## Supporting Information SI1

**Gottwald and Melbourne**

**SI Text 1**

### Lattice models for diffusion

As discussed in the main text, it is standard in the Physics literature to consider deterministic lattice models for diffusion. We refer to [1, 2, 3, 4] and also the survey article [5]. The advantage of this approach is that there is a straightforward correspondence between the equations, and their solutions, for the underlying models and the skew-product systems.

In particular, [1, 2, 3] consider deterministic models for diffusion and anomalous diffusion on the real line, by considering 1-periodic maps $\tilde{f} : \mathbb{R} \to \mathbb{R}$. The periodicity defines cells of length 1; the map $\tilde{f}$ may map outside the cell which causes diffusion into other cells. Such systems have discrete translation symmetry $\mathbb{Z}$. We consider extensions of their models to higher dimensions and carry out numerical simulations that confirm the predictions in the main part of our paper. There is also a reflection symmetry in the work of [1, 2, 3] that plays no role here and is suppressed throughout. (Though see the second to last paragraph in the *Summary and discussion* section of the main text.)

The class of dynamical systems with $\mathbb{Z}$ symmetry on the line is in one-to-one correspondence with the class of skew-product systems on $X \times \mathbb{Z}$ where $X = [0, 1)$. The identification $X \times \mathbb{Z} \cong \mathbb{R}$ is given by $(x, k) \mapsto x + k$. Similarly, we can write $\tilde{f}(y) \in \mathbb{R}$ as $\tilde{f}(y) = f(y) + v(y)$ where $f(y) \in X$, $v(y) \in \mathbb{Z}$.

In this way, we obtain the skew product on $X \times \mathbb{Z}$ given by $(x, k) \mapsto (f(x), k + v(x))$ where $f : X \to X$, $v : X \to \mathbb{Z}$ are given by $f(x) \equiv \tilde{f}(x) \bmod 1$ and $v(x) = \tilde{f}(x) - f(x)$.

Note that passing from $\tilde{f}$ to $(f, v)$ introduces discontinuities as does the reverse procedure. Whereas [1, 2, 3] initially specify $\tilde{f}$ and then derive $(f, v)$, we take the equivalent approach of specifying $(f, v)$ from the outset (which implies then a choice of $\tilde{f}$). This means that we can focus on the fundamental domain $X$ for the action of the symmetry group $\mathbb{Z}$ on $\mathbb{R}$. From this point of view, a convenient choice of map is to take $f$ to be the Pomeau-Manneville intermittency map from the main text (eq **[10]**) and to take $v$ to be any integer-valued map that is continuous (hence constant) and nonzero for $x$ near the neutral fixed point at zero. This corresponds exactly to the approach in [3]. The mechanism for superdiffusion in the skew product formulation is as follows: The dynamics spends very long times near the neutral fixed point for $f$, corresponding to ballistic propagation under $\tilde{f} = f + v$ along the axis. This leads to a process on $\mathbb{R}$ that is asymptotically a linear drift (typically nonzero) superimposed with Brownian motion for $\gamma < 1/2$ and a stable Lévy process for $\gamma \in (1/2, 1)$.

**Deterministic model for planar diffusion.** Proceeding to two dimensions, we replace the Euclidean group of planar rotations and translations by the discrete group $G = \mathbb{Z}_4 \ltimes \mathbb{Z}^2$ where $\mathbb{Z}^2$ consists of translations $(x_1, x_2) \mapsto (x_1 + k_1, x_2 + k_2)$ for $k_1, k_2 \in \mathbb{Z}$, and $\mathbb{Z}_4$ consists of rotations by angle $0, \pi/2, \pi, 3\pi/2$ about the origin. The action of $\mathbb{Z}_4$ on $\mathbb{R}^2$ is generated by $(y_1, y_2) \mapsto (-y_2, y_1)$. A fundamental domain for the action of $G$ on $\mathbb{R}^2$ is given by $X = [0, \frac{1}{2}) \times (0, \frac{1}{2}]$ and the identification $X \times G \cong \mathbb{R}^2$ is given by $(x, A, k) \mapsto Ax + k$ where $x \in X$, $A \in \mathbb{Z}_4$, $k \in \mathbb{Z}^2$.

