrak{p}[f_b^{-(m-k+i)}(A_j)] \cap U \neq \emptyset$. □

Using Lemma 5.2, we now construct a parametric family of sections.

Step 4 (Construction of a parametric family $\widehat{\Sigma}(\vec{t})$ of sections): For each $1 \leq i \leq m+n$, choose finitely many families $\{A_{i,j}(t)\}_{|t| \leq 1}$, $1 \leq j \leq N_i$. Given $\vec{t} = (t_{i,j})_{\substack{1 \leq i \leq m+n \\ 1 \leq j \leq N_i}}$, we require that $B_i(\vec{t_i}) = B_i(t_{i,1}, \ldots, t_{i,N_i})$ is a topological disc whose boundary is the polygon defined by $\{A_{i,j}(t_{i,j})\}_{1 \leq j \leq N_i}$ and such that $D_a(x_i) \subset B_i(\vec{t_i}) \subset D_b(x_i)$ for $i = 1, \ldots, m+n$. See Figure 7. We require that

$$\widehat{\Sigma}(\vec{t}) = \biguplus_{1 \leq i \leq m+n} B_i(\vec{t_i})$$

defines a cross-section to the flow satisfying the following conditions:

(H1) $\{A_{i,j}(t)\}_{|t|\leq 1}$ satisfies Lemma 5.2 for every $1\leq i\leq m+n$, $1\leq j\leq N_i$.

(H2) $A_{i_1,j_1}(t)\pitchfork f_b^{-k}(A_{i_2,j_2}(t'))$ for all distinct pairs $(i_1,j_1),(i_2,j_2)\in\{(i,j):1\leq i\leq m+n, 1\leq j\leq N_i\}$, $|t|,|t'|\leq 1$ and $0\leq k\leq \ell_0 n_0$. In particular, $A_{i_1,j_1}(t)\cap f_b^{-k}(A_{i_2,j_2}(t'))$ is at most countable.

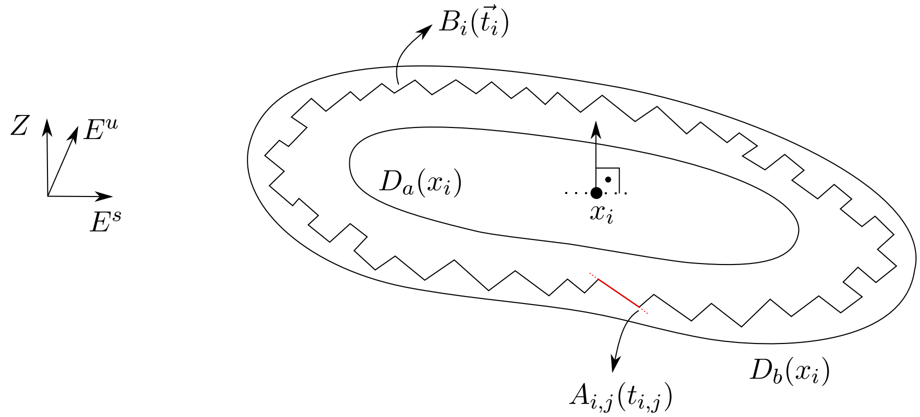

FIGURE 7. Construction of $B_i(\vec{t_i})$.

Here is how we guarantee (H2). Since $\{A_{i,j}(t)\}_{|t|\leq 1}$ satisfies Lemma 5.2, it is transverse to E^s,E^u. The pre-iterates of pieces of curves that make no LBT are associated to bounded flow times, hence can be perturbed to satisfy (H2). Now, if A makes its first LBT at time i, then up to a compact component the infinite curves composing $f_b^{-(i+1)}(A)$ belong to the stable cone, and so are transverse to every $A_{i,j}(t)$ after a small perturbation. The transversality implies that each compact component of $f_b^{-k}(A_{i_2,j_2}(t'))$ intersects $A_{i_1,j_1}(t)$ in finitely many points, therefore the intersection $A_{i_1,j_1}(t)\cap f_b^{-k}(A_{i_2,j_2}(t'))$ is at most countable.

To finish the construction of $\widehat{\Sigma}$, we show that the space of parameters $\vec{t}$ such that $\mathscr{S}^+(\widehat{\Sigma}(\vec{t}))$ has a triple intersection up to the $\ell_0 n_0$'th pre-iterate of f_b has zero Lebesgue measure. For that, we analyze all possible triple intersections. Recall that $\mathscr{S}^+(\widehat{\Sigma}(\vec{t}))=\partial\widehat{\Sigma}(\vec{t})\cup\{\tau_{\widehat{\Sigma}(\vec{t})}=\infty\}$. Since $\{\tau_{\widehat{\Sigma}(\vec{t})}=\infty\}$ is asymptotic to γ, there are not even double intersections associated to it. We have six remaining possibilities for triple intersections:

- $A_{i,j}(t_{i,j})\cap f_b^{-k}[A_{i,j}(t_{i,j})]\cap f_b^{-p}[A_{i,j}(t_{i,j})]$: this intersection is empty, by (H1).
- $A_{i_1,j_1}(t_{i_1,j_1})\cap f_b^{-k}[A_{i_2,j_2}(t_{i_2,j_2})]\cap f_b^{-p}[A_{i_3,j_3}(t_{i_3,j_3})]$ with $(i_3,j_3)\neq(i_1,j_1)$ or $(i_3,j_3)\neq(i_2,j_2)$: by (H2), $A_{i_1,j_1}(t_{i_1,j_1})\cap f_b^{-k}[A_{i_2,j_2}(t_{i_2,j_2})]$ is at most countable. If we fix all parameters except t_{i_3,j_3}, there are at most countably many choices for t_{i_3,j_3} such that the triple intersection is non-empty.
- $A_{i,j}(t_{i,j})\cap f_b^{-k}[A_{i,j}(t_{i,j})]\cap f_b^{-p}[\{\tau_{\widehat{\Sigma}(\vec{t})}=\infty\}]$: this intersection is empty, by (H1).
- $A_{i_1,j_1}(t_{i_1,j_1})\cap f_b^{-k}[A_{i_2,j_2}(t_{i_2,j_2})]\cap f_b^{-p}[\{\tau_{\widehat{\Sigma}(\vec{t})}=\infty\}]$ with $(i_1,j_1)\neq(i_2,j_2)$: the intersection $A_{i_1,j_1}(t_{i_1,j_1})\cap f_b^{-p}[\{\tau_{\widehat{\Sigma}(\vec{t})}=\infty\}]$ is at most countable, since every compact component of $f_b^{-p}[\{\tau_{\widehat{\Sigma}(\vec{t})}=\infty\}]$ is transverse to $A_{i_1,j_1}(t_{i_1,j_1})$ and hence intersects it in finitely many points. Thus, if we fix all parameters except t_{i_2,j_2}, there are at most countably many choices for t_{i_2,j_2} such that the triple intersection is non-empty.

- $\{\tau_{\widehat{\Sigma}(\vec{t})}=\infty\}\cap f_b^{-k}[A_{i,j}(t_{i,j})]\cap f_b^{-p}[A_{i,j}(t_{i,j})]$: this intersection is empty, by (H1).
- $\{\tau_{\widehat{\Sigma}(\vec{t})}=\infty\}\cap f_b^{-k}[A_{i_2,j_2}(t_{i_2,j_2})]\cap f_b^{-p}[A_{i_3,j_3}(t_{i_3,j_3})]$ with $(i_2,j_2)\neq(i_3,j_3)$: this case is similar to the fourth one, since again $\{\tau_{\widehat{\Sigma}(\vec{t})}=\infty\}\cap f_b^{-k}[A_{i_2,j_2}(t_{i_2,j_2})]$ is at most countable.

THE SECTION $\widehat{\Sigma}$: Define $\widehat{\Sigma}=\widehat{\Sigma}(\vec{t})$, where $\vec{t}$ is any parameter such that $\mathscr{S}^+(\widehat{\Sigma}(\vec{t}))$ has no triple intersections up to the $\ell_0 n_0$'th pre-iterate under f_b.

5.3. **Construction of Σ_0.** The final step in the construction, which leads to the section Σ_0, is to make small flow displacements in $\widehat{\Sigma}$ so that the Poincaré return time of Σ_0 is at least $T_\chi>0$ and Σ_0 is almost perpendicular to the flow direction. We measure the perpendicularity using a new parameter $\epsilon\ll\chi$. For each $1\leq i\leq m+n$, let $\mathfrak{p}_i: g_{[-\chi,\chi]}D_b(x_i)\to D_b(x_i)$ be the flow projection. Write $\widehat{\Sigma}=\bigcup_{1\leq i\leq m+n}B_i$, and let $t_{\min}$ be the minimal flow time defined by f_b.

STEP 5 (REFINEMENT OF $\widehat{\Sigma}$): For each $1\leq i\leq m+n$, construct a family $\mathcal{Q}_i=\{Q\}$ such that:

- each $Q\in\mathcal{Q}_i$ is contained in a su–disc of radius $<\epsilon$ and
$$B_i\subset\bigcup_{Q\in\mathcal{Q}_i}g_{[-\frac{1}{3}t_{\min},\frac{1}{3}t_{\min}]}Q;$$
- $Q\cap g_{[-\frac{1}{100}t_{\min},\frac{1}{100}t_{\min}]}Q'=\emptyset$ for all distinct $Q,Q'\in\mathcal{Q}_i$.
- $\mathfrak{p}_i[\partial Q]$ is transverse to E^s,E^u for all $Q\in\mathcal{Q}_i$;
- there are no triple intersections between $\mathscr{S}^+(\widehat{\Sigma})$ and $\bigcup_{\substack{1\leq i\leq m+n\\ Q\in\mathcal{Q}_i}}\mathfrak{p}_i[\partial Q]$ up to the $\ell_0 n_0$'th pre-iterate under f_b.

Observe that the smaller ϵ is, the more perpendicular is Q to the flow. To create $\mathcal{Q}_i$, first consider a refinement $\widehat{\mathscr{R}_i}=\{\widehat{R}\}$ of B_i by finitely many compact curves transverse to E^s,E^u, see Figure 8. Proceeding similarly to the proof of [LS19, Lemma 2.7], displace each $\widehat{R}$ in the flow direction to obtain R, so that $\mathscr{R}_i=\{R\}$ is a disjoint family and the displacements of neighbor $\widehat{R}$'s differ at least $t_{\min}/50$. For each $R\in\mathscr{R}_i$, choose $y_R\in R$ and apply Step 4 to construct $\overline{R}\subset Q\subset D_{\mathrm{diam}(R)}(y_R)$ satisfying the above conditions, where $\overline{R}$ is the flow projection of R to $D_{\mathrm{diam}(R)}(y_R)$.

THE SECTION Σ_0: Define $\Sigma_0=\bigcup_{\substack{1\leq i\leq m+n\\ Q\in\mathcal{Q}_i}}Q$.

By the first two conditions in Step 5, the corresponding Poincaré return time of Σ_0 is bounded below by a constant $T_\chi>0$ which is independent of ϵ.

