Necessary and sufficient condition for $\mathcal{M}_2$-convergence to a Lévy process for billiards with cusps at flat points

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Abstract

We consider a class of planar dispersing billiards with a cusp at a point of vanishing curvature. Convergence to a stable law and to the corresponding Lévy process in the $\mathcal{M}_1$ and $\mathcal{M}_2$ Skorohod topologies has been studied in recent work. Here we show that certain sufficient conditions for $\mathcal{M}_2$-convergence are also necessary.

1 Introduction

Bálint, Chernov & Dolgopyat [1] proved anomalous diffusion for planar dispersing billiards with cusps, namely that the central limit theorem holds with nonstandard normalization $(n \log n)^{1/2}$ instead of $n^{1/2}$. They also established the functional version, proving weak convergence to Brownian motion (again with the anomalous normalization $(n \log n)^{1/2}$). Their results hold for all Hölder observables with zero mean, and throughout we restrict attention to such observables.

Recently, Zhang [10] introduced a class of billiards with cusps where the boundary has vanishing curvature at a cusp. For definiteness, we suppose throughout that there is a single symmetric cusp; more general examples are considered in [2]. Jung &

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Zhang [3] proved convergence to an α-stable law (with normalization $n^{1/\alpha}$) for such billiards; any $\alpha \in (1, 2)$ can be achieved depending on the flatness at the cusp. Then in two papers [2, 4] written independently and using different methods, we obtained functional versions, yielding weak convergence to the corresponding α-stable Lévy process.

For convergence to a Lévy process, there is the question as to which topology to use on the Skorohod space of càdlàg paths. For Hölder (and hence bounded) observables, it is easy to see that convergence in the standard $\mathcal{J}_1$ Skorohod topology [7] fails. Convergence in the $\mathcal{M}_1$ Skorohod topology is proved in [2] for observables that have constant sign near the cusp. In [4], necessary and sufficient conditions for $\mathcal{M}_1$-convergence, and sufficient conditions for convergence in the $\mathcal{M}_2$ topology, are given. In particular, by [4] there are observables for which convergence fails in $\mathcal{M}_1$ but holds in the weaker $\mathcal{M}_2$ topology. Moreover, it is conjectured in [4] that the sufficient conditions for $\mathcal{M}_2$-convergence are necessary and hence that there are observables for which convergence fails in all of the Skorohod topologies [7]. (For more details about the various Skorohod topologies, we refer to [2, 4, 5, 7, 9].)

In this paper, we prove the conjecture in [4]: the conditions therein for $\mathcal{M}_2$-convergence are indeed necessary and sufficient. To prove this we need extra information from the proof in [2]. There it is shown that a certain first return process, denoted U_n in Lemma 5 below, converges in $\mathcal{J}_1$ (the setup in [4] yields only $\mathcal{M}_1$-convergence for U_n which seems insufficient for proving necessity of conditions for $\mathcal{M}_2$-convergence for the full process).

In the remainder of the introduction, we describe the example in [3] and state our main result. For the sake of simplicity, we concentrate on the setting of a single cusp which is the example studied in [3, 4]. As explained in Section 3, our main result extends straightforwardly to more general billiard tables with finitely many cusps with vanishing curvature as studied in [2].

The Jung & Zhang example [10, 3] is a billiard with a table $Q \subset \mathbb{R}^2$ whose boundary consists of a finite number of C^3 curves Γ_i, $i = 1, \dots, n_0$, where $n_0 \geq 3$, with a cusp formed by two of these curves Γ_1, Γ_2. The other intersection points correspond to corners. In coordinates $(s, z) \in \mathbb{R}^2$, the cusp lies at $(0, 0)$ and Γ_1, Γ_2 are tangent to the s-axis at $(0, 0)$. Moreover, in a small neighborhood of $(0, 0)$, the curves Γ_1 and Γ_2 can be represented as the graph of $z = \beta^{-1}s^\beta$ and $z = -\beta^{-1}s^\beta$ respectively, where $\beta > 2$ is a parameter.

