Good inducing schemes for uniformly hyperbolic maps

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18 June 2025

Abstract

These notes are not intended for publication. However, it seemed useful to have the arguments in Section 6 of L.-S. Young, Statistical properties of dynamical systems with some hyperbolicity, *Ann. of Math.* **147** (1998) 585–650 written in a simplified form, in preparation for the case of flows (I. Melbourne and P Varandas, Good inducing schemes for uniformly hyperbolic flows, and applications to exponential decay of correlations, *Ann. Henri Poincaré* **26** (2025) 921–945.) One definition is simplified, therefore bypassing an unclear argument.

1 Uniformly expanding maps

Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a C^{1+} map[1] and let $\Lambda \subset \mathbb{R}^d$ be a compact f-invariant subset with $\overline{\operatorname{Int}\Lambda} = \Lambda$. We assume that $f : \Lambda \to \Lambda$ is a transitive uniformly expanding map with adapted norm $|\ |$, so there exists $\lambda \in (0, 1)$ such that $|Df(x)^{-1}| \le \lambda$ for all $x \in \Lambda$.

Theorem 1.1 (Young [3]) *There exists an open disk $Y \subset \operatorname{Int}\Lambda$ and a function $R : Y \to \mathbb{Z}^+ \cup \{\infty\}$ such that*

(i) $\operatorname{Leb}(R > n) = O(\gamma^n)$ *for some $\gamma \in (0, 1)$;*

(ii) Each connected component of $\{R = n\}$ is mapped diffeomorphically by f^n onto Y.[2]

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[1]A map is C^{1+} if it is C^1 and Df is Hölder continuous.

[2]This is in the sense that f^n is defined on an open neighbourhood of the connected component, which gets mapped diffeomorphically by f^n onto an open neighbourhood of Y.

Remark 1.2 Let $\mathcal{P}$ be the partition of Y consisting of connected components of $\{R = n\}$ for $n \geq 1$. (It follows from Theorem 1.1(i) that $\mathcal{P}$ is a partition of Y mod 0.) The induced map $F = f^R : Y \to Y$ is a full-branch Gibbs-Markov map with partition $\mathcal{P}$ and the partition elements are C^{1+}-diffeomorphic to disks.

Our proof of Theorem 1.1 is essentially the same as in [3, Section 6] but we closely follow the treatment in [1] which provides many of the details of arguments sketched in [3]. Moreover, the definition of U^L_{nj} below is slightly modified, leading to a simplified proof, see Remark 1.5.

Remark 1.3 The combinatorics in the proof of Theorem 1.1 are identical to those in [3, Section 6], [1] and [2]. Generally, the accompanying arguments are simpler here since the setting of uniformly expanding maps is simpler. The only aspect that is more complicated (also in [1]) is the need to consider invertibility domains since the dynamics is not invertible.

Choice of constants Let $B_r(y) = \{z \in \Lambda : |y - z| < r\}$ for $y \in \Lambda$, $r > 0$. Choose $\delta_0 > 0$ corresponding to invertibility domains. This means that for any $y \in \Lambda$, $x \in f^{-1}y$, there exists an inverse branch

$$f_x^{-1} : B_{\delta_0}(y) \to f_x^{-1} B_{\delta_0}(y) \subset B_{\delta_0}(x).$$

As f is uniformly expanding and $\log|\det Df|$ is Hölder continuous, we can choose δ_0 so that the following bounded distortion property holds: there exists $C_1 \geq 1$ so that

$$\frac{|\det Df^n(x)|}{|\det Df^n(y)|} \leq C_1 \tag{1.1}$$

for every $n \geq 1$ and all $x, y \in \Lambda$ such that $|f^j x - f^j y| < 4\delta_0$ for all $0 \leq j \leq n$. Fix $L > 11$ sufficiently large that

$$C_1 \frac{2^d - 1}{(L-1)^d} < \frac{1}{4} \tag{1.2}$$

and set $\delta = \delta_0/(L+1)$. By the assumptions on f, we can choose $p \in \operatorname{Int}\Lambda$ such that $\bigcup_{i\geq 1} f^{-i}p$ is dense in Λ. If necessary, shrink δ_0 so that $B_{L\delta}(p) \subset \Lambda$. Choose $N_1 \geq 1$ such that $\bigcup_{i=1}^{N_1} f^{-i}p$ is δ-dense in Λ. Finally, fix

$$\varepsilon \in (0, \delta_0/4) \cap (0, \delta(\lambda^{-1} - 1)). \tag{1.3}$$