5.4. **The first return map f.** We now define a first return map f which will eventually be shown to satisfy the Chernov axioms. Recall that, by construction, all flow trajectories intersect Σ_0 infinitely often except those forward and backward asymptotic to γ.

RETURN TIME FUNCTION: The *return time function* of Σ_0 is $\tau=\tau_+:\Sigma_0\to(0,\infty]$ such that $\tau(x)=\inf\{t>0:g_t(x)\in\Sigma_0\}$. Define also $\tau_-:\Sigma_0\to[-\infty,0)$ by $\tau_-(x)=\sup\{t<0:g_t(x)\in\Sigma_0\}$.

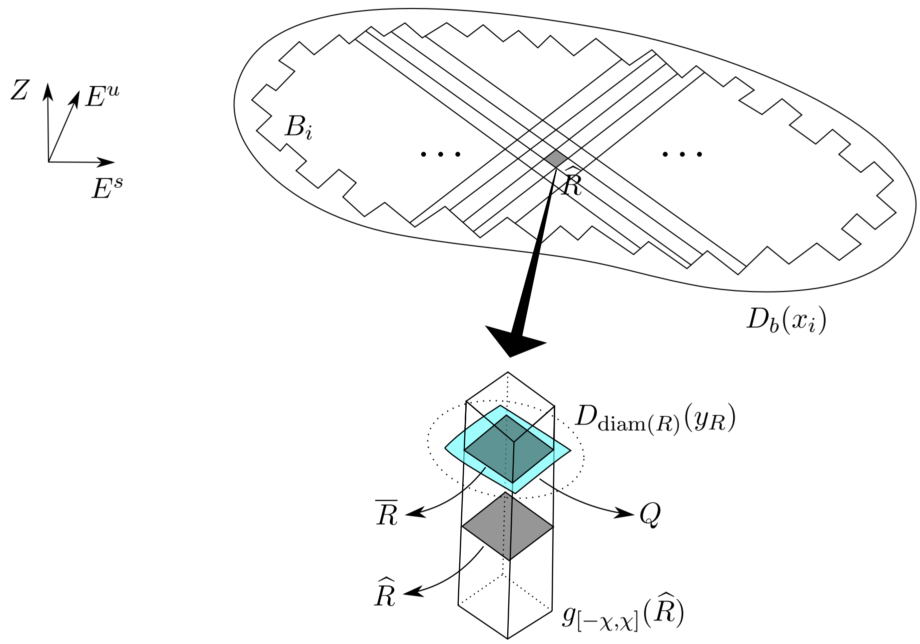

FIGURE 8. Construction of Σ_0: refine $\widehat{\Sigma}$ displacing each component to improve perpendicularity and preserving no triple intersections.

We have $\tau(x) = \infty$ if and only if x is an asymptotic vector, hence $\{\tau = \infty\}$ is a finite union of compact curves, each of them contained in $g_{[-\chi,\chi]}\Omega_{+}^{=}$.

PRIMARY SINGULAR SETS $\mathscr{S}_{\mathrm{P}}^{\pm}$: The *primary singular set* $\mathscr{S}_{\mathrm{P}} = \mathscr{S}_{\mathrm{P}}^{+}$ is defined as

$$\mathscr{S}_{\mathrm{P}} = \{x \in \Sigma_0 : \tau(x) < \infty \text{ and } g_{\tau(x)}(x) \in \partial\Sigma_0\} \cup \{\tau = \infty\}.$$

Similarly, the *primary singular set* $\mathscr{S}_{\mathrm{P}}^{-}$ is defined as

$$\mathscr{S}_{\mathrm{P}}^{-} = \{x \in \Sigma_0 : \tau_{-}(x) > -\infty \text{ and } g_{\tau_{-}(x)}(x) \in \partial\Sigma_0\} \cup \{\tau_{-} = -\infty\}.$$

Note that $\mathscr{S}_{\mathrm{P}}^{\pm}$ are closed sets. Proceeding similarly to Section 4.3, we partition a neighborhood of $\{\tau = \infty\}$ into homogeneity bands. For that, fix a sufficiently large integer n_0 (how large n_0 is will depend on a finite number of conditions, which include the validity of Lemma 5.9 and the estimates in Section 6).

HOMOGENEITY BANDS ON Σ_0: For each $n \geq n_0$, the *homogeneity band* with index n is $\mathscr{D}_n = \mathscr{D}_n^{>} \cup \mathscr{D}_n^{<}$ where

$$\mathscr{D}_n^{>} = \left\{x \in \mathrm{int}(\Sigma_0) \cap g_{[-\chi,\chi]}\Omega_{+} : 1 + \tfrac{1}{(n+1)^2} < |c(x)| < 1 + \tfrac{1}{n^2}\right\}$$
$$\mathscr{D}_n^{<} = \left\{x \in \mathrm{int}(\Sigma_0) \cap g_{[-\chi,\chi]}\Omega_{+} : 1 - \tfrac{1}{n^2} < |c(x)| < 1 - \tfrac{1}{(n+1)^2}\right\}.$$

SECONDARY SINGULAR SETS $\mathscr{S}_{\mathrm{S}}^{\pm}$: The *secondary singular set* $\mathscr{S}_{\mathrm{S}} = \mathscr{S}_{\mathrm{S}}^{+}$ is

$$\mathscr{S}_{\mathrm{S}} = \bigcup_{n \geq n_0} \partial\mathscr{D}_n.$$

The *secondary singular set* $\mathscr{S}_{\rm S}^-$ is

$$\mathscr{S}_{\rm S}^- = \{g_{\tau(x)}(x) : x \in \mathscr{S}_{\rm S}\}.$$

Let $X_0 = {\rm int}(\Sigma_0)$, $\mathscr{S}^+ = \mathscr{S}_{\rm P} \cup \mathscr{S}_{\rm S}$, and $\mathscr{S}^- = \mathscr{S}_{\rm P}^- \cup \mathscr{S}_{\rm S}^-$.

First return map f: Define the map $f : X_0 \backslash \mathscr{S}^+ \to X_0 \backslash \mathscr{S}^-$ to be the first return map of the flow to Σ_0, i.e. $f(x) = g_{\tau(x)}(x)$.

The map f has the same regularity of g_t, hence it is C^2. It has uniformly bounded derivatives away from $\{\tau = \infty\}$. Since Σ_0 is almost perpendicular to the flow direction, the hyperbolicity properties of f away from $\{\tau = \infty\}$ are almost the same as those of the flow. We lose control as we approach $\{\tau = \infty\}$, and the homogeneity bands $\mathscr{D}_n$ enable us to recover this control. Since Σ_0 is obtained from small flow displacements of $\widehat{\Sigma} \subset \widetilde{\Sigma}$, the first return map f is a small perturbation (in the flow direction) of (a restriction of) f_b and, for trajectories near asymptotic vectors, f_b is a small perturbation of f_0. Hence, we can understand f inside homogeneity bands by studying f_0 inside homogeneity bands. Let us be more specific in the relation between f and f_0. Since we are interested in the transitions in the neck, define

$$\Sigma_+ = \left\{x \in \Sigma_0 \cap g_{[-\chi,\chi]}\Omega_+ : ||c(x)| - 1| < \tfrac{1}{n_0^2} \text{ and } fx \in g_{[-\chi,\chi]}\Omega_-\right\}$$
$$\Sigma_- = f(\Sigma_+) = \left\{x \in \Sigma_0 \cap g_{[-\chi,\chi]}\Omega_- : ||c(x)| - 1| < \tfrac{1}{n_0^2} \text{ and } f^{-1}x \in g_{[-\chi,\chi]}\Omega_+\right\}$$

It is clear that $f \restriction_{\Sigma_+} : \Sigma_+ \to \Sigma_-$.

Coordinate maps $\mathfrak{p}_\pm$ and $\mathfrak{t}_\pm$: The *coordinate maps* $\mathfrak{p}_+ : \Sigma_+ \to \Omega_+$ and $\mathfrak{t}_+ : \Sigma_+ \to [-\chi,\chi]$ are defined by the equality $z = g_{\mathfrak{t}_+(z)}[\mathfrak{p}_+(z)]$ for $z \in \Sigma_+$. The *coordinate maps* $\mathfrak{p}_- : \Sigma_- \to \Omega_-$ and $\mathfrak{t}_- : \Sigma_- \to [-\chi,\chi]$ are defined analogously.

The coordinate maps have the same regularity of g_t, hence they are C^2. Since Σ_0 and Ω are uniformly transversal to the flow direction, we have that $\|d\mathfrak{p}_\pm^{\pm1}\| \approx 1$. By the first inclusion of (5.1), $\mathfrak{p}_+$ is surjective. It is also injective, because if $x \in \Sigma_+$ then x, fx are uniquely characterized as being the starting/ending point of the transition in the neck. Therefore, $\mathfrak{p}_+$ is a bijection. By symmetry, the same holds for $\mathfrak{p}_-$. Recall the definition of $\mathscr{C}_n$ in Section 4.3. We note that:

- $\circ$ $\mathscr{C}_n = \mathfrak{p}_+(\mathscr{D}_n)$ for all $n \geq n_0$.
- $\circ$ $f = \mathfrak{p}_-^{-1} \circ f_0 \circ \mathfrak{p}_+$ in Σ_+.

Since $E^{s,u}$ are defined in $\Sigma_\pm, \Omega_\pm$ as the projections of $\widehat{E}^{s,u}$ onto the respective tangent spaces, the maps $\mathfrak{p}_\pm$ preserves these subspaces. Using that $\|d\mathfrak{p}_\pm^{\pm1}\| \approx 1$, we obtain that $\|df \restriction_{E_x^{s,u}}\| \approx \|df_0 \restriction_{E^{s,u}_{\mathfrak{p}_+(x)}}\|$ for $x \in \Sigma_+$. In the next two subsections, we will estimate $\|df_0 \restriction_{E^{s,u}}\|$ inside homogeneity bands and some related bounds.

5.5. **Excursion times in the neck.** In the notation of Section 4.2, let $\mathbf{x} = \mathbf{x}(t)$ be a bouncing/crossing geodesic undergoing an excursion in the neck. Recall from Section 4.2 that Υ_0 (more precisely $2\Upsilon_0$) is the transition time from Ω_+ to Ω_-. Similarly, we define Υ

to be the time taken to pass from Σ_+ to Σ_-. Since $f = \mathfrak{p}_-^{-1} \circ f_0 \circ \mathfrak{p}_+$ on Σ_+, we have the relation

$$\Upsilon(\mathbf{x}) = -\mathfrak{t}_+(x) + 2\Upsilon_0(\mathbf{x}) + \mathfrak{t}_-(f(x)), \tag{5.2}$$

where x is the starting point of $\mathbf{x}$ in Σ^+. Since $|\mathfrak{t}_\pm| \le \chi$, Lemma 4.4 implies the following estimate.

Lemma 5.3. *If $\mathbf{x}$ is a geodesic with entry vector in $\mathscr{D}_n$ then $\Upsilon(\mathbf{x}) \approx n^{\frac{r-2}{r}}$.*

Next, we estimate the tail of Υ. Let Leb denote the Lebesgue measure on Σ_+ in Clairaut coordinates.