The phase space of the billiard map (or collision map) T is given by $\Lambda = \partial Q \times [0, \pi]$, with coordinates (r, θ) where r denotes arc length along ∂Q and θ is the angle between the tangent line of the boundary and the collision vector in the clockwise direction. There is a natural ergodic invariant probability measure $d\mu = (2|\partial Q|)^{-1} \sin\theta \, dr \, d\theta$ on Λ, where $|\partial Q|$ is the length of ∂Q.

In configuration space, the cusp is a single point $(0, 0) = \Gamma_1 \cap \Gamma_2$. Let $r' \in \Gamma_1$ and $r'' \in \Gamma_2$ be the arc length coordinates of $(0, 0)$. Then in phase space Λ, the cusp is the union of two line segments

$$\mathcal{C} = \{(r', \theta) : 0 \le \theta \le \pi\} \cup \{(r'', \theta) : 0 \le \theta \le \pi\}.$$

Let $\alpha = \frac{\beta}{\beta-1} \in (1, 2)$ where $\beta > 2$ is the curvature at the flat cusp as described above. Given $v : \Lambda \to \mathbb{R}$ continuous, define

$$I_v(s) = \frac{1}{2} \int_0^s \{v(r', \theta) + v(r'', \pi - \theta)\} (\sin\theta)^{1/\alpha}\, d\theta, \quad s \in [0, \pi].$$

Now suppose that $v : \Lambda \to \mathbb{R}$ is a Hölder mean zero observable. By [3], convergence to a totally skewed α-stable law holds provided $I_v(\pi) \ne 0$. We restrict from now on to the case $I_v(\pi) > 0$. (The case $I_v(\pi) < 0$ is similar with obvious modifications.) Then $n^{-1/\alpha} \sum_{j=0}^{n-1} v \circ T^j \to_d G$ where G has characteristic function

$$\mathbb{E}(e^{iuG}) = \exp\{-|u|^\alpha \sigma^\alpha (1 - i \operatorname{sgn} u \tan \tfrac{\pi\alpha}{2})\}, \quad \sigma^\alpha = \frac{I_v(\pi)^\alpha \Gamma(1-\alpha) \cos\frac{\pi\alpha}{2}}{2^{\alpha-1}\beta|\partial Q|}.$$

We remark that the constant $\Gamma(1-\alpha)\cos\frac{\pi\alpha}{2}$ was missing from the statement of the main result in Jung & Zhang [3], but this error was corrected in the papers [2, 4]. There is also a difference in the scale parameter between [2] and [4], by a factor of $2^{-\alpha}$, due to a difference of a factor of 2 in the definitions of $I_v(s)$ in those two papers. We use the version of $I_v(s)$ in [4].

The references [2, 4] study the corresponding functional limit law $W_n \to_w W$ in $D([0, \infty), \mathcal{M}_1)$ where

$$W_n(t) = n^{-1/\alpha} \sum_{j=0}^{[nt]-1} v \circ T^j,$$

and W is the α-stable Lévy process with $W(1) =_d G$. In particular, by [4, Theorem 1.3] a necessary and sufficient condition for convergence in $\mathcal{M}_1$ is that $s \mapsto I_v(s)$ is monotone. When convergence in $\mathcal{M}_1$ breaks down, [4, Theorem 1.4] gives a sufficient condition for convergence in $\mathcal{M}_2$, namely that $I_v(s) \in [0, I_v(\pi)]$ for all $s \in [0, \pi]$. Our main result is that this condition is also necessary:

Theorem 1 *Let $v : \Lambda \to \mathbb{R}$ be a Hölder mean zero observable with $I_v(\pi) > 0$. Then $W_n \to_w W$ in $(D[0, \infty), \mathcal{M}_2)$ as $n \to \infty$ if and only if $I_v(s) \in [0, I_v(\pi)]$ for all $s \in [0, \pi]$.*

2 Proof of Theorem 1

As in [3] and [4] we consider the first return map $f = T^\varphi : X \to X$ where X is a region bounded away from the cusp. Specifically, let $X = (\Gamma_3 \cup \cdots \cup \Gamma_{n_0}) \times [0, \pi]$ and define the first return time $\varphi : X \to \mathbb{Z}^+$ and first return map $f = T^\varphi : X \to X$,

$$\varphi(x) = \inf\{n \ge 1 : T^n x \in X\}, \qquad f(x) = T^{\varphi(x)} x.$$

Define $\varphi_k = \sum_{j=0}^{k-1} \varphi \circ f^j$ and $v_k = \sum_{j=0}^{k-1} v \circ T^j$ for $k \geq 0$. Also, set $\bar\varphi = \frac{1}{\mu(X)} \int_X \varphi \, d\mu$.