Construction of the partition We consider various small neighbourhoods $\mathcal{D}_e = B_{e\delta}(p)$ with $e \in \{1, 2, L-1, L\}$. Take $Y = \mathcal{D}_1$ to be the inducing set. Define a partition $\{I_k : k \geq 1\}$ of $\mathcal{D}_2 \setminus \mathcal{D}_1$,

$$I_k = \big\{ y \in \mathcal{D}_2 : \delta(1 + \lambda^k) \leq |y - p| < \delta(1 + \lambda^{k-1}) \big\}.$$

We define sets Y_n and functions $t_n : Y_n \to \mathbb{N}$, and $R : Y \to \mathbb{Z}^+$ inductively, with $Y_n = \{R > n\}$. Define $Y_0 = Y$ and $t_0 \equiv 0$. Inductively, suppose that $Y_{n-1} = Y \setminus \{R < n\}$ and that $t_{n-1} : Y_{n-1} \to \mathbb{N}$ is given. Write $Y_{n-1} = A_{n-1} \dot\cup B_{n-1}$ where

$$A_{n-1} = \{t_{n-1} = 0\}, \qquad B_{n-1} = \{t_{n-1} \geq 1\}.$$

Consider the neighbourhood

$$A_{n-1}^{(\varepsilon)} = \big\{y \in Y_{n-1} : \mathrm{dist}(f^n y, f^n A_{n-1}) < \varepsilon\big\}$$

of the set A_{n-1} and define U_{nj}^L, $j \geq 1$, to be the connected components of $A_{n-1}^{(\varepsilon)} \cap f^{-n}\mathcal{D}_L$ that are mapped diffeomorphically onto $\mathcal{D}_L$ and satisfy

$$(f^n | U_{nj}^L)^{-1}(\mathcal{D}_{L-1}) \subset A_{n-1}.$$

Let

$$U_{nj}^e = U_{nj}^L \cap f^{-n}\mathcal{D}_e \quad \text{for } e = 1, 2, L-1.$$

(By construction, $U_{nj}^{L-1} \subset A_{n-1}$.) Define $R|U_{nj}^1 = n$ for each U_{nj}^1 and take $Y_n = Y_{n-1} \setminus \bigcup_j U_{nj}^1$. Finally, define $t_n : Y_n \to \mathbb{N}$ as

$$t_n(y) = \begin{cases} k, & y \in \bigcup_j U_{nj}^2 \text{ and } f^n y \in I_k \text{ for some } k \geq 1 \\ 0, & y \in A_{n-1} \setminus \bigcup_j U_{nj}^2 \\ t_{n-1}(y) - 1, & y \in B_{n-1} \end{cases}$$

and take $A_n = \{t_n = 0\}$, $B_n = \{t_n \geq 1\}$ and $Y_n = A_n \dot\cup B_n$.

Remark 1.4 By construction, property (ii) of Theorem 1.1 is satisfied. It remains to verify that $\mathrm{Leb}(R > n)$ decays exponentially.

Remark 1.5 Our definition of U_{nj}^L is different from what is written in [3]: we stipulate that U_{nj}^{L-1} is contained in A_{n-1} whereas in [3] (see also [1]) it is seen as being an immediate consequence of the other definitions for ε sufficiently small. As pointed out in [2, Figure 2], this is not particularly obvious. In [2, Proposition 2.4], we attempted to write a proof but this is also flawed. In the end, it turns out that the slightly modified definition given here (which may well be what was intended in the first place) sidesteps the issue altogether.

Visualisation of B_n. The set B_n is a disjoint union $B_n = \bigcup_{m=1}^n C_n(m)$ where $C_n(m)$ is a disjoint union of *collars* around each component of $\{R = m\}$. Each collar in $C_n(m)$ is homeomorphic under f^m to $\bigcup_{k \geq n-m+1} I_k$ with outer ring homeomorphic under f^m to I_{n-m+1}, and the union of outer rings is the set $\{t_n = 1\}$. Since $U_{nj}^2 \subset U_{nj}^{L-1} \subset A_{n-1}$, each new generation of collars $C_n(n)$ does not intersect the set $\bigcup_{1 \leq m \leq n-1} C_{n-1}(m)$ of collars in the previous generations. A sample visualisation after 7 generations is shown in Figure 1.