Lemma 5.4. $\mathrm{Leb}[\{x \in \Sigma_+ : \Upsilon(\mathbf{x}) > n\}] \approx n^{-\frac{2r}{r-2}}$.

Proof. The push-forward of Leb under $\mathfrak{p}_+$ is equivalent to the Lebesgue measure of Ω_+, hence we just need to estimate the Lebesgue measure of $\mathfrak{p}_+\{x \in \Sigma_+ : \Upsilon(\mathbf{x}) > n\}$. Since $|\mathfrak{t}_\pm| \le \chi$, equation (5.2) gives that

$$\mathrm{Leb}\left[\mathfrak{p}_+\{x \in \Sigma_+ : \Upsilon(\mathbf{x}) > n\}\right] \approx \mathrm{Leb}\left[\{x \in \Omega_+ : 2\Upsilon_0(\mathbf{x}) > n\}\right].$$

By Lemma 4.5, $\mathrm{Leb}[\mathscr{C}_n^<] \approx n^{-3}$ and $\mathrm{Leb}[\mathscr{C}_n^>] \approx n^{-3}$, therefore $\mathrm{Leb}[\mathscr{C}_n] \approx n^{-3}$. Letting $\mathscr{U}_k = \bigcup_{\ell > k} \mathscr{C}_\ell$, it follows that

$$\mathrm{Leb}[\mathscr{U}_k] \approx \sum_{\ell > k} \ell^{-3} \approx k^{-2}.$$

By Lemma 4.4, there exists $C > 1$ independent of k such that $C^{-1}k^{\frac{r-2}{r}} \le 2\Upsilon_0(\mathbf{x}) \le Ck^{\frac{r-2}{r}}$ for every geodesic $\mathbf{x}$ with entry vector in $\mathscr{C}_k$. For such k, we have the following:

- If $k > (Cn)^{\frac{r}{r-2}}$ then $2\Upsilon_0(\mathbf{x}) \ge C^{-1}k^{\frac{r-2}{r}} > n$.
- If $k \le \left(C^{-1}n\right)^{\frac{r}{r-2}}$ then $2\Upsilon_0(\mathbf{x}) \le Ck^{\frac{r-2}{r}} \le n$.

This implies the inclusions

$$\mathscr{U}_{(Cn)^{\frac{r}{r-2}}} \subset \{x \in \Omega_+ : 2\Upsilon_0(\mathbf{x}) > n\} \subset \mathscr{U}_{(C^{-1}n)^{\frac{r}{r-2}}}$$

and, since $\mathrm{Leb}\left[\mathscr{U}_{(C^{\pm1}n)^{\frac{r}{r-2}}}\right] \approx \left(n^{\frac{r}{r-2}}\right)^{-2} = n^{-\frac{2r}{r-2}}$, the proof is complete. □

Remark 5.5. From Remark 5.1, f is conjugate to the extended transition map f_0, and Υ is cohomologous to the extended function $2\Upsilon_0$. More specifically, if $h : \Sigma_0 \to \widetilde{\Omega}$ is the conjugacy with $h \circ f = f_0 \circ h$ then $\Upsilon - 2\Upsilon_0 \circ h$ is a coboundary for f.

5.6. **Hyperbolicity properties of f on Σ_+.** We now establish some hyperbolicity properties of $f \restriction_{\Sigma_+}$. Our reference metric is the δ–Sasaki metric $\|\cdot\| = \|\cdot\|_{\delta-\mathrm{Sas}}$ for a small $\delta > 0$. Recall that this metric is equivalent to the Sasaki metric $\|\cdot\|_{\mathrm{Sas}}$ and also to the Clairaut metric $\|\cdot\|_{\mathrm{C}}$. Among the Chernov axioms, the only one that requires a precise multiplicative constant is (A2.5). Recall that $\widehat{E}^{s/u}$ are the stable/unstable subspaces for g_t, which project to directions $E^{s/u}$ on Σ_0 and Ω. Let P denote such projection. Since Σ_0, Ω are nowhere perpendicular to $\widehat{E}^{s,u}$, we have $\|P^{\pm1}\| \approx 1$.

Lemma 5.6. *Lebesgue almost every $x \in \Sigma_0$ has an LSM/LUM $W_x^{s/u}$ for f.*

Proof. We prove the statement for LSM (the argument for LUM is the same, by time reversion). Since every $x \in \Sigma_0$ has invariant directions $E^{s/u}$, it is enough to show that there is no fast convergence of trajectories to $\mathscr{S}^+$, i.e. that $\frac{1}{n}\log d(f^n x, \mathscr{S}^+) \not\to 0$ for a.e. $x \in \Sigma_0$. To see this, let $\alpha > 0$ and consider the set

$$A_{\alpha,n} = \{x \in \Sigma_0 : d(f^n x, \mathscr{S}^+) < e^{-\alpha n}\}.$$

By the Borel-Cantelli lemma, it is enough to show that $\sum\limits_{n\geq 1} \mathrm{Leb}[A_{\alpha,n}] < \infty$ for every $\alpha > 0$, in which case $\limsup A_{\alpha,n}$ has zero Lebesgue measure and so does the set $\{x \in \Sigma_0 : \frac{1}{n}\log d(f^n x, \mathscr{S}^+) \to 0\}$. We have

$$A_{\alpha,n} = \{x \in \Sigma_0 : d(f^n x, \mathscr{S}_{\mathrm{P}}) < e^{-\alpha n}\} \cup \{x \in \Sigma_0 : d(f^n x, \mathscr{S}_{\mathrm{S}}) < e^{-\alpha n}\}.$$

If B is the first set in the above union, then $f^n(B)$ is covered using finitely many sets of measure $\approx e^{-\alpha n}$, one for each curve of $\mathscr{S}_{\mathrm{P}}$. Hence $\mathrm{Leb}(f^n(B)) \ll e^{-\alpha n}$ and so, by f–invariance, we have $\mathrm{Leb}(B) \ll e^{-\alpha n}$. If C is the second set in the above union, then $f^n(C)$ is covered using $\lceil e^{\alpha n/2} \rceil$ sets of measure $\approx e^{-\alpha n}$, namely one to cover all of $\bigcup_{k \geq e^{\alpha n/2}} \mathscr{D}_k$ and the others to cover $\mathscr{D}_{n_0}, \ldots, \mathscr{D}_{\lceil e^{\alpha n/2} \rceil - 1}$. Again by f–invariance, we get that $\mathrm{Leb}(C) \ll e^{\alpha n/2} \cdot e^{-\alpha n} = e^{-\alpha n/2}$. Hence

$$\mathrm{Leb}(A_{\alpha,n}) \ll e^{-\alpha n} + e^{-\alpha n/2} \ll e^{-\alpha n/2}.$$

The proof is complete. □

By Lemma 2.4(3), there are continuous functions $x \in \Omega_+ \mapsto e^s_x, e^u_x$ such that $e^{s/u}_x \in T_x\Omega_+$ is a unit vector spanning $E^{s/u}_x$. Since our analysis is local, we write $f_0(x) = f_0(\theta, \psi) = (\theta \pm \zeta(\psi), \pm\psi)$ omitting the entry $\pm\varepsilon_0$. We focus on the case $f_0(x) = (\theta + \zeta(\psi), \pm\psi)$ since the other sign is treated the same way. In this notation,

$$(df_0)_x = (df_0)_{(\theta,\psi)} = \begin{bmatrix} 1 & \zeta'(\psi) \\ 0 & \pm 1 \end{bmatrix}.$$

If $\mathbf{x}$ is the geodesic defined by $x \in \Omega_+ \setminus \Omega_+^=$, then $(df_0)_x \circ P = P \circ (dg_{2\Upsilon_0(\mathbf{x})})_x$ and so $\|(df_0)_x\| \approx \|(dg_{2\Upsilon_0(\mathbf{x})})_x\|$. By equation (2.1), we have $\|df_0 e^s_x\| \ll 1$ for $x \in \Omega_+ \setminus \Omega_+^=$.

Lemma 5.7. *The following are true.*

(1) *There is a Hölder continuous function $a : \Omega_+ \to \mathbb{R}$ such that E^u_x is spanned by $\begin{bmatrix} a(x) \\ 1 \end{bmatrix}$ for all $x \in \Omega_+$.*
(2) *For all $x \in \Omega_+$, there a $C^{1+\mathrm{Lip}}$ function Θ such that W^u_x is locally the graph $\{(\Theta(\psi), \psi)\}$ of Θ.*

Part (1) says that E^u is not horizontal in the (θ, ψ) coordinates.

Proof. Recall that the definition of Ω_+ depends on the small parameter χ; hence the homogeneity bands $\mathscr{C}_n$ are only defined for large n. Let $x \in \Omega_+ \setminus \Omega_+^=$. Writing $e^s_x = \begin{bmatrix} e_1(x) \\ e_2(x) \end{bmatrix}$, we have

$$df_0 e^s_x = \begin{bmatrix} e_1(x) + \zeta'(x) e_2(x) \\ \pm e_2(x) \end{bmatrix}$$

where $\zeta' \approx n^{3-\frac{2}{r}}$ in $\mathscr{C}_n$ by Lemma 4.6. Since $\|df_0 e_x^s\| \ll 1$, we get that $|e_2| \ll n^{-3+\frac{2}{r}}$ and in particular $e_2 \to 0$ as $n \to \infty$. In other words, $e_x^s \to (1,0)$ as $n \to \infty$, see Figure 9. Since

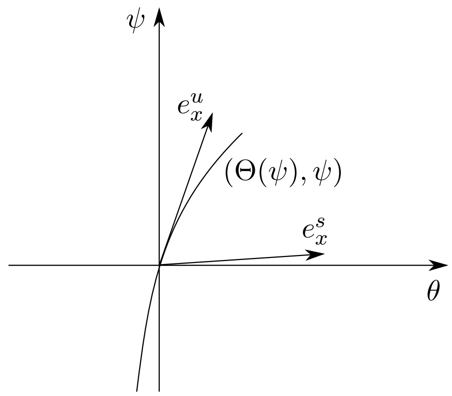

FIGURE 9. We have $e_x^s \to (1,0)$ as $n \to \infty$, hence W_x^u is not horizontal.

$x \mapsto E_x^u$ is Hölder continuous (Theorem 2.6) and E^u is transverse to E^s, part (1) follows.