Proposition 2 $\lim_{n\to\infty} n^{-1} \max_{j \leq n} \varphi \circ f^j = 0$ *almost surely.*

Proof Since φ is integrable, it follows from the ergodic theorem that $\lim_{n\to\infty} n^{-1}\varphi_n = \bar\varphi$ a.e. and so $\lim_{n\to\infty} n^{-1}\varphi \circ f^n = 0$ a.e. The result follows easily. ∎

Turning to the proof of Theorem 1, we continue to assume that $v : \Lambda \to \mathbb{R}$ is a Hölder mean zero observable with $I_v(\pi) > 0$. By [4], it suffices to consider the case $I_v(s) \notin [0, I_v(\pi)]$ for some s. From now on, we suppose that

$$\max_{s\in[0,\pi]} I_v(s) > I_v(\pi).$$

(The case $\min_{s\in[0,\pi]} I_v(s) < 0$, is treated similarly.)

Let $I_v^* = \max_{s\in[0,\pi]} I_v(s)$ and choose s^* such that $I_v^* = I_v(s^*)$. Define

$$\ell^* : X \to \mathbb{N}, \qquad \ell^*(x) = [\varphi(x)\Psi(s^*)],$$

where $\Psi : [0,\pi] \to [0,1]$ is the diffeomorphism $\Psi(s) = I_1(\pi)^{-1} I_1(s)$.

Proposition 3 *There exist $C, \delta > 0$ such that*

$$\Big| \frac{W_n((\varphi_k + \ell^* \circ f^k)/n) - W_n(\varphi_k/n)}{W_n(\varphi_{k+1}/n) - W_n(\varphi_k/n)} - \frac{I_v^*}{I_v(\pi)} \Big| \leq C\varphi^{-\delta} \circ f^k,$$

for all $k \geq 0$, $n \geq 1$.

Proof We have $W_n(\varphi_{k+1}/n) - W_n(\varphi_k/n) = n^{-1/\alpha} v_\varphi \circ f^k$ for $k \geq 0$. Also,

$$W_n((\varphi_k + \ell^* \circ f^k)/n) - W_n(\varphi_k/n) = n^{-1/\alpha} v_{\ell^*} \circ f^k.$$

By [4, Proposition 8.1], there exists $\delta > 0$ such that

$$v_\ell = \varphi I_1(\pi)^{-1} I_v \circ \Psi^{-1}(\ell/\varphi) + O(\varphi^{1-\delta}),$$

for $0 \leq \ell \leq \varphi$. In particular, we have

$$n^{1/\alpha}\big\{W_n(\varphi_{k+1}/n) - W_n(\varphi_k/n)\big\} = \varphi \circ f^k I_1(\pi)^{-1} I_v(\pi) + O(\varphi^{1-\delta} \circ f^k),$$

$$\begin{aligned} &n^{1/\alpha}\big\{W_n((\varphi_k + \ell^* \circ f^k)/n) - W_n(\varphi_k/n)\big\} \\ &\qquad = \varphi \circ f^k I_1(\pi)^{-1} I_v \circ \Psi^{-1}((\ell^*/\varphi) \circ f^k) + O(\varphi^{1-\delta} \circ f^k). \end{aligned}$$

Since $I_v \circ \Psi^{-1}$ is C^1,

$$\begin{aligned} I_v \circ \Psi^{-1}(\ell^*/\varphi) &= I_v \circ \Psi^{-1}([\varphi\Psi(s^*)]/\varphi) \\ &= I_v \circ \Psi^{-1}(\Psi(s^*)) + O(\varphi^{-1}) = I_v^* + O(\varphi^{-1}). \end{aligned}$$