Again there is a one-to-one correspondence between $G$-equivariant deterministic diffusion models on $\mathbb{R}^2$ and skew product maps on $X \times G$ of the form $(x, A, k) \mapsto (f(x), Ah(x), k + Av(x))$ where $f : X \to X$, $h : X \to \mathbb{Z}_4$, $v : X \to \mathbb{Z}^2$. To obtain strongly/weakly chaotic dynamics on $X$, a simple choice is to take

$$f(x_1, x_2) = (f_1(x_1), \tfrac{1}{2}x_2)\,, \tag{1}$$

with

$$f_1(x_1) = \begin{cases} x_1(1 + 4^\gamma x_1^\gamma), & 0 \le x_1 < \frac{1}{4} \\ 2x_1 - \frac{1}{2}, & \frac{1}{4} \le x_1 < \frac{1}{2} \end{cases}. \tag{2}$$

This map has a neutral fixed point (a nonhyperbolic saddle) at $(0, 0)$ and the dynamics is strongly/weakly chaotic for $\gamma \in [0, \frac{1}{2})$ and $\gamma \in (\frac{1}{2}, 1)$ respectively.

In the strongly chaotic case, we predict normal diffusion. In the anisotropic case (so $h \equiv I_2$) this will be superimposed on a linear drift; in the isotropic case where rotation symmetry is present, typically the drift term will vanish.

In the weakly chaotic case, we predict superdiffusion superimposed on a linear drift in the anisotropic case. In the isotropic case, again the linear drift vanishes but moreover we predict that the anomalous diffusion is suppressed in favour of Brownian motion. These predictions are borne out by the numerical experiments described below.

**Deterministic model for three-dimensional diffusion.** Next, we consider the three-dimensional case, replacing the Euclidean group of rotations and translations by the discrete group $G = \mathbb{O} \ltimes \mathbb{Z}^3$ where $\mathbb{Z}^3$ consists of translations $(x_1, x_2, x_3) \mapsto (x_1 + k_1, x_2 + k_2, x_3 + k_3)$ for $k_1, k_2, k_3 \in \mathbb{Z}$, and $\mathbb{O}$ is the 24 element group consisting of rotation symmetries of the cube. The action of $\mathbb{O}$ on $\mathbb{R}^3$ is generated by $(x_1, x_2, x_3) \mapsto (-x_2, x_1, x_3)$ and $(x_1, x_2, x_3) \mapsto (x_1, -x_3, x_2)$. A fundamental domain for the action of $G$ on $\mathbb{R}^3$ is given by $X = \{x \in [0, \frac{1}{2}]^2 : x_2 \le x_1, x_3 \le x_1\}$ (we choose to be imprecise with regard to the boundaries; this is unimportant since the dynamics sees the boundary only for a set of initial conditions of measure zero) and the identification $X \times G \cong \mathbb{R}^3$ is given by $(x, A, k) \mapsto Ax + k$ where $x \in X$, $A \in \mathbb{O}$, $k \in \mathbb{Z}^3$.

Once again there is a one-to-one correspondence between $G$-equivariant deterministic diffusion models on $\mathbb{R}^3$ and skew product maps on $X \times G$ of the form $(x, A, k) \mapsto (f(x), Ah(x), k + Av(x))$ where $f : X \to X$, $h : X \to \mathbb{O}$, $v : X \to \mathbb{Z}^3$. An example of a map that generates strongly/weakly chaotic dynamics is

$$f(x_1, x_2, x_3) = \begin{cases} \begin{pmatrix} x_1(1 + 4^\gamma x_1^\gamma) \\ \frac{1}{2}x_2 \\ \frac{1}{2}x_3 \end{pmatrix} & 0 \le x_1 < \frac{1}{4} \\[2ex] \begin{pmatrix} 2x_1 - \frac{1}{2} \\ \min(2x_1 - \frac{1}{2}, \frac{1}{2}x_2) \\ \min(2x_1 - \frac{1}{2}, \frac{1}{2}x_3)) \end{pmatrix} & \frac{1}{4} \le x_1 < \frac{1}{2} \end{cases}.$$

In the strongly chaotic case $\gamma \in [0, \frac{1}{2})$ and in the anisotropic case, our predictions are the same as in two dimensions. However for weak chaos $\gamma \in (\frac{1}{2}, 1)$ in the isotropic case, we predict that the anomalous diffusion persists despite the rotation symmetry and that there is a stable Lévy process.