For part (2), recall that by Theorem 2.6 the leaves $\widehat{W}_x^u$ are uniformly $C^{1+\mathrm{Lip}}$. This implies that W_x^u are uniformly $C^{1+\mathrm{Lip}}$ for $x \in \Omega_+$. Since W_x^u is transverse to the horizontal direction $(1,0)$, the implicit function theorem implies that we can locally write W_x^u as the graph $\{(\Theta(\psi), \psi)\}$ of a $C^{1+\mathrm{Lip}}$ function Θ. This completes the proof. $\square$

Lemma 5.8 (Growth bounds)**.** *If $x \in \mathscr{D}_n$ then $\|df \restriction_{E_x^u}\| \approx n^{3-\frac{2}{r}}$.*

Proof. Since $\|df \restriction_{E_x^u}\| \approx \|df_0 \restriction_{E^u_{\mathfrak{p}_+(x)}}\|$, we need to prove that $\|df_0 \restriction_{E_x^u}\| \approx n^{3-\frac{2}{r}}$ for $x \in \mathscr{C}_n$. We use the Clairaut metric. By Lemma 5.7(1), there exists $a : \Omega_+ \to \mathbb{R}$ continuous such that E_x^u is spanned by $\begin{bmatrix} a(x) \\ 1 \end{bmatrix}$ for $x \in \Omega_+$. If $x = (\theta, \psi) \in \mathscr{C}_n$, Lemma 4.6 gives that $df_0 \begin{bmatrix} a(x) \\ 1 \end{bmatrix} = \begin{bmatrix} a(x) + \zeta'(x) \\ \pm 1 \end{bmatrix}$ where $\zeta'(x) \approx n^{3-\frac{2}{r}}$. Hence

$$\|df_0 \restriction_{E_x^u}\|_{\mathrm{C}} = \left\| \begin{bmatrix} a(x) + \zeta'(x) \\ \pm 1 \end{bmatrix} \right\|_{\mathrm{C}} \Bigg/ \left\| \begin{bmatrix} a(x) \\ 1 \end{bmatrix} \right\|_{\mathrm{C}} = \frac{1 + |a(x) + \zeta'(x)|}{1 + |a(x)|} \tag{5.3}$$

and, since a is bounded, we obtain that $\|df_0 \restriction_{E_x^u}\|_{\mathrm{C}} \approx n^{3-\frac{2}{r}}$. $\square$

Lemma 5.9 (Distortion bounds)**.** *If $x, \overline{x} \in \mathscr{D}_n^>$ or $x, \overline{x} \in \mathscr{D}_n^<$ with $\overline{x} \in W_x^u$, then*

$$\big| \log \|df \restriction_{E_x^u}\| - \log \|df \restriction_{E_{\overline{x}}^u}\| \big| \ll d(fx, f\overline{x})^{\frac{1}{3}}.$$

Proof. Again, it is enough to prove the estimate for the map f_0 in the Clairaut metric. Performing a calculation analogous to the one before Lemma 5.7 and using the equivalence between metrics, we have $\|df_0 \restriction_{E_x^u}\|_{\mathrm{C}} \gg 1$ for $x \in \Omega_+ \setminus \Omega_+^=$ and so $d_{\mathrm{C}}(x, \overline{x}) \ll d_{\mathrm{C}}(f_0 x, f_0 \overline{x})$ whenever $\overline{x} \in W_x^u$.

Write $x=(\theta,\psi)$ and $\overline{x}=(\overline{\theta},\overline{\psi})$. By equation (5.3),

$$\begin{aligned}\big|\log\|df_0\restriction_{E^u_x}\|_{\rm C}-\log\|df_0\restriction_{E^u_{\overline{x}}}\|_{\rm C}\big| &= \left|\log\left(\frac{1+|a(x)+\zeta'(x)|}{1+|a(x)|}\right)-\log\left(\frac{1+|a(\overline{x})+\zeta'(\overline{x})|}{1+|a(\overline{x})|}\right)\right|\\ &=\left|\log\left(\frac{1+|a(x)+\zeta'(x)|}{1+|a(\overline{x})+\zeta'(\overline{x})|}\right)-\log\left(\frac{1+|a(x)|}{1+|a(\overline{x})|}\right)\right|.\end{aligned}$$

Using that $\log\left(\frac{|a|}{|b|}\right)=\log\left(1+\frac{|a|-|b|}{|b|}\right)\leq\frac{|a-b|}{|b|}$, we obtain

$$\begin{aligned}&\big|\log\|df_0\restriction_{E^u_x}\|_{\rm C}-\log\|df_0\restriction_{E^u_{\overline{x}}}\|_{\rm C}\big|\leq\frac{\big||a(x)+\zeta'(x)|-|a(\overline{x})+\zeta'(\overline{x})|\big|}{1+|a(\overline{x})+\zeta'(\overline{x})|}+\frac{\big||a(x)|-|a(\overline{x})|\big|}{1+|a(\overline{x})|}\\ &\leq\frac{|\zeta'(x)-\zeta'(\overline{x})|}{1+|a(\overline{x})+\zeta'(\overline{x})|}+\frac{|a(x)-a(\overline{x})|}{1+|a(\overline{x})+\zeta'(\overline{x})|}+\frac{|a(x)-a(\overline{x})|}{1+|a(\overline{x})|}.\end{aligned}$$

Since a is bounded and n_0 is large, Lemma 4.6 implies that $|a(\overline{x})+\zeta'(\overline{x})|\geq\frac{1}{2}|\zeta'(\overline{x})|$ and so

$$\big|\log\|df_0\restriction_{E^u_x}\|_{\rm C}-\log\|df_0\restriction_{E^u_{\overline{x}}}\|_{\rm C}\big|\leq 2\frac{|\zeta'(x)-\zeta'(\overline{x})|}{|\zeta'(\overline{x})|}+2|a(x)-a(\overline{x})|. \tag{5.4}$$

This estimate holds for all $x,\overline{x}\in\mathscr{C}_n^<$ or $x,\overline{x}\in\mathscr{C}_n^>$. Assuming that $\overline{x}\in W^u_x$, then by Lemma 5.7(2) we can write the latter expression as a function of ψ and apply the mean value theorem to get that

$$\begin{aligned}\big|\log\|df_0\restriction_{E^u_x}\|_{\rm C}-\log\|df_0\restriction_{E^u_{\overline{x}}}\|_{\rm C}\big|&\leq 2\frac{\big|\zeta'(\psi)-\zeta'(\overline{\psi})\big|}{\big|\zeta'(\overline{\psi})\big|}+2|a(\psi)-a(\overline{\psi})|\\ &\leq 2\tfrac{\|\zeta''\|_\infty}{|\zeta'(\overline{\psi})|}|\psi-\overline{\psi}|+2|a(\psi)-a(\overline{\psi})|.\end{aligned}$$

Also by Lemma 5.7(2), the restriction of a to W^u_x is $C^{\rm Lip}$, i.e. the map $\psi\mapsto a(\psi)$ is $C^{\rm Lip}$, hence $|a(\psi)-a(\overline{\psi})|\ll|\psi-\overline{\psi}|\leq d_{\rm C}(x,\overline{x})\ll d_{\rm C}(f_0x,f_0\overline{x})$. To estimate the other term, apply Lemma 4.6 to get that $\frac{\|\zeta''\|_\infty}{|\zeta'(\overline{\psi})|}\ll n^2$ and so by Lemma 4.5

$$\tfrac{\|\zeta''\|_\infty}{|\zeta'(\overline{\psi})|}|\psi-\overline{\psi}|\ll\left[n^2|\psi-\overline{\psi}|^{\frac{2}{3}}\right]|\psi-\overline{\psi}|^{\frac{1}{3}}\ll|\psi-\overline{\psi}|^{\frac{1}{3}}\leq d_{\rm C}(x,\overline{x})^{\frac{1}{3}}\ll d_{\rm C}(f_0x,f_0\overline{x})^{\frac{1}{3}}.$$

Combining these estimates, we conclude the proof of the lemma. □

Lemma 5.10 (Uniform bounds on jacobian of holonomies)**.** *There is a constant $\kappa>0$ such that if $x,\overline{x}\in\mathscr{D}_n^>$ or $x,\overline{x}\in\mathscr{D}_n^<$ with $\overline{x}\in W^s_x$ then*

$$\big|\log\|df\restriction_{E^u_x}\|-\log\|df\restriction_{E^u_{\overline{x}}}\|\big|\ll d(x,\overline{x})^\kappa.$$

Proof. As usual, we prove the corresponding estimate for f_0, taking the Clairaut metric and assuming that $f_0(\theta,\psi)=(\theta+\zeta,\pm\psi)$. Let $x=(\theta,\psi)$ and $\overline{x}=(\overline{\theta},\overline{\psi})$. Fix $\kappa_0>0$ such that $x\mapsto a(x)$ is κ_0–Hölder continuous (see Lemma 5.7(2)), i.e. $|a(x)-a(\overline{x})|\ll d_{\rm C}(x,\overline{x})^{\kappa_0}$. By (5.4),

$$\begin{aligned}\big|\log\|df_0\restriction_{E^u_x}\|_{\rm C}-\log\|df_0\restriction_{E^u_{\overline{x}}}\|_{\rm C}\big|&\leq 2\frac{\big|\zeta'(\psi)-\zeta'(\overline{\psi})\big|}{\big|\zeta'(\overline{\psi})\big|}+2|a(x)-a(\overline{x})|\\ &\leq 2\tfrac{\|\zeta''\|_\infty}{|\zeta'(\overline{\psi})|}|\psi-\overline{\psi}|+2|a(x)-a(\overline{x})|.\end{aligned}$$

Again, $\frac{\|\zeta''\|_\infty}{|\zeta'(\overline{\psi})|}|\psi-\overline{\psi}| \ll \left[n^2|\psi-\overline{\psi}|^{\frac{2}{3}}\right]|\psi-\overline{\psi}|^{\frac{1}{3}} \ll |\psi-\overline{\psi}|^{\frac{1}{3}} \le d_{\mathrm{C}}(x,\overline{x})^{\frac{1}{3}}$. The lemma follows with $\kappa = \min\left\{\frac{1}{3},\kappa_0\right\}$. ☐

We end this section with an estimate of the variation of Υ on stable/unstable manifolds.

Lemma 5.11. *If $\mathbf{x},\overline{\mathbf{x}}$ are both bouncing/crossing geodesics with entry vectors $x,\overline{x}\in\Sigma^+$ such that $\overline{x}\in W^{s/u}_x$ then*

$$|\Upsilon(\mathbf{x})-\Upsilon(\overline{\mathbf{x}})| \ll d(x,\overline{x})+d(fx,f\overline{x}).$$

Proof. For $\chi>0$ small enough, let $\mathfrak{p}_{\Sigma_0}: g_{[-\chi,\chi]}\Sigma_0\to\Sigma_0$ and $\mathfrak{t}_{\Sigma_0}: g_{[-\chi,\chi]}\Sigma_0\to[-\chi,\chi]$ be the coordinate maps of the flow box $g_{[-\chi,\chi]}\Sigma_0$. These maps have the same regularity as the flow g_t, hence in particular are Lipschitz. Since the leaves $\widehat{W}^{s/u}$ are uniformly transverse to the flow direction, $d(x,y)\approx d(x,\mathfrak{p}_{\Sigma_0}(y))$ whenever $x\in\Sigma_0$ and $y\in\widehat{W}^{s/u}_x$ is close to x.