Hence we obtain

$$n^{1/\alpha}\big\{W_n((\varphi_k + \ell^* \circ f^k)/n) - W_n(\varphi_k/n)\big\} = \varphi \circ f^k I_1(\pi)^{-1} I_v^* + O(\varphi^{1-\delta} \circ f^k)$$

and the result follows. ∎

Corollary 4 *Let $\epsilon > 0$. There exists $n_0 \geq 1$ such that for $k \geq 0$, $n \geq n_0$, if $W_n(\varphi_{k+1}/n) - W_n(\varphi_k/n) \geq 1$, then*

$$\frac{W_n((\varphi_k + \ell^* \circ f^k)/n) - W_n(\varphi_k/n)}{W_n(\varphi_{k+1}/n) - W_n(\varphi_k/n)} \in \Big[\frac{I_v^*}{I_v(\pi)} - \epsilon, \frac{I_v^*}{I_v(\pi)} + \epsilon\Big].$$

Proof We have

$$1 \leq W_n(\varphi_{k+1}/n) - W_n(\varphi_k/n) = n^{-1/\alpha} v_\varphi \circ f^k \leq n_0^{-1/\alpha} |v|_\infty \varphi \circ f^k.$$

This implies that

$$\varphi \circ f^k \geq |v|_\infty^{-1} n_0^{1/\alpha}.$$

Hence, we can choose n_0 so large that $C\varphi^{-\delta} \circ f^k \leq \epsilon$. The result now follows from Proposition 3. ∎

Following [5] (see also [4, Section 4]), we write $W_n = U_n + R_n$, where

$$U_n(t) = n^{-1/\alpha} \sum_{j=0}^{N_{[nt]}-1} v_\varphi \circ f^j \quad \text{and} \quad R_n(t) = n^{-1/\alpha} \Bigg(\sum_{\ell=0}^{[nt]-\varphi_{N_{[nt]}}-1} v \circ T^\ell \Bigg) \circ f^{N_{[nt]}}.$$

Here, $N_k(x) = \max\{\ell \geq 1 : \varphi_\ell(x) \leq k\}$ is the number of returns of x to the set X, under iteration by the map T, up to time k.

Lemma 5 *$U_n \to_w W$ in $D([0,\infty), \mathcal{J}_1)$ as $n \to \infty$.*

Proof Define the induced process

$$\widetilde{W}_n(t) = n^{-1/\alpha} \sum_{j=0}^{[nt]-1} v_\varphi \circ f^j, \;\; n \geq 1.$$

Proceeding as in [5, Lemma 3.4], we note that $U_n = \widetilde{W}_n \circ g_n$, where $g_n(t) = n^{-1} N_{[nt]}$.

Now, by [2, Theorem 3.1], $\widetilde{W}_n \to_w \bar{\varphi}^{1/\alpha} W$ in $D([0,\infty), \mathcal{J}_1)$. Also, $g_n \to g$ uniformly on compact subsets of $[0,\infty)$ where $g(t) = t/\bar{\varphi}$.

Let D_0 be the space of elements of $D[0,\infty)$ that are nonnegative and nondecreasing. Then $(\widetilde{W}_n, g_n) \in D[0,\infty) \times D_0$. Since g is continuous and deterministic, it follows from [8, Theorem 3.1] that

$$U_n = \widetilde{W}_n \circ g_n \to_w \bar{\varphi}^{1/\alpha} W \circ g = W$$

in $D([0,\infty, \mathcal{J}_1)$. ∎

Given $u \in D[0,1]$, we define $\Delta u(t) = u(t) - u(t-)$. Let Π_α denote the Lévy measure with density $\alpha x^{-(\alpha+1)} 1_{(0,\infty)}(x)$. Then $\#\{t \in [0,1] : \Delta W(t) \in B\}$ has a Poisson distribution with mean $\Pi_\alpha(c^{-1/\alpha} B)$ for each open interval $B \subset (0,\infty)$ (see for example [6, Chapter 4]). Here, $c > 0$ is a scaling constant determined by G (and hence $I_v(\pi)$, α and $|\partial Q|$).