**Numerical experiment.** We carry out a numerical verification of our predictions in the case of weakly chaotic dynamics for two-dimensional systems. For the base dynamics $f : X \to X$, we use

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the map defined in (1), (2). In the numerics, we compute a dynamical orbit $(x_n, A_n, k_n)$ in the skew product and plot the sequence of points $y_n = A_n x_n + k_n$ on the $\mathbb{R}^2$-plane. We do this for both the anisotropic case ($h \equiv I_2$) and the isotropic case (where we choose $h$ to be rotation by $\pi/2$ independent of $x$). In both cases, we take $v = (v_1, v_2)$ with

$$v_1(x) = \begin{cases} 1 & 0 \le x_1 \le 0.15 \\ -2 & 0.15 < x_1 \le 0.5 \end{cases}$$

$$v_2(x) = \begin{cases} 3 & 0 \le x_1 \le 0.33 \\ 1 & 0.33 < x_1 \le 0.5 \end{cases}$$

The results for the anisotropic and isotropic cases are shown in Figures 1 and 2 respectively confirming our theoretical results. In the anisotropic case, the Lévy process is completely anti-symmetric for $f$ an intermittent map with a single neutral fixed point (just as in the one dimension case (see Figure 2 of the main text). Hence the Lévy flights are concentrated along a single direction in the plane.

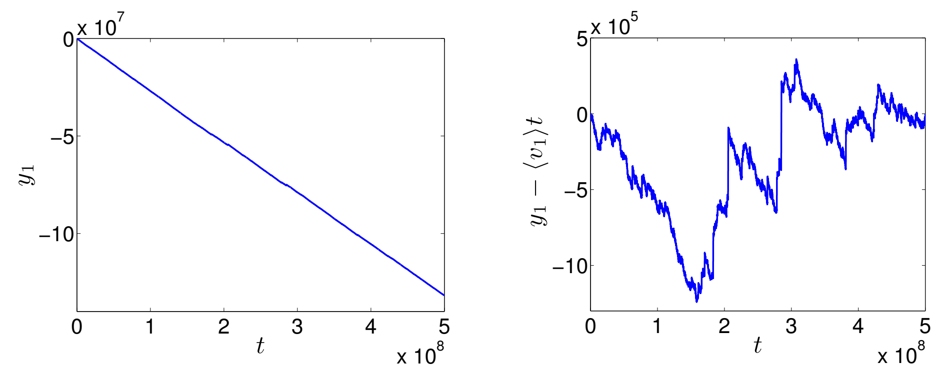


**Fig. 1.** Anisotropic case: Coordinate $y_1$ as a function of time for a $\mathbb{Z}^2$ skew product driven by the Pomeau-Manneville map (1), (2) with $\gamma = 0.7$. Shown are the full dynamics including the linear drift (left) and with the linear drift eliminated by subtracting the mean from the data (right).

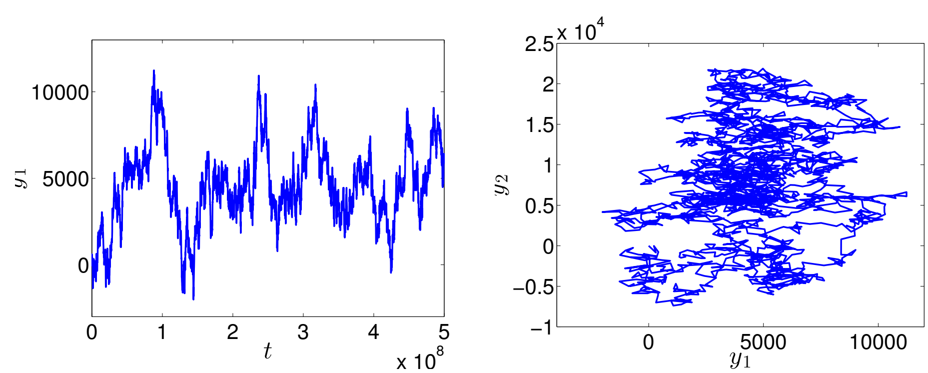


**Fig. 2.** Isotropic case: Coordinates $y_1$ (left) and $(y_1, y_2)$ (right) as functions of time for a $\mathbb{Z}_4 \ltimes \mathbb{Z}^2$ skew product driven by the Pomeau-Manneville map (1), (2) with $\gamma = 0.7$.

1. Geisel T, Nierwetberg J (1982) Onset of diffusion and universal scaling in chaotic systems. *Phys. Rev. Lett.* 48:7–10.
2. Geisel T, Thomae S (1984) Anomalous diffusion in intermittent chaotic systems. *Phys. Rev. Lett.* 52:1936–1939.
3. Geisel T, Nierwetberg J, Zacherl A (1985) Accelerated diffusion in Josephson junctions and related chaotic systems. *Phys. Rev. Lett.* 54:16–19.
4. Pierrehumbert RT (2000) Lattice models of advection-diffusion. *Chaos* 10:61–74.
5. Klages R (2013) Weak chaos, infinite ergodic theory, and anomalous dynamics. In *From Hamiltonian Chaos to Complex Systems*, eds Leoncini X, Leonetti M (Springer, Berlin).