Fix $x\in\Sigma^+$ and $\overline{x}\in W^{s/u}_x$. By definition, $\overline{x}=\mathfrak{p}_{\Sigma_0}(y)$ for some $y\in\widehat{W}^{s/u}_x$, i.e. $y=g_{\mathfrak{t}_{\Sigma_0}(y)}(\overline{x})$. We also have $fx=g_{\Upsilon(\mathbf{x})}(x)$ and $z:=g_{\Upsilon(\mathbf{x})}(y)\in\widehat{W}^{s/u}_{fx}$. Hence $f(\overline{x})=\mathfrak{p}_{\Sigma_0}(z)=g_{-\mathfrak{t}_{\Sigma_0}(z)}(z)=g_{\mathfrak{t}_{\Sigma_0}(y)+\Upsilon(\mathbf{x})-\mathfrak{t}_{\Sigma_0}(z)}(\overline{x})$, so

$$\Upsilon(\overline{\mathbf{x}})=\Upsilon(\mathbf{x})+\mathfrak{t}_{\Sigma_0}(y)-\mathfrak{t}_{\Sigma_0}(z).$$

Since

- $|\mathfrak{t}_{\Sigma_0}(y)|=|\mathfrak{t}_{\Sigma_0}(y)-\mathfrak{t}_{\Sigma_0}(x)|\ll d(x,y)\approx d(x,\overline{x})$ and
- $|\mathfrak{t}_{\Sigma_0}(z)|=|\mathfrak{t}_{\Sigma_0}(z)-\mathfrak{t}_{\Sigma_0}(fx)|\ll d(z,fx)\approx d(f(\overline{x}),fx)$,

we conclude that $|\Upsilon(\overline{\mathbf{x}})-\Upsilon(\mathbf{x})|\le|\mathfrak{t}_{\Sigma_0}(y)|+|\mathfrak{t}_{\Sigma_0}(z)|\ll d(x,\overline{x})+d(fx,f\overline{x})$. ☐

6. The first return map f satisfies the Chernov axioms

In Section 5.4, we defined the first return map $f: X_0\backslash\mathscr{S}^+\to X_0\backslash\mathscr{S}^-$, and in Sections 5.5 and 5.6 we obtained precise estimates for trajectories that approach the degenerate closed geodesic γ. Even though the rate of hyperbolicity of g_t in a neighborhood of γ is weak, geodesics spend a long time during the transition (Lemma 5.3) and so the accumulated hyperbolicity is large (Lemma 5.8). We now prove that f satisfies the Chernov axioms (A1)–(A8) stated in Section 3.1.

Theorem 6.1. *The first return map f satisfies the Chernov axioms* (A1)–(A8).

Proof. Recall that the parameters $r\ge4$, $\varepsilon_0>0$ are fixed and we are choosing χ, δ, $\eta>0$ small enough, and $n_0\in\mathbb{N}$ large. We consider the δ-Sasaki metric, cf. Section 2.1, which is equivalent to the Clairaut metric on T^1A, cf. Section 2.2. Let $\mathrm{T}: X_0\setminus\mathscr{S}^+\to(0,\infty)$ be the Poincaré return time defined by f. We have $\inf(\mathrm{T})\ge T_\chi>0$, cf. Section 5.3.

Verification of (A1): Recall that $X_0=\mathrm{int}(\Sigma_0)$, cf. Section 5.4. Take $\widehat{X}=X=\Sigma_0$, which is a compact Riemannian surface with the metric induced by the δ–Sasaki metric. We have that $\mathscr{S}^+,\mathscr{S}^-$ are closed subsets of X, cf. Section 5.4. The regularity of f is the same of g_t, hence it is a C^2 diffeomorphism. This proves axiom (A1).

Verification of (A2): Recall that the directions $E^{s/u}_x$ are the flow projections of $\widehat{E}^{s/u}_x$, defined for all $x\in X$. By Lemma 2.4(1), $\{E^{s/u}_x\}$ are df–invariant. On Σ_+, f has high rate of hyperbolicity by Lemma 5.8, hence we just need to estimate the hyperbolicity of f on

$X_0 \setminus \Sigma_+$. Since each su–disc is tangent to $\widehat{E}^{s/u}$ at its center and the su–discs used in the construction of Σ_0 have radii smaller than η (see Step 5 in Section 5.3), there is $C = C(\eta) > 1$ with $\lim_{\eta\to 0} C(\eta) = 1$ such that $C^{-1}\|dg_{\mathrm{T}(x)} \restriction_{\widehat{E}^{s/u}_x}\| \leq \|df \restriction_{E^{s/u}_x}\| \leq C\|dg_{\mathrm{T}(x)} \restriction_{\widehat{E}^{s/u}_x}\|$ for all $x \in X_0 \setminus \mathscr{S}^+$. We estimate the hyperbolicity along E^u_x. The set

$$Y = \{g_t(x) : x \in X_0 \setminus \Sigma_+ \text{ and } 0 \leq t \leq \mathrm{T}(x)\}$$

is at distance $> \varepsilon_0/2$ from γ. Letting $m_1 = \inf(u_+ \restriction_Y)$, Proposition 2.5 implies that $0 < m_1 < \infty$ and so equation (2.1) for E^u gives that

$$\|dg_{\mathrm{T}(x)} \restriction_{\widehat{E}^u_x}\| \geq C_\delta^{-1} \exp\left[\int_0^{\mathrm{T}(x)} u_+(g_t x)dt\right] \geq C_\delta^{-1} \exp\left[\inf(\mathrm{T})m_1\right] =: \Lambda_u.$$

If $\delta > 0$ is small enough then $\Lambda_u > 1$. Hence, for δ, η small enough we have $\|df \restriction_{E^u}\| \geq C^{-1}\|dg_{\mathrm{T}(x)} \restriction_{\widehat{E}^u}\| \geq C^{-1}\Lambda_u > 1$. Arguing similarly with E^s_x, we find $\Lambda_s < 1$ such that $\|df \restriction_{E^s}\| \leq C\Lambda_s < 1$. Choosing $1 < \Lambda < C^{-1}\min\{\Lambda_s^{-1}, \Lambda_u\}$, we have that $\|df \restriction_{E^u}\| > \Lambda$ and $\|df^{-1} \restriction_{E^s}\| > \Lambda$. Finally, choose $\alpha = \alpha(\Lambda) > 0$ small enough, and for $x \in X$ define the cones

$$C^s_x = \{v^s + v^u : v^{s,u} \in E^{s,u}_x \text{ and } \|v^u\| < \alpha\|v^s\|\}$$
$$C^u_x = \{v^s + v^u : v^{s,u} \in E^{s,u}_x \text{ and } \|v^s\| < \alpha\|v^u\|\}.$$

Hence:

- ∘ Condition (A2.1) follows from Lemma 2.4(3).
- ∘ Condition (A2.2) follows from Lemma 2.4(2).
- ∘ Condition (A2.3) follows from Lemma 2.4(4), since $X \cap \mathrm{Deg} = \emptyset$.
- ∘ Condition (A2.4) follows from Lemma 2.4(1).
- ∘ Condition (A2.5) follows because $\|df \restriction_{E^u}\|, \|df^{-1} \restriction_{E^s}\| > \Lambda$ and $\alpha > 0$ is small.

Verification of (A3). Recall that $\mathscr{S}^+ = \mathscr{S}_{\mathrm{P}} \cup \mathscr{S}_{\mathrm{S}}$, where

$$\mathscr{S}_{\mathrm{P}} = \{x \in \Sigma_0 : \tau(x) < \infty \text{ and } g_{\tau(x)}(x) \in \partial\Sigma_0\} \cup \{\tau = \infty\},$$
$$\mathscr{S}_{\mathrm{S}} = \bigcup_{n \geq n_0} \partial\mathscr{D}_n.$$

By construction, $\partial\Sigma_0$ is transverse to E^u, hence by df–invariance the set $\{x \in \Sigma_0 : \tau(x) < \infty \text{ and } g_{\tau(x)}(x) \in \partial\Sigma_0\}$ is as well. We also know that $\{\tau = \infty\}$ is contained in the stable manifold of Deg, hence it is transverse to E^u. This shows that $\mathscr{S}_{\mathrm{P}}$ is transverse to E^u, but also $\mathscr{S}_{\mathrm{S}}$ since the curves $\partial\mathscr{D}_n$ converge in the C^1 norm to $\{\tau = \infty\}$. Another way to see this later property is observing that, in Clairaut coordinates, $\{\tau = \infty\}$ is contained in $\{\psi = \psi_0\}$ on Ω_1 (with suitable modifications on the remainder of Ω_+) while the two curves composing $\partial\mathscr{D}_n$ are of the form $\{\psi = \psi_n\}$ with $\lim_{n\to\infty} \psi_n = \psi_0$.

Verification of (A4). The Liouville measure μ on M induces a Liouville measure μ_{Σ_0} on Σ_0, invariant under f. We consider the ergodicity of f^n with respect to μ_{Σ_0}. Since the invariant manifolds of Deg have zero Liouville measure, the suspension flow (f, T) induced by f is isomorphic to the flow g_t, hence ergodic. The suspension flows (f, T) and (f^n, T_n) are isomorphic (T_n denotes the n–th Birkhoff sum), hence (f^n, T_n) is ergodic and so f^n is ergodic.

Verification of (A5). The leaves $W_x^{s/u}$ are flow projections of the leaves $\widehat{W}_x^{s/u}$ which, by Theorem 2.6, are uniformly $C^{1+\mathrm{Lip}}$. Hence the same holds for the leaves $W_x^{s/u}$.

Verification of (A6). We claim that $\psi(s) \approx s^{\frac{1}{3}}$ satisfies (A6). To see that, observe that if x, y belong to the same connected component of $\Sigma_0 \setminus \Sigma_+$ then

$$\Big|\log \|Df \restriction_{E_x^u} \| - \log \|Df \restriction_{E_y^u} \|\Big| \ll d(fx, fy)^{\frac{1}{3}},$$

since outside Σ_+ the map f has the same regularity as g_t. Inside Σ_+, Lemma 5.9 gives the same estimate if both $x, y \in \mathscr{D}_n^>$ or $x, y \in \mathscr{D}_n^<$. If W is a LUM and x, y belong to a same connected component of $W \cap \mathscr{S}_{n-1}$, then for all $0 \leq k < n$ either $f^k x, f^k y$ belong to the same connected component of $\Sigma_0 \setminus \Sigma_+$ or $f^k x, f^k y \in \mathscr{D}_{n_k}^>$ or $f^k x, f^k y \in \mathscr{D}_{n_k}^<$ for some $n_k \geq n_0$. Since we also have $d(f^k x, f^k y) \leq \Lambda^{k-n} d(f^n x, f^n y)$ for $0 \leq k < n$, we conclude that

$$\begin{aligned}
\big|\log \|Df^n \restriction_{E_x^u} \| - \log \|Df^n \restriction_{E_y^u} \|\big| &\leq \sum_{k=0}^{n-1} \big|\log \|Df \restriction_{E_{f^k x}^u} \| - \log \|Df \restriction_{E_{f^k y}^u} \| \\
\ll \sum_{k=1}^{n} d(f^k x, f^k y)^{\frac{1}{3}} &\ll \sum_{k=1}^{n} \Lambda^{\frac{1}{3}(k-n)} d(f^n x, f^n y)^{\frac{1}{3}} \ll d(f^n x, f^n y)^{\frac{1}{3}}.
\end{aligned}$$