For $b > 0$, define

$$E(b) = \{u \in D[0,1] : \Delta u(t) > b \text{ for some } t \in [0,1]\}.$$

By the above discussion, $\mathbb{P}(W \in E(b)) = 1 - e^{-cb^{-\alpha}}$. Also $U_n \to_w W$ in $\mathcal{J}_1$ by Lemma 5, so

$$\lim_{n\to\infty} \mu(U_n \in E_n(b)) = 1 - e^{-cb^{-\alpha}}.$$

Similarly, if we suppose for contradiction that $W_n \to_w W$ in $\mathcal{M}_2$, then for any $\epsilon > 0$, $b > 0$, there exists $\delta > 0$, $n_0 \geq 1$, such that for $n \geq n_0$,

$$\mu\big\{W_n(t) - W_n(t') > b \text{ for some } 0 \leq t' < t < (t' + \delta) \wedge 1\big\} < 1 - e^{-cb^{-\alpha}} + \epsilon.$$

Now, $U_n \in E(1)$ if and only if $W_n(\varphi_{k+1}/n) - W_n(\varphi_k/n) > 1$ for some $0 \leq k \leq N_{[nt]}$. By Corollary 4, this implies for n sufficiently large that

$$W_n((\varphi_k + \ell^* \circ f^k)/n) - W_n(\varphi_k/n) \geq \frac{I_v^*}{I_v(\pi)} - \epsilon,$$

for some $0 \leq k \leq N_{[nt]}$. Hence we obtain that

$$W_n(t) - W_n(t') > \frac{I_v^*}{I_v(\pi)} - \epsilon,$$

for some $0 \leq t' < t \leq 1$ with $t - t' < n^{-1} \max_{k \leq n} \varphi \circ f^k$.

Putting these observations together, we obtain

$$\begin{aligned} 1 - e^{-c} = \mathbb{P}(W \in E(1)) &= \lim_{n\to\infty} \mu(U_n \in E(1)) \\ &\leq \lim_{n\to\infty} \mu\Big\{W_n(t) - W_n(t') > \frac{I_v^*}{I_v(\pi)} - \epsilon \text{ for some } 0 \leq t' < t < (t' + \delta) \wedge 1\Big\} \\ &\qquad + \lim_{n\to\infty} \mu\big(n^{-1} \max_{k \leq n} \varphi \circ f^k > \delta\big) \\ &< 1 - \exp\Big\{ - c\Big(\frac{I_v^*}{I_v(\pi)} - \epsilon\Big)^{-\alpha}\Big\} + \epsilon + \lim_{n\to\infty} \mu\big(n^{-1} \max_{k \leq n} \varphi \circ f^k > \delta\big). \end{aligned}$$

By Proposition 2, $1-e^{-c} \le 1-\exp\big\{-c\big(\frac{I_v^*}{I_v(\pi)}-\epsilon\big)^{-\alpha}\big\}+\epsilon$. Also, ϵ is arbitrary so

$$1-e^{-c} \le 1-\exp\Big\{-c\Big(\frac{I_v(\pi)}{I_v^*}\Big)^{\alpha}\Big\}.$$

Since $I_v^* > I_v(\pi)$, this is the desired contradiction and the proof of Theorem 1 is complete.

3 Billiards with several cusps

Our main result, Theorem 1, is formulated for the case of a single cusp but extends straightforwardly to billiards with several cusps with vanishing curvature as studied in [2]. In this section, we sketch the arguments for proving this extended result.