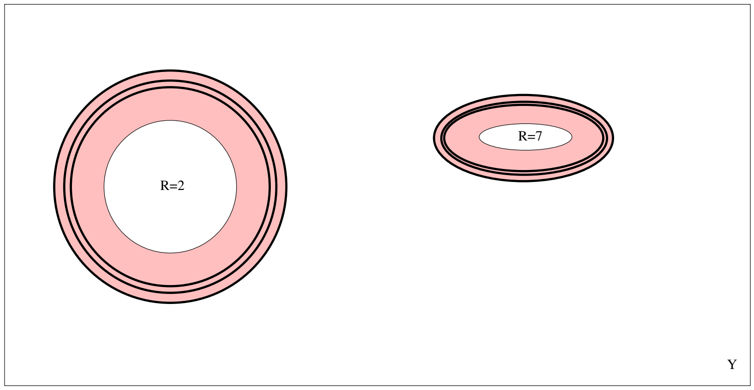

Figure 1: Visualisation of Y after 7 generations where there is one return at time 2 and one return at time 7. The pink region B_7 consists of collars around the sets $\{R = 2\}$ and $\{R = 7\}$ that have made a return. The two outermost shells $\{t_7 = 1\}$ and $\{t_7 = 2\}$ of each collar are shown. The collars in B_7 are diffeomorphic by f^2 and f^7 respectively to an annulus; in reality the collar around $\{R = 2\}$ should be slightly distorted and the collar around $\{R = 7\}$ more so (and smaller).

Proposition 1.6 *For all $n \geq 1$,*

(a) $A^{(\varepsilon)}_{n-1} \subset \{y \in Y_{n-1} : t_{n-1}(y) \leq 1\}$; and

(b) $f^{-n}B_\varepsilon(f^n x) \subset A^{(\varepsilon)}_{n-1}$ for all $x \in A_{n-1}$.

Proof (a) Suppose for contradiction that $y \in A^{(\varepsilon)}_{n-1}$ but $t_{n-1}(y) \geq 2$. In particular, y is contained in a collar in $C_{n-1}(n-k)$ from the $(n-k)$'th generation for some $k \geq 1$. Let Q denote the outer ring of this collar with outer boundary Q_1 and inner boundary Q_2. Then $t_{n-1}|Q \equiv 1$ and $t_{n-1}(y) \geq 2$, so y lies inside the region bounded by Q_2.

Since $y \in A^{(\varepsilon)}_{n-1}$, we can choose $x \in A_{n-1}$ with $|f^n x - f^n y| < \varepsilon$. Let ℓ be the line segment connecting $f^n x$ to $f^n y$ and define $q_j \in Q_j \cap f^{-n}\ell$ for $j = 1, 2$.

Recall that Q is homeomorphic under f^{n-k} to I_k. Moreover, $f^{n-k}q_j$ lie in distinct components of the boundary of I_k, so by (1.3)

$$|f^{n-k}q_1 - f^{n-k}q_2| \geq \delta(\lambda^{k-1} - \lambda^k) = \delta(\lambda^{-1} - 1)\lambda^k > \varepsilon\lambda^k.$$

Hence

$$|f^n q_1 - f^n q_2| \geq \lambda^{-k}|f^{n-k}q_1 - f^{n-k}q_2| > \varepsilon.$$

But $|f^n q_1 - f^n q_2| \leq |f^n y - f^n x| < \varepsilon$ so we obtain the desired contradiction.

(b) Let $x \in A_{n-1}$ and $y \in f^{-n}B_\varepsilon(f^n x)$. Note that $y \in A^{(\varepsilon)}_{n-1}$ if and only if $y \in Y_{n-1}$. Hence we must show that $y \in Y_{n-1}$. If not, then there exists $k \geq 1$ such that

$y \in \{R = n-k\}$. Define $Q \subset C_{n-1}(n-k)$ to be the outer ring of the corresponding collar. Choosing q_1 and q_2 as in part (a) we again obtain a contradiction. ∎

Lemma 1.7 *There exists $a_1 > 0$ such that for all $n \geq 1$,*

(a) $\mathrm{Leb}(B_{n-1} \cap A_n) \geq a_1 \,\mathrm{Leb}(B_{n-1})$.

(b) $\mathrm{Leb}(A_{n-1} \cap B_n) \leq \frac{1}{4}\,\mathrm{Leb}(A_{n-1})$.