Verification of (A7). As in the verification of (A6), if $x, \overline{x}$ belong to the same connected component of $\Sigma_0 \setminus \Sigma_+$ or both $x, \overline{x} \in \mathscr{D}_n^>$ or both $x, \overline{x} \in \mathscr{D}_n^<$ then

$$\big|\log \|Df \restriction_{E_x^u} \| - \log \|Df \restriction_{E_{\overline{x}}^u} \|\big| \leq d(x, y)^{\kappa}.$$

Indeed, the first case holds because the restriction of f to $\Sigma_0 \setminus \Sigma_+$ has bounded C^2 norm, and the latter cases follow from Lemma 5.10. By symmetry, it is enough to verify (A7) for unstable holonomies, so let W_1, W_2 be sufficiently small and close enough LSM's, and let $H : W_1 \to W_2$ be the (unstable) holonomy map. By classical Pesin theory (see e.g. [BP07, Theorem 8.6.13]), the Jacobian JH of H is given by the equation

$$\log JH(x) = \sum_{i=0}^{\infty} \Big(\log \big\|Df \restriction_{E_{f^{-i}x}^u} \big\| - \log \big\|Df \restriction_{E_{f^{-i}H(x)}^u} \big\| \Big).$$

By the uniform hyperbolicity of f, we have $d(f^{-i}x, f^{-i}H(x)) \leq \Lambda^{-i} d(x, H(x))$ for all $i \geq 0$ and, since $f^{-i}x, f^{-i}H(x)$ belong to the same connected component of $\Sigma_0 \setminus \Sigma_+$ or are both in the same $\mathscr{D}_n^>$ or $\mathscr{D}_n^<$, we conclude that

$$\begin{aligned}
|\log JH(x)| &\leq \sum_{i=0}^{\infty} \Big|\log \big\|Df \restriction_{E_{f^{-i}x}^u} \big\| - \log \big\|Df \restriction_{E_{f^{-i}H(x)}^u} \big\|\Big| \\
&\ll \sum_{i=0}^{\infty} d(f^{-i}x, f^{-i}H(x))^{\kappa} \ll \sum_{i=0}^{\infty} \Lambda^{-\kappa i} d(x, H(x))^{\kappa} \ll 1.
\end{aligned}$$

Verification of (A8). Let W be a LUM. By (A3), E^u is transverse to the boundaries of homogeneity bands and so $W \cap \mathscr{S}_1$ is at most countable. When it is countable, we can write $W \cap \mathscr{S}_1 = \{x_n, x_{n+1}, \ldots\}$ where $|c(x_n)| = 1 \pm \frac{1}{n^2}$ and $x_n \to x_\infty \in \Sigma_0 \cap g_{[-\delta,\delta]}\Omega_+^=$.

Writing $x_n = (\varepsilon_n, \theta_n, \psi_n)$ in Clairaut coordinates, we obtain that $\rho(x_n, x_\infty) \approx |\psi_n - \psi_\infty| \approx |c(x_n) - c(x_\infty)| \approx n^{-2}$, which proves (A8.1). For (A8.2), note that by Lemma 5.8

$$\theta_0 := \liminf_{\delta\to 0} \sup_{|W|<\delta} \sum_{n\geq n_0} \tfrac{1}{\Lambda_n} \approx \sum_{n\geq n_0} \tfrac{1}{n^{3-\frac{2}{r}}} \approx \tfrac{1}{n_0^{2-\frac{2}{r}}} < 1,$$

for n_0 large enough, so (A8.2) follows. Finally, (A8.3) follows from our construction of Σ_0, which was made to prevent triple intersections of $\mathscr{S}_{\rm P}$ up to the n_0'th pre-iterate under f. In the terminology of Section 3.1, this gives that $K_{P,n_0} = 3$, which is obviously smaller than $\min\{\theta_0^{-1}, \Lambda\}^{n_0}$ for n_0 large enough. ☐

7. Proof of Theorem 1.1 and statistical limit laws

In this section, we prove the results mentioned in the introduction, Theorem 1.1 and Remark 1.2 as well as various statistical limit laws.

In the previous sections, we constructed a first return map $f : \Sigma_0 \to \Sigma_0$ that satisfies the Chernov axioms. The return time function $\tau : \Sigma_0 \to (0,\infty)$ defined in Section 5.4 is bounded below but not above. We have $f = g_\tau$ (i.e. $f(x) = g_{\tau(x)}(x)$).

By Theorem 3.1, f is modelled by a Young tower with exponential tails: there is a subset $Y \subset \Sigma_0$ with $\text{Leb}(Y) > 0$ and a function $\sigma : Y \to \mathbb{N}$ such that $F = f^\sigma : Y \to Y$ is "nice" (uniformly hyperbolic with product structure and bounded distortion) and $\text{Leb}(\sigma > n) \to 0$ exponentially quickly as $n \to \infty$. Note that

$$F = f^\sigma = (g_\tau)^\sigma = g_\varphi \quad \text{where} \quad \varphi = \sum_{\ell=0}^{\sigma-1} \tau \circ f^\ell.$$

Hence, we have shown that the geodesic flow g_t is modelled by a suspension flow over $F : Y \to Y$ with roof function φ.

Next we estimate the tails of φ. Recalling the definition of Σ_+ in Section 5.4, $\tau = \Upsilon$ is unbounded on Σ_+, while τ is bounded on $\Sigma_0 \setminus \Sigma_+$. By Lemma 5.4, $\mu_{\Sigma_0}(\tau > n) \approx \text{Leb}(\Upsilon > n) \approx n^{-(a+1)}$ where $a = \frac{r+2}{r-2}$. Since σ has exponential tails, a standard argument (see for example [Mar04, CZ05a]) shows that $\mu_{\Sigma_0}(\varphi > n) \ll (\log n)^{a+1} n^{-(a+1)}$. In particular, $\mu_{\Sigma_0}(\varphi > n) \ll n^{-(a+1-\epsilon)}$ for any $\epsilon > 0$.

To apply the recent work of [BBM19], we also require the following "bounded Hölder constants" property for φ.

Lemma 7.1. *We have $|\varphi(x) - \varphi(\overline{x})| \ll d(x, \overline{x})$ for $x, \overline{x} \in Y$ with $\overline{x} \in W^s_x$, and $|\varphi(x) - \varphi(\overline{x})| \ll d(Fx, F\overline{x})$ for $x, \overline{x} \in Y$ with $\overline{x} \in W^u_x$.*

Proof. This is essentially Lemma 5.11. Assume first that $\overline{x} \in W^u_x$. If $x, \overline{x} \in \Sigma_+$ then $\tau = \Upsilon$, and so by Lemma 5.11 we have that $|\tau(x) - \tau(\overline{x})| \ll d(fx, f\overline{x})$. Since the same estimate is trivially true for $x, \overline{x} \in \Sigma_0 \setminus \Sigma_+$, it follows that $|\tau(x) - \tau(\overline{x})| \ll d(fx, f\overline{x})$ for all $x, \overline{x} \in \Sigma_0$ with $\overline{x} \in W^u_x$. By the hyperbolicity of f, this estimate implies that $|\varphi(x) - \varphi(\overline{x})| \ll d(Fx, F\overline{x})$ for all $x, \overline{x} \in Y$ with $\overline{x} \in W^u_x$. The argument along stable manifolds is similar. ☐

In the terminology of [BBM19, Section 6], we have shown that g_t is a "Gibbs-Markov flow". The condition (H) in [BBM19, Section 6] follows from Lemma 7.1 by [BBM19, Lemma 8.3].[3] The main remaining hypothesis in [BBM19] is "absence of approximate eigenfunctions". This can be verified using ideas from [BBM19, Section 8.4]. The general setup there applies by Lemma 7.1. Since g_t is a geodesic flow and hence has a contact structure, it follows from [BBM19, Remark 8.11] that absence of approximate eigenfunctions is automatic.

By [BBM19, Theorem 6.4], we can now deduce decay of correlations for the geodesic flow g_t with rate $t^{-(a-\epsilon)}$ as claimed in Theorem 1.1 for a certain class of observables. However, these observables belong to a regularity class defined in terms of the abstract suspension flow over F with unbounded roof function φ. In order to work with sufficiently smooth observables on the underlying phase space M, it is necessary to introduce a new Poincaré map g with bounded roof function.

Remark 7.2. In the corresponding decay questions for billiards, we would often take g to be the billiard collision map. However, for the geodesic flow, there is no such natural candidate for g.

To construct g, we adjoin the cross-section Ω_0 to Σ_0. Recall that Σ_0 is constructed by small flow displacements of $\widehat{\Sigma}$. Since $\widehat{\Sigma} \cup \Omega_0$ is a global Poincaré section for the geodesic flow g_t, the same holds for $\Sigma = \Sigma_0 \cup \Omega_0$. Let $g : \Sigma \to \Sigma$ be the corresponding Poincaré return map. Also, let $h : \Sigma_0 \to (0, \infty)$ be the first return time to Σ under g_t. Then $g = g_h$ and h is bounded above and below.

Let $\partial_t \phi = \frac{d}{dt}(\phi \circ g_t)|_{t=0}$ denote the derivative in the flow direction. An observable $\phi : M \to \mathbb{R}$ is "sufficiently smooth" in Theorem 1.1 if $\partial_t^j \phi$ is Hölder for $j = 0, \ldots, k$, for some k sufficiently large independent of ϵ and ϕ. (Actually, it suffices that ϕ and ψ are both Hölder and that one of them is sufficiently smooth.)

In particular, when $r \geq 4$ is an even integer, an observable ϕ is sufficiently smooth if it is C^k for k sufficiently large. Otherwise, the flow is not smooth (nor is M) and we require in addition that the observable is sufficiently flat at the degenerate geodesic γ.

Proof of Theorem 1.1. We are now in the situation of [BBM19, Sections 7.1 and 7.2] (where g_t and g are called T_t and f, and there is no counterpart of our f). Most of the assumptions therein follow from the existence of the Young tower, and the other assumptions (7.2), (7.4), (7.5) are immediate (see [BBM19, Remark 7.2] for extra information). The remaining assumptions of [BBM19, Corollary 8.1] (i.e. condition (H) and absence of approximate eigenfunctions) have been dealt with above. Hence we conclude from [BBM19, Corollary 8.1] the desired polynomial decay for Hölder observables that are sufficiently smooth. □

Turning to decay of correlations for the global Poincaré map g, we can write $F = g^{\varphi^*}$ where $\varphi^* : Y \to \mathbb{N}$ is the return time to Y under g. Hence

$$g_\varphi = F = g^{\varphi^*} = (g_h)^{\varphi^*}$$

[3] Condition (8.2) in [BBM19, Lemma 8.3] is stated more generally in terms of a separation time s. It is standard in the Young tower set up that the estimate in terms of the metric d in Lemma 7.1 is stronger, e.g. apply condition (7.3) from [BBM19] with $n = 1$.

so it follows that $\varphi = \sum_{\ell=0}^{\varphi^*-1} h \circ g^\ell$. Since h is bounded above and below, φ^* has the same tails as φ, hence [BMT21, Proposition 5.1] gives that

$$\frac{1}{n^{a+1}\log n} \ll \mu_{\Sigma_0}(\varphi^* > n) \ll \frac{(\log n)^{a+1}}{n^{a+1}}.$$

This says that $g : \Sigma \to \Sigma$ is modelled by a Young tower with polynomial tails with tail rate essentially of order $n^{-(a+1)}$. The upper and lower bounds in Remark 1.2 now follow from [You99] and [BMT21, Theorem 7.4(a)] respectively.