In particular, in [2], the billiard table Q is such that there are multiple cusps with bounding curves locally of the form

$$\Gamma_i = \{(s, C_i s^{\beta_i} + \mathcal{O}(s^{2\beta_i-1}))\}, \quad \Gamma_i' = \{(s, -C_i' s^{\beta_i} + \mathcal{O}(s^{2\beta_i-1}))\},$$

with tangent vectors $(1, \beta_i C_i s^{\beta_i-1} + \mathcal{O}(s^{2\beta_i-2}))$ and $(1, -\beta_i C_i' s^{\beta_i-1} + \mathcal{O}(s^{2\beta_i-2}))$, where $\max_i \beta_i > 2$ and where $C_i > 0$ and $C_i' \ge 0$. It is assumed that any two cusps are not diametrically opposite, in the sense that the tangent trajectory coming out of a cusp does not match up precisely with the tangent trajectory coming out of any another cusp. Let $\beta = \max_i \beta_i$. The cusps with $\beta_i < \beta$ play no role in the subsequent analysis.

For each i with $\beta_i = \beta$, define

$$I_v^{(i)}(s) = \frac{1}{2}\int_0^s \{v(r_i', \theta) + v(r_i'', \pi-\theta)\}(\sin\theta)^{1/\alpha}\, d\theta, \quad s \in [0,\pi],$$

where $\alpha = \frac{\beta}{\beta-1}$.

Following [2], we consider the first return map to a region X bounded away from all the cusps. The induced process $\widetilde{W}_n$ defined as in the proof of Lemma 5 converges to an α-stable Lévy process in the $\mathcal{J}_1$ topology by [2, Theorem 3.1]. In addition [2, Theorem 2.2] gives sufficient conditions for convergence of the full process W_n in the $\mathcal{M}_1$ topology to a rescaled α-stable Lévy process W (we refer to [2] for the definition of W).

We can now formulate necessary and sufficient conditions for convergence $W_n \to_w W$ in the $\mathcal{M}_1$ and $\mathcal{M}_2$ topology. By [4], convergence in $\mathcal{M}_1$ holds if and only if $s \mapsto I_v^{(i)}(s)$ is monotone for each i. By [4] and the argument in Section 2 of the current paper, convergence in $\mathcal{M}_2$ holds if and only if $I_v^{(i)}(s)$ lies between 0 and $I_v^{(i)}(\pi)$ for all $s \in [0,\pi]$ and for each i.

References

[1] P. Bálint, N. Chernov and D. Dolgopyat. Limit theorems for dispersing billiards with cusps. *Comm. Math. Phys.* **308** (2011) 479–510.

[2] P. Jung, F. Pène and H.-K. Zhang. Convergence to α-stable Lévy motion for chaotic billiards with several cusps at flat points. *Nonlinearity* **33** (2020) 807–839.

[3] P. Jung and H.-K. Zhang. Stable laws for chaotic billiards with cusps at flat points. *Annales Henri Poincaré* **19** (2018) 3815–3853.

[4] I. Melbourne and P. Varandas. Convergence to a Lévy process in the Skorohod $\mathcal{M}_1$ and $\mathcal{M}_2$ topologies for nonuniformly hyperbolic systems, including billiards with cusps. *Comm. Math. Phys.* **375** (2020) 653–678.

[5] I. Melbourne and R. Zweimüller. Weak convergence to stable Lévy processes for nonuniformly hyperbolic dynamical systems. *Ann Inst. H. Poincaré (B) Probab. Statist.* **51** (2015) 545–556.

[6] K.-I. Sato. *Lévy processes and infinitely divisible distributions.* Cambridge Studies in Advanced Mathematics **68**, Cambridge University Press, Cambridge, 1999. Translated from the 1990 Japanese original. Revised by the author.

[7] A. V. Skorohod. Limit theorems for stochastic processes. *Teor. Veroyatnost. i Primenen.* **1** (1956) 289–319.

[8] W. Whitt. Some useful functions for functional limit theorems. *Math. Oper. Res.* **5** (1980) 67–85.

[9] W. Whitt. *Stochastic-process limits.* Springer Series in Operations Research, Springer-Verlag, New York, 2002. An introduction to stochastic-process limits and their application to queues.

[10] H.-K. Zhang. Decay of correlations for billiards with flat points II: cusps effect. *Dynamical systems, ergodic theory, and probability: in memory of Kolya Chernov.* Contemp. Math. **698**, Amer. Math. Soc., Providence, RI, 2017, pp. 287–316.