Nonexistence of spectral gaps in Hölder spaces for continuous time dynamical systems

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Abstract

We show that there is a natural restriction on the smoothness of spaces where the transfer operator for a continuous dynamical system has a spectral gap. Such a space cannot be embedded in a Hölder space with Hölder exponent greater than $\frac{1}{2}$ unless it consists entirely of coboundaries.

1 Introduction

Decay of correlations (rates of mixing) and strong statistical properties are well-understood for Axiom A diffeomorphisms since the work of [2, 9, 10]. Mixing rates are computed with respect to any equilibrium measure with Hölder potential. Up to a finite cycle, such diffeomorphisms have exponential decay of correlations for Hölder observables. In the one-sided (uniformly expanding) setting, this is typically proved by establishing quasicompactness and a spectral gap for the associated transfer operator L. Such a spectral gap yields a decay rate $\|L^n v - \int v\| \le C_v e^{-an}$ for v Hölder, where $\|\ \|$ is a suitable Hölder norm and a, C_v are positive constants. Decay of correlations for Hölder observables is an immediate consequence of the decay for L^n. This philosophy has been extended to large classes of nonuniformly expanding dynamical systems with exponential [13] and subexponential decay of correlations [14].

For continuous time dynamical systems, the usual techniques [5, 7, 8] bypass spectral gaps; the only exceptions that we know of being Tsujii [11, 12]. However, the result in [11] is for suspension semiflows over the doubling map with a C^3 roof function, where the smoothness of the roof function is crucial and very restrictive. A spectral gap for contact Anosov flows is obtained in [12]; unfortunately it seems nontrivial to extend this to nonuniformly hyperbolic contact flows (or uniformly hyperbolic

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contact flows with unbounded distortion), see [1] which proves exponential decay of correlations for billiard flows with a contact structure but does not establish a spectral gap. Indeed, apart from [11, 12], there are no results on spectral gaps of transfer operators for semiflows and flows.

The results of Tsujii [11, 12] provide a spectral gap in an anisotropic Banach space. In this paper we obtain a restriction on the Banach spaces that can yield a spectral gap. We work in the following general setting:

Let (Λ, d) be a bounded metric space with Borel probability measure μ, and let $F_t : \Lambda \to \Lambda$ be a measure-preserving semiflow. We suppose that $t \to F_t$ is Lipschitz a.e. on Λ. Let $L_t : L^1(\Lambda) \to L^1(\Lambda)$ denote the transfer operator corresponding to F_t (so $\int_\Lambda L_t v\, w\, d\mu = \int_\Lambda v\, w \circ F_t\, d\mu$ for all $v \in L^1(\Lambda)$, $w \in L^\infty(\Lambda)$, $t > 0$). Let $v \in L^\infty(\Lambda)$ and define $v_t = \int_0^t v \circ F_r\, dr$ for $t \ge 0$.

Theorem 1.1 *Let $\eta \in (\frac{1}{2}, 1)$. Suppose that $L_t v \in C^\eta(\Lambda)$ for all $t > 0$ and that $\int_0^\infty \|L_t v\|_\eta\, dt < \infty$. Then v_t is a coboundary:*

$$v_t = \chi \circ F_t - \chi \quad \text{for all } t \ge 0, \text{ a.e. on } \Lambda$$

where $\chi = \int_0^\infty L_t v\, dt \in C^\eta(\Lambda)$. In particular, $\sup_{t \ge 0} |v_t|_\infty < \infty$.

Here, $|g|_\infty = \operatorname{ess\,sup}_\Lambda |g|$ and $\|g\|_\eta = |g|_\infty + \sup_{x \ne y} |g(x) - g(y)|/d(x,y)^\eta$.

Theorem 1.1 implies that any Banach space admitting a spectral gap and embedded in $C^\eta(\Lambda)$ for some $\eta > \frac{1}{2}$ is cohomologically trivial. However, for (non)uniformly expanding semiflows and (non)uniformly hyperbolic flows of the type in the aforementioned references, coboundaries are known to be exceedingly rare, see for example [3, Section 2.3.3]. Hence, Theorem 1.1 can be viewed as an "anti-spectral gap" result for such continuous time dynamical systems.

2 Proof of Theorem 1.1

Let $v \in L^\infty(\Lambda)$, with $L_t v \in C^\eta(\Lambda)$ for all $t > 0$ and $\int_0^\infty \|L_t v\|_\eta\, dt < \infty$ where $\eta \in (\frac{1}{2}, 1)$. Following Gordin [6] we consider a martingale-coboundary decomposition. Define $\chi = \int_0^\infty L_t v\, dt \in C^\eta(\Lambda)$, and

$$v_t = \int_0^t v \circ F_r\, dr, \qquad m_t = v_t - \chi \circ F_t + \chi,$$

for $t \ge 0$. Let $\mathcal{B}$ denote the Borel σ-algebra on Λ.

Proposition 2.1 *(i) $t \to m_t$ is C^η a.e. on Λ.*

(ii) $\mathbb{E}(m_t | F_t^{-1}\mathcal{B}) = 0$ for all $t \ge 0$.

Proof (i) For $0 \le s \le t \le 1$ and $x \in \Lambda$,

$$\begin{aligned}|m_s(x) - m_t(x)| &\le |v_s(x) - v_t(x)| + |\chi(F_s x) - \chi(F_t x)| \\ &\le |s-t||v|_\infty + |\chi|_\eta\, d(F_s x, F_t x)^\eta.\end{aligned}$$

Since $t \mapsto F_t$ is a.e. Lipschitz, it follows that $t \mapsto m_t$ is a.e. C^η.

(ii) Let $U_t v = v \circ F_t$, and recall that $L_t U_t = I$ and $\mathbb{E}(\cdot|F_t^{-1}\mathcal{B}) = U_t L_t$. Then

$$\begin{aligned}L_t m_t &= L_t(v_t - U_t\chi + \chi) = \textstyle\int_0^t L_t U_r v\,dr - \chi + \int_0^\infty L_t L_r v\,dr \\ &= \textstyle\int_0^t L_{t-r} v\,dr - \chi + \int_0^\infty L_{t+r} v\,dr = \int_0^t L_r v\,dr - \chi + \int_t^\infty L_r v\,dr = 0.\end{aligned}$$

Hence $\mathbb{E}(m_t|F_t^{-1}\mathcal{B}) = U_t L_t m_t = 0$. ∎