7.1. **Statistical limit laws.** Since $g : \Sigma \to \Sigma$ is modelled by a Young tower with polynomials tails, it is possible to read off numerous statistical limit laws for Hölder observables on Σ. Many of these pass over to the flow. The results in this subsection do not rely on Theorem 1.1 and hence are not restricted to sufficiently smooth observables.

Since $a = \frac{r+2}{r-2} > 1$, the return time function $\varphi^* : Y \to \mathbb{N}$ lies in L^2. Hence it follows from [MV16, Corollary 2.1] that the central limit theorem (CLT) holds for the map g. Namely, let $\phi : \Sigma \to \mathbb{R}$ be Hölder with $\int_\Sigma \phi \, d\mu_\Sigma = 0$. Then $n^{-1/2}\sum_{j=0}^{n-1} \phi \circ g^j$ converges in distribution (with respect to μ_Σ) to a (typically nondegenerate) normal distribution. (At the level of the one-sided Young tower obtained by quotienting stable leaves, the CLT was proved by [You99, Theorem 4].) By [Zwe07], the convergence in distribution can equally be taken with respect to Lebesgue measure on Σ.

Statistical limit laws for the flow g_t follow by inducing from those for maps, see e.g. [MT04, MZ15, KM16]. Here it is convenient to apply [BM18, Theorem 5.5]. (The roof function φ and return times τ and σ are denoted by H, h and τ respectively in [BM18].) Again $\varphi \in L^2$. The underlying assumptions on Y and conditions (5.3)–(5.4) at the beginning of [BM18, Section 5] are automatic consequences of the fact that Y is the base of a Young tower. The assumptions on g_t in conditions (5.1)–(5.2) of [BM18] follow from Proposition 2.5 and (2.1). Finally, the assumptions on φ in conditions (5.1)–(5.2) of [BM18] were verified in Lemma 7.1. By [BM18, Theorem 5.5] the CLT holds for the flow. Namely, let $\phi : M \to \mathbb{R}$ be Hölder with $\int_M \phi \, d\mu = 0$. Then $t^{-1/2}\int_0^t \phi \circ g_s \, ds$ converges in distribution (with respect to μ or Lebesgue measure) to a (typically nondegenerate) normal distribution.

A refinement of the CLT is the functional CLT or weak invariance principle (WIP). Given $\phi : M \to \mathbb{R}$ Hölder with $\int_M \phi \, d\mu = 0$, we define $W_n(t) = n^{-1/2}\int_0^{nt} \phi \circ g_s \, ds$. Then W_n converges weakly (with respect to μ or Lebesgue measure) in $C([0,1])$ to Brownian motion W by [BM18, Theorem 5.5].

Finally, we briefly mention applications to homogenization of deterministic fast-slow systems where the aim is to prove convergence, as the time separation goes to infinity, to a stochastic differential equation driven by the Brownian motion W. See [CFK$^+$19] for a recent survey. Using rough path theory, it is sufficient [KM16, KM17] to check that the fast dynamics satisfies certain statistical limit laws. As we now explain, our geodesic flow examples g_t satisfy all of these requirements for all $r \geq 4$.

First, we require a multidimensional version of the WIP (for observables $v : M \to \mathbb{R}^d$). Again this holds for g by [MV16] and for the flow by [MZ15]. However, the WIP does not suffice to specify stochastic integrals, and for this one requires the so-called iterated WIP. Once more, this holds for g by [MV16, Corollary 2.3] and for the flow by [BM18, Theorem 5.5]. The remaining ingredients needed for homogenization are moment bounds and iterated moment bounds. Since h is bounded above and below, such bounds for the flow follow by [KM16, Proposition 7.5] from the corresponding bounds for the map g. For two-sided Young towers, such as we have here, optimal bounds for moments and iterated moments were very recently obtained by [FV22]; in particular they hold in the full range $r \in [4, \infty)$.

Appendix A. On two theorems of Gerber & Wilkinson

In this appendix we show how to adapt the results of Gerber & Wilkinson [GW99] to prove Theorem 2.6. More specifically, we show how to obtain [GW99, Theorems I and II] for surfaces with degenerate closed geodesic, see Section 2.3 for the definition.

In [GW99], Theorems I and II are proved for C^r metrics of nonpositive curvature, where $r \geq 4$ is an integer, under two assumptions on the surface:

(1) If γ is a geodesic that is not closed, then there is no infinite time interval I for which $K(\gamma(t)) = 0$, for all $t \in I$.
(2) If γ is a closed geodesic, then there is a t such that K vanishes to order at most $r - 3$.

See the statements of the theorems and the remark in [GW99, p. 43]. It is clear that surfaces with degenerate closed geodesic satisfy assumption (1), regardless of the value of $r \in [4, \infty)$ (integer or not). But assumption (2), for non-integer r, makes no sense. Our goal is to check that, even though surfaces with degenerate closed geodesic do not satisfy (2), all estimates of Gerber & Wilkinson remain true, and so does Theorem 2.6. The reason is that (2) is used to obtain a control on how the curvature approaches zero: if γ is a closed geodesic of zero curvature, then there are constants $C_1, C_2 > 0$ such that

$$- C_1 \mathrm{dist}(p, \gamma)^{r-2} \leq K(p) \leq -C_2 \mathrm{dist}(p, \gamma)^{r-2} \tag{A.1}$$

in a neighborhood of γ. This estimate does hold for surfaces with degenerate closed geodesic, as we now explain. Let S be such a surface. The region containing the closed geodesic γ with zero curvature is the surface of revolution $\mathcal{N}$, which we call the *neck*. The profile function is $\xi(s) = 1 + |s|^r$, hence by the curvature formula given in Section 2.2 we have:

(2)' There are constants $C_1, C_2 > 0$ such that

$$-C_1 |s|^{r-2} \leq K(s, \theta) \leq -C_2 |s|^{r-2},$$

where (s, θ) are the Clairaut coordinates on the neck $\mathcal{N}$.

It is clear that (2)' is (A.1) in our context. In the sequel, we check that [GW99, Theorems I and II] hold under assumptions (1) and (2)'. We warn the reader that, while [GW99] uses Fermi coordinates (s, a), we will maintain our use of the Clairaut coordinates (s, θ, ψ). Let $\mathcal{H}^-$ and $\mathcal{H}^+$ be the stable and unstable horocycle foliations of S.

Theorem A.1. *Let S be a surface with degenerate closed geodesic. Then:*

(i) *The leaves of $\mathcal{H}^-$ and $\mathcal{H}^+$ are uniformly $C^{1+\mathrm{Lip}}$.*
(ii) *The tangent distributions $T\mathcal{H}^-$ and $T\mathcal{H}^+$ are Hölder continuous.*

The proof of Theorem A.1 requires two general lemmas [GW99, Lemmas 3.1 and 3.2], which we reproduce below (only the items that we explicitly refer to are listed).

Lemma A.2 (Lemma 3.1 of [GW99])**.** *Let $K, K_0, K_1 : [A,B] \to \mathbb{R}$ be continuous functions and $u, u_0, u_1 : [A,B] \to \mathbb{R}$ be solutions of the Riccati equations $u'+u^2+K=0$, $u_i'+u_i^2+K_i=0$, $i=0,1$. Let $y=u_1-u_0$ and $\widehat{j}_i(t)=\exp\left[-\int_t^B u_i(\tau)\,d\tau\right]$, $i=0,1$, and let j_0, j_1 be solutions of the Jacobi equations $j_i''+K_ij_i=0$, $i=0,1$. Then the following hold:*

(ii) *$y(B)=y(A)\widehat{j}_0(A)\widehat{j}_1(A)+\int_A^B[K_0(t)-K_1(t)]\widehat{j}_0(t)\widehat{j}_1(t)dt$.*

(iii) *$\widehat{j}_i(B)=1$ and $\widehat{j}_i$ satisfies the Jacobi equation $\widehat{j}_i''+K_i\widehat{j}_i=0$ for $i=0,1$; moreover, if $u_1\geq 0$ on $[A,B]$, then $0\leq\widehat{j}_1\leq 1$ on $[A,B]$ and $\widehat{j}_1'(A)\leq 1/(B-A)$.*

(v) *If K is nonpositive and $u(A)\geq 0$, then*

$$u(B)\geq\frac{u(A)}{(B-A)u(A)+1}$$

and this estimate is an equality whenever K is identically zero.

(vi) *If $0\leq j_0(A)\leq j_1(A)$, $0\leq j_0'(A)\leq j_1'(A)$ and $K_1(t)\leq K_0(t)\leq 0$ for all $t\in[A,B]$, then $j_0(B)\leq j_1(B)$.*

Lemma A.3 (Lemma 3.2 of [GW99])**.** *Let $f : S\to\mathbb{R}$ be a nonpositive C^2 function on a C^2 compact surface S. Define $L := \sup\left\{\left|\frac{d^2}{dt^2}f(\sigma(t))\right| : \sigma \text{ geodesic and } t\in\mathbb{R}\right\}$. Then*

$$|f(p)-f(q)|\leq\sqrt{2L}\sqrt{-f(p)}d(p,q)+\frac{L}{2}d(p,q)^2$$

for all $p,q\in S$.

Given $v\in T^1S$, let $k_-(v)$ and $k_+(v)$ be the curvature at v of the stable and unstable horocycles. Recall from Section 2.1 that $k_\pm=u_\pm$, and that $k_\pm=0$ only at $T^1\gamma$. The next result is a lower bound on the curvatures of horocycles, which is [GW99, Lemma 3.3] in our context.

Lemma A.4. *Let S be a surface with degenerate closed geodesic. There is a constant $C_3>0$ with the following property: if $v=(s,\theta,\psi)\in T^1\mathcal{N}$ in Clairaut coordinates, then*

$$k_\pm(v)\geq C_3\max\{|s|^{(r-2)/2},|\psi|^{(r-2)/r}\}.$$

Proof. In [GW99], this lemma is proved in Section 4. It considers a geodesic that visits the flat region of S in the time interval $[-T,0]$, and decomposes $[-T,0]$ according to whether an estimate as in assumption (2)' holds or not. In our case, the estimate always holds, hence we do not decompose $[-T,0]$. The other ingredient is [GW99, Lemma 4.2], a lemma due to K. Burns, which holds in our case in Clairaut coordinates, also due to assumption (2)'. □

Using the above lemma, we can control the curvature of the horocycles in terms of the Gaussian curvature at the basepoint. This estimate is [GW99, Lemma 3.4] in our context.