(c) $\mathrm{Leb}(A_{n-1} \cap \{R = n\}) \leq \frac{1}{4}\,\mathrm{Leb}(A_{n-1})$.

Proof (a) By construction, $B_{n-1} \cap A_n = \{t_{n-1} = 1\}$. Let $Q \subset C_{n-1}(n-k) \subset B_{n-1}$ be a collar $(1 \leq k \leq n)$ with outer ring $Q \cap A_n = Q \cap \{t_{n-1} = 1\}$. Then f^{n-k} maps Q diffeomorphically onto $\bigcup_{i \geq k} I_i$ and $Q \cap \{t_{n-1} = 1\}$ diffeomorphically onto I_k. In particular, $\mathrm{diam}\, f^j Q \leq \mathrm{diam}\, f^{n-k} Q \leq \mathrm{diam}\, \mathcal{D}_2 = 4\delta < 4\delta_0$ for all $0 \leq j \leq n-k$. By (1.1),

$$\frac{\mathrm{Leb}(Q)}{\mathrm{Leb}(Q \cap A_n)} = \frac{\mathrm{Leb}(Q)}{\mathrm{Leb}(Q \cap \{t_{n-1} = 1\})} \leq C_1 \frac{\mathrm{Leb}(\bigcup_{i \geq k} I_i)}{\mathrm{Leb}(I_k)} = C_1 D(d, \lambda, k)$$

where $D(d,\lambda,k) = \dfrac{(1+\lambda^{k-1})^d - 1}{(1+\lambda^{k-1})^d - (1+\lambda^k)^d}$. Since $\lim_{k\to\infty} D(d,\lambda,k) = (1-\lambda)^{-1}$, we obtain that $\mathrm{Leb}(Q) \leq C_1 D \,\mathrm{Leb}(Q \cap A_n)$ where $D = \sup_{k \geq 1} D(d,\lambda,k)$ is a constant depending only on d and λ. Summing over collars Q, it follows that $\mathrm{Leb}(B_{n-1}) \leq C_1 D\,\mathrm{Leb}(B_{n-1} \cap A_n)$.

(b) By construction, $U_{nj}^2 \subset U_{nj}^{L-1} \subset A_{n-1}$ for each j. It follows that $A_{n-1} \cap B_n = \bigcup_j U_{nj}^2 \setminus U_{nj}^1$. Again, $\mathrm{diam}\, f^m U_{nj}^{L-1} \leq \mathrm{diam}\, f^n U_{nj}^{L-1} = \mathrm{diam}\, \mathcal{D}_{L-1} \leq 2(L-1)\delta < 2\delta_0$ for all $0 \leq m \leq n-k$, so by (1.1) and (1.2),

$$\frac{\mathrm{Leb}(U_{nj}^2 \setminus U_{nj}^1)}{\mathrm{Leb}(U_{nj}^{L-1})} \leq C_1 \frac{\mathrm{Leb}(\mathcal{D}_2 \setminus \mathcal{D}_1)}{\mathrm{Leb}(\mathcal{D}_{L-1})} = C_1 \frac{2^d - 1}{(L-1)^d} < \frac{1}{4}.$$

Hence

$$\frac{\mathrm{Leb}(A_{n-1} \cap B_n)}{\mathrm{Leb}(A_{n-1})} \leq \frac{\sum_j \mathrm{Leb}(U_{nj}^2 \setminus U_{nj}^1)}{\sum_j \mathrm{Leb}(U_{nj}^{L-1})} < \frac{1}{4}.$$

(c) Proceeding as in part (b) with $U_{nj}^2 \setminus U_{nj}^1$ replaced by U_{nj}^1, leads to the estimate

$$\frac{\mathrm{Leb}(A_{n-1} \cap \{R = n\})}{\mathrm{Leb}(A_{n-1})} \leq \frac{\sum_j \mathrm{Leb}(U_{nj}^1)}{\sum_j \mathrm{Leb}(U_{nj}^{L-1})} \leq \frac{C_1}{(L-1)^d} < \frac{1}{4}.$$ ∎

Corollary 1.8 *For all $n \geq 1$,*

(a) $\mathrm{Leb}(A_{n-1} \cap A_n) \geq \frac{1}{2}\,\mathrm{Leb}(A_{n-1})$.

(b) $\mathrm{Leb}(B_{n-1}\cap B_n)\le(1-a_1)\,\mathrm{Leb}(B_{n-1})$.

(c) $\mathrm{Leb}(B_n)\le\frac14\,\mathrm{Leb}(A_{n-1})+(1-a_1)\,\mathrm{Leb}(B_{n-1})$.

(d) $\mathrm{Leb}(A_n)\ge\frac12\,\mathrm{Leb}(A_{n-1})+a_1\,\mathrm{Leb}(B_{n-1})$.

Proof Recall that $A_{n-1}\subset Y_{n-1}=Y_n\,\dot\cup\,\{R=n\}=A_n\,\dot\cup\,B_n\,\dot\cup\,\{R=n\}$. Hence by Lemma 1.7(b,c),

$$\begin{aligned}\mathrm{Leb}(A_{n-1})&=\mathrm{Leb}(A_{n-1}\cap A_n)+\mathrm{Leb}(A_{n-1}\cap B_n)+\mathrm{Leb}(A_{n-1}\cap\{R=n\})\\&\le\mathrm{Leb}(A_{n-1}\cap A_n)+\tfrac12\,\mathrm{Leb}(A_{n-1}),\end{aligned}$$

proving (a). Similarly, by Lemma 1.7(a),

$$\begin{aligned}\mathrm{Leb}(B_{n-1})&=\mathrm{Leb}(B_{n-1}\cap A_n)+\mathrm{Leb}(B_{n-1}\cap B_n)+\mathrm{Leb}(B_{n-1}\cap\{R=n\})\\&\ge a_1\,\mathrm{Leb}(B_{n-1})+\mathrm{Leb}(B_{n-1}\cap B_n),\end{aligned}$$

proving (b).

Next, recall that $B_n=B_n\cap Y_{n-1}=B_n\cap\big(A_{n-1}\,\dot\cup\,B_{n-1}\big)$. Hence part (c) follows from Lemma 1.7(b) and part (b). Similarly, $A_n=A_n\cap\big(A_{n-1}\,\dot\cup\,B_{n-1}\big)$ and part (d) follows from Lemma 1.7(a) and part (a). ∎

Corollary 1.9 *There exists $a_0>0$ such that $\mathrm{Leb}(B_n)\le a_0\,\mathrm{Leb}(A_n)$ for all $n\ge0$.*

Proof Let $a_0=\dfrac{2+a_1}{2a_1}$. We prove the result by induction. The case $n=0$ is trivial since $B_0=\emptyset$. For the induction step from $n-1$ to n, we consider separately the cases $\mathrm{Leb}(B_{n-1})>\frac{1}{2a_1}\,\mathrm{Leb}(A_{n-1})$ and $\mathrm{Leb}(B_{n-1})\le\frac{1}{2a_1}\,\mathrm{Leb}(A_{n-1})$.

Suppose first that $\mathrm{Leb}(B_{n-1})>\frac{1}{2a_1}\,\mathrm{Leb}(A_{n-1})$. By Corollary 1.8(c),

$$\mathrm{Leb}(B_n)<\big\{\tfrac12a_1+(1-a_1)\big\}\,\mathrm{Leb}(B_{n-1})=(1-\tfrac12a_1)\,\mathrm{Leb}(B_{n-1})<\mathrm{Leb}(B_{n-1}).$$

By Corollary 1.8(d),

$$\mathrm{Leb}(A_n)>(\tfrac12+a_1\tfrac{1}{2a_1})\,\mathrm{Leb}(A_{n-1})=\mathrm{Leb}(A_{n-1}).$$

Hence by the induction hypothesis,

$$\mathrm{Leb}(B_n)<\mathrm{Leb}(B_{n-1})\le a_0\,\mathrm{Leb}(A_{n-1})<a_0\,\mathrm{Leb}(A_n),$$

establishing the result at time n.

Finally, suppose that $\mathrm{Leb}(B_{n-1})\le\frac{1}{2a_1}\,\mathrm{Leb}(A_{n-1})$. By Corollary 1.8(a,c),

$$\begin{aligned}\mathrm{Leb}(B_n)&\le\tfrac14\,\mathrm{Leb}(A_{n-1})+\mathrm{Leb}(B_{n-1})\le(\tfrac14+\tfrac{1}{2a_1})\,\mathrm{Leb}(A_{n-1})\\&\le(\tfrac12+\tfrac{1}{a_1})\,\mathrm{Leb}(A_n)=a_0\,\mathrm{Leb}(A_n),\end{aligned}$$

completing the proof. ∎

Lemma 1.10 *There exist $c_1 > 0$ and $N \geq 1$ such that*

$$\mathrm{Leb}\Big(\bigcup_{i=0}^{N}\{R = n+i\}\Big) \geq c_1\,\mathrm{Leb}(A_{n-1}) \quad \textit{for all } n \geq 1.$$

Proof Fix $\lambda \in (0,1)$, $L > 11$, $\delta > 0$, $\delta_0 = (L+1)\delta$, $N_1 \geq 1$ and $\varepsilon > 0$ as defined from the outset. Choose $N_2 \geq 1$ such that $\lambda^{N_2} < \varepsilon/\delta_0$ and take $N = N_1 + N_2$.

We claim that

(*) For all $z \in \Lambda$, there exists $i \in \{1, \dots, N_1\}$ such that $f^{i+N_2}B_\varepsilon(z) \supset \mathcal{D}_L$.

Fix $z \in \Lambda$. By the definition of N_1, there exists $1 \leq i \leq N_1$ and $q \in f^{-i}p$ such that $|q - f^{N_2}z| < \delta$. Let $y \in \mathcal{D}_L$. By the choice of δ_0 and backward contraction, there exists $x \in f^{-i}y$ such that $|x - q| \leq |f^i x - f^i q| = |y - p| < L\delta$. Hence

$$|x - f^{N_2}z| \leq |x - q| + |q - f^{N_2}z| < (L+1)\delta = \delta_0.$$

By the definition of N_2,

$$y = f^i x \in f^i B_{\delta_0}(f^{N_2}z) \subset f^{i+N_2}B_\varepsilon(z).$$

This means that $\mathcal{D}_L \subset f^{i+N_2}B_\varepsilon(z)$ proving (*).

Next, we claim that

(**) For all $z \in f^n A_{n-1}$, $n \geq 1$, there exist $i \in \{0, \dots, N\}$ and j such that $U^1_{n+i,j} \subset f^{-n}B_{\delta_0/2}(z)$.

To prove (**), choose $x \in A_{n-1}$ with $f^n x = z$. Fix an invertibility domain V_ε with $z \in V_\varepsilon \subset \Lambda$, diffeomorphic under f^n to $B_\varepsilon(z)$.

By Proposition 1.6(b), $V_\varepsilon \subset f^{-n}B_\varepsilon(f^n x) \subset A^{(\varepsilon)}_{n-1}$. We now consider two possible cases.

Suppose first that $V_\varepsilon \subset A_{n+i}$ for all $0 \leq i \leq N$. By claim (*), there exists $1 \leq i \leq N = N_1 + N_2$ such that $f^{n+i}V_\varepsilon = f^i B_\varepsilon(z) \supset \mathcal{D}_L$, while $V_\varepsilon \subset A_{n+i-1}$ by assumption. This means that $V_\varepsilon \supset U^L_{n+i,j}$ for some j. Hence

$$U^1_{n+i,j} \subset U^L_{n+i,j} \subset V_\varepsilon \subset f^{-n}B_\varepsilon(z) \subset f^{-n}B_{\delta_0/4}(z),$$

and we are done.

In this way, we reduce to the second case where there exists $0 \leq i \leq N$ least such that $V_\varepsilon \not\subset A_{n+i}$. Since i is least, $V_\varepsilon \subset A^{(\varepsilon)}_{n+i-1}$. (The ε is required in case $i = 0$.) By Proposition 1.6(a), $V_\varepsilon \subset \{t_{n+i-1} \leq 1\}$. Hence

$$\begin{aligned} V_\varepsilon \setminus A_{n+i} &= (V_\varepsilon \cap B_{n+i}) \cup (V_\varepsilon \cap \{R = n+i\}) \\ &\subset \{t_{n+i-1} \leq 1,\, t_{n+i} \geq 1\} \cup \{R = n+i\} \subset \bigcup_j U^2_{n+i,j}. \end{aligned}$$

Since $V_\varepsilon \setminus A_{n+i} \neq \emptyset$, this means that there exists j so that V_ε intersects $U^2_{n+i,j}$.

Recall that $f^{n+i}U^e_{n+i,j} = \mathcal{D}_e$ for $e = 1, 2$. In particular, $f^i B_\varepsilon(z) = f^{n+i}V_\varepsilon$ intersects $\mathcal{D}_2$. Choose $a \in f^i B_\varepsilon(z) \cap \mathcal{D}_2$ and let $b \in \mathcal{D}_1$. Since $|a - b| < 3\delta < \delta_0$, we can choose preimages a',b' of a,b under f^i such that $|a' - b'| \leq |a - b| < 3\delta = 3\delta_0/(L + 1) < \delta_0/4$ and $|a' - z| < \varepsilon < \delta_0/4$. It follows that $|b' - z| < \delta_0/2$ and so $b \in f^i B_{\delta_0/2}(z)$. This proves that $\mathcal{D}_1 \subset f^i B_{\delta_0/2}(z)$. Hence $U^1_{n+i,j} \subset f^{-(n+i)}\mathcal{D}_1 \subset f^{-n}B_{\delta_0/2}(z)$ verifying claim (**).

We are now in a position to complete the proof of the lemma. Let $n \geq 1$, and let $Z \subset f^n A_{n-1}$ be a maximal set of points such that the balls $B_{\delta_0/2}(z)$ are disjoint for $z \in Z$. If $x \in f^n A_{n-1}$, then $B_{\delta_0/2}(x)$ intersects at least one $B_{\delta_0/2}(z)$, $z \in Z$, by maximality of the set Z. Hence $f^n A_{n-1} \subset \bigcup_{z\in Z} B_{\delta_0}(z)$. It follows that

$$A_{n-1} \subset \bigcup_{z\in Z} f^{-n}B_{\delta_0}(z).$$

Let $z \in Z$ and let $U_z = U^1_{n+i,j}$ be as in claim (**). In particular, $f^{n+i}U_z = \mathcal{D}_1 = B_\delta(p)$. Hence

$$\mathrm{Leb}(B_\delta(p)) = \mathrm{Leb}(f^{n+i}U_z) \leq |Df|^{id}_\infty \,\mathrm{Leb}(f^n U_z).$$

By (1.1),

$$\frac{\mathrm{Leb}(f^{-n}B_{\delta_0}(z))}{\mathrm{Leb}(U_z)} \leq C_1 \frac{\mathrm{Leb}(B_{\delta_0}(z))}{\mathrm{Leb}(f^n U_z)} \leq K,$$

where $K = C_1|Df|^{Nd}_\infty \dfrac{\mathrm{Leb}(B_{\delta_0}(z))}{\mathrm{Leb}(B_\delta(p))} = C_1|Df|^{Nd}_\infty(\delta_0/\delta)^d$.

Finally, the sets U_z are connected components of $\bigcup_{0\leq i\leq N}\{R = n + i\}$ lying in distinct disjoint sets $f^{-n}B_{\delta_0/2}(z)$. Hence

$$\mathrm{Leb}(A_{n-1}) \leq \sum_{z\in Z} \mathrm{Leb}(f^{-n}B_{\delta_0}(z)) \leq K \sum_{z\in Z} \mathrm{Leb}(U_z) \leq K\,\mathrm{Leb}\Big(\bigcup_{0\leq i\leq N}\{R = n + i\}\Big),$$

as required. ∎

We can now complete the proof of Theorem 1.1.

Corollary 1.11 $\mathrm{Leb}(R > n) = O(\gamma^n)$ *for some* $\gamma \in (0, 1)$.

Proof By Corollary 1.9 and Lemma 1.10,

$$\begin{aligned}\mathrm{Leb}(R \geq n) &= \mathrm{Leb}(A_{n-1}) + \mathrm{Leb}(B_{n-1}) \\ &\leq (1 + a_0)\,\mathrm{Leb}(A_{n-1}) \leq d_2\,\mathrm{Leb}\Big(\bigcup_{i=0}^{N}\{R = n + i\}\Big)\end{aligned}$$

where $d_2 = c_1^{-1}(1 + a_0)$. It follows that

$$\begin{aligned}d_2^{-1}\,\mathrm{Leb}(R \geq n) &\leq \mathrm{Leb}(R = n) + \cdots + \mathrm{Leb}(R = n + N) \\ &= \mathrm{Leb}(R \geq n) - \mathrm{Leb}(R > n + N).\end{aligned}$$

Hence

$$\mathrm{Leb}(R > n + N) \le (1 - d_2^{-1})\,\mathrm{Leb}(R \ge n).$$

In particular, $\mathrm{Leb}(R > kN) \le \gamma^{kN}$ with $\gamma = (1 - d_2^{-1})^{1/N}$ and the result follows. ∎

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