Lemma A.5. *Let S be a surface with degenerate closed geodesic. There is a constant $C_4>0$ such that for any $v\in T^1S$ with basepoint $p\in S$ it holds*

$$k_\pm(v)\geq C_4\sqrt{-K(p)}.$$

Proof. We first assume that $v = (p, \psi) = (s, \theta, \psi) \in T^1\mathcal{N}$. By assumption (2)', $-K(p) \leq C_1|s|^{r-2}$. By Lemma A.4, we have $k_\pm(v) \geq C_3 C_1^{-1/2}\sqrt{-K(p)}$ for $v \in T^1\mathcal{N}$. Now, since $k_\pm(v)$ is continuous and positive outside $T^1\mathcal{N}$, there is $C > 0$ such that

$$k_\pm(v) \geq C\sqrt{-K(p)}$$

for $v \notin T^1\mathcal{N}$. Taking $C_4 = \min\{C_3 C_1^{-1/2}, C\}$, the proof is complete. $\square$

As done in [GW99, pp. 51 and 52], Lemmas A.2, A.3 and A.5 imply the next result.

Lemma A.6. *Let S be a surface with degenerate closed geodesic. There are constants $C_5, C_6 > 0$ with the following property. Let γ_0, γ_1 be geodesics, let $K_i(t) = K(\gamma_i(t))$ and $u_i : [A, B] \to \mathbb{R}$ be a solution of the Riccati equation $u_i' + u_i^2 + K_i = 0$, $i = 0, 1$. If $u_0(t) \geq k_+(\gamma_0(t))$ for all $A \leq t \leq B$ and $u_1(A) \geq 0$, then*

$$|u_1(B) - u_0(B)| \leq C_5\varepsilon + C_6(B - A)\varepsilon^2 + |u_1(A) - u_0(A)|\widehat{j}_0(A)\widehat{j}_1(A),$$

where $\varepsilon := \max\{d(\gamma_0(t), \gamma_1(t)) : t \in [A, B]\}$ and $\widehat{j}_i(t) := \exp\left(-\int_t^B u_i(\tau)\,d\tau\right)$ as defined in Lemma A.2*, $i = 0, 1$.*

Proof of part (i) *of Theorem* A.1. Let S be a surface with degenerate closed geodesic, and let γ_0, γ_1 be geodesics on the same unstable horocycle $\mathcal{W} \subset S$. We estimate $|k_+(\gamma_0'(0)) - k_+(\gamma_1'(0))|$ in terms of $\varepsilon := d(\gamma_0(0), \gamma_1(0))$. Since $t \mapsto d(\gamma_0(0), \gamma_1(t))$ is convex ($K \leq 0$) and $d(\gamma_0(t), \gamma_1(t)) \to 0$ as $t \to +\infty$, we have $d(\gamma_0(t), \gamma_1(t)) \leq \varepsilon$ for all $t \leq 0$. Let u_i be the unstable Riccati solutions along γ_i, $i = 0, 1$. Applying Lemma A.6 with $A = -1/\varepsilon$ and $B = 0$, we have

$$|k_+(\gamma_1'(0)) - k_+(\gamma_0'(0))| = |u_1(0) - u_0(0)| \leq C_5\varepsilon + C_6\varepsilon + |u_1(A) - u_0(A)|\widehat{j}_0(A)\widehat{j}_1(A).$$

Since $u_i \geq 0$, we have $0 \leq \widehat{j}_i \leq 1$ and, by Lemma A.2(iii), $\widehat{j}_i'(A) \leq \frac{1}{B-A} = \varepsilon$, hence

$$|k_+(\gamma_1'(0)) - k_+(\gamma_0'(0))| \leq C_5\varepsilon + C_6\varepsilon + \max_{i=0,1} u_i(A)\widehat{j}_i(A) \leq (C_5 + C_6 + 1)\varepsilon.$$

$\square$

Next, we prove part (ii) of Theorem A.1. We will need two auxiliary lemma, the first being [GW99, Lemma 3.6] in our context. Recall that k_+, k_- are the curvatures of the stable and unstable horocycles.

Lemma A.7. *Let S be a surface with degenerate closed geodesic. There is a constant $C_7 > 0$ such that for all $v \in T^1 S$ it holds*

$$\frac{1}{C_7}k_+(v) \leq k_-(v) \leq C_7 k_+(v).$$

Proof. As in Lemma A.5, we divide the proof into two cases. Start assuming $v \in T^1\mathcal{N}$. Assumptions (1) and (2)' allow to apply a result of Gerber & Nitica [GN99, Theorem 3.1], and obtain the upper bound

$$k_\pm(v) \leq C\max\{|s|^{(r-2)/2}, |\psi|^{(r-2)/r}\}$$

for $v = (s, \theta, \psi) \in T^1\mathcal{N}$ in Clairaut coordinates. This and Lemma A.4 imply that

$$\frac{1}{C'}k_+(v) \leq k_-(v) \leq C'k_+(v).$$

for some $C' > 0$. The case $v \notin T^1\mathcal{N}$ follows as in the proof of Lemma A.5. $\square$

The second auxiliary lemma is a simple estimate on solutions of the Riccati equation.

Lemma A.8 (Lemma 3.7 of [GW99])**.** *Let $K : [A, B] \to (-\infty, 0]$ continuous. If u_0, u_1 are solutions of the Riccati equation $u' + u^2 + K = 0$ and $u_1(t) \geq u_0(t) > 0$ for $t \in [A, B]$, then*

$$\frac{\exp\left[\int_A^B u_1(t)\,dt\right]}{\exp\left[\int_A^B u_0(t)\,dt\right]} \leq \frac{u_1(A)}{u_0(A)}.$$

Proof of part (ii) *of Theorem* A.1. We wish to show that $|k_+(v_0) - k_+(v_1)| \leq Cd(v_0, v_1)^\alpha$ for all $v_0, v_1 \in T^1S$. As in [GW99], we divide the proof into five steps.

Step 1. It is enough to show that $|k_+(v_0) - k_+(v_1)| \leq Cd(v_0, v_1)^\alpha$ for v_0, v_1 with the same basepoint. Indeed, given v_0, v_1 with basepoints p_0, p_1, let $v_1' \in T^1_{p_1}S$ be the vector spanning a geodesic negatively asymptotic to the geodesic spanned by v_0. By part (i), we have $|k_+(v_1') - k_+(v_0)| \leq Cd(p_0, p_1)$. Since S has nonpositive curvature, Busemann functions are (uniformly) C^2 and so $d(v_1', v_1) \leq Cd(v_0, v_1)$. Hence, if $|k_+(v_1') - k_+(v_1)| \leq Cd(v_1', v_1)^\alpha$ then

$$|k_+(v_0) - k_+(v_1)| \leq Cd(p_0, p_1) + Cd(v_0, v_1)^\alpha \leq 2Cd(v_0, v_1)^\alpha.$$

Step 2. Given $p \in S$ and $v_0, v_1 \in T^1_pS$, let ω be the angle between v_0 and v_1. We can assume that $|\omega| < \omega_0$ for ω_0 small. Let $v_r \in T^1_pS$, $0 \leq r \leq 1$, be the continuous family of unit vectors making angle $r\omega$ with v_0, and let γ_r be the variation of geodesics with $\gamma_r(0) = p$ and $\gamma_r'(0) = -v_r$. Define

$$T := \max\{T_0 : \text{ the curve } r \in [0, 1] \mapsto \gamma_r(t) \text{ has length} \leq \sqrt{\omega} \text{ for all } 0 \leq t \leq T_0\},$$

and consider the scalar function $j_r(t)$ associated to the perperdicular Jacobi field generated by the variation of geodesics γ_r. By definition, $j_r(0) = 0$, $j_r'(0) = \omega$, and

$$\int_0^1 j_r(T)\,dr = \sqrt{\omega}.$$

Comparing j_r with the solution of the Jacobi equation in zero curvature (cf. Lemma A.2(vi)), we also have $j_r(T) \geq \omega T$. Therefore $T \leq 1/\sqrt{\omega}$. Similarly, comparing with the case of constant curvature $K_{\min} = \inf K$, we obtain that $j_r(T) \leq \frac{\omega}{\sqrt{-K_{\min}}} \sinh(\sqrt{-K_{\min}}T)$. If ω_0 is small enough, then $T > 1$. Now, as in [GW99, pp. 55], applying Lemma A.6 and Lemma A.7 we obtain

$$|k_+(v_0) - k_+(v_1)| \quad \leq \quad C_8\sqrt{\omega} + C_9\left(\exp\left[-\int_1^T u_0(t)dt\right]\right)^\beta \left(\exp\left[-\int_1^T u_1(t)dt\right]\right)^\beta$$

for constants $\beta, C_8, C_9 > 0$.

Step 3. We estimate $\exp\left[\int_1^T w_0(t)dt\right]$, where $w_0 = j_0'/j_0$ satisfies $w_0' + w_0^2 + K \circ \gamma_0 = 0$ with $w_0(0) = \infty$ and $w_0(t) > 0$ for $t > 0$. Proceeding as in [GW99, pp. 56–57], there is a constant $C_{10} > 0$ such that

$$\exp\left[-\int_1^T w_0(t)dt\right] \leq C_{10}\sqrt{\omega}.$$

STEP 4. Assume that $k_+(v_0) \neq 0$. Since $w_0(1) > u_0(1)$, Lemma A.8 implies that

$$\frac{\exp\left[\int_1^T w_0(t)\,dt\right]}{\exp\left[\int_1^T u_0(t)\,dt\right]} \leq \frac{w_0(1)}{u_0(1)}.$$

By Lemma A.2(v), we have $u_0(1) \geq \frac{u_0(0)}{u_0(0)+1}$ and so $\frac{w_0(1)}{u_0(1)} \leq \frac{w_0(1)(u_0(0)+1)}{u_0(0)} = \frac{w_0(1)(u_0(0)+1)}{k_+(v_0)}$. Since K is bounded from below, $u_0(0)$ and $w_0(1)$ are bounded from above and so there is a constant $C_{11} > 0$ such that

$$\exp\left[-\int_1^T u_0(t)\,dt\right] \leq \frac{C_{11}}{k_+(v_0)} \exp\left[-\int_1^T w_0(t)\,dt\right].$$

STEP 5. By Steps 3 and 4, if $k_+(v_i) \neq 0$ then

$$\exp\left(-\int_1^T u_i(t)\,dt\right)^\beta \leq \frac{C_{12}}{k_+(v_i)^\beta}\omega^{\beta/2}.$$

for some $C_{12} > 0$. By Step 2,

$$|k_+(v_0) - k_+(v_1)| \leq C_8\sqrt{\omega} + C_9 C_{12} \min\{k_+(v_0), k_+(v_1)\}^{-\beta}\omega^{\beta/2}.$$

Recalling that $\omega := \mathrm{dist}(v_0, v_1)$, we conclude the proof, since:

- if $\min\{k_+(v_0), k_+(v_1)\} \leq \omega^{1/4}$, then $|k_+(v_0) - k_+(v_1)| \leq k_+(v_0) + k_+(v_1) \leq 2\omega^{1/4}$;
- if $\min\{k_+(v_0), k_+(v_1)\} > \omega^{1/4}$, then $|k_+(v_0) - k_+(v_1)| \leq C_8\sqrt{\omega} + C_9 C_{12}\omega^{\beta/4}$.

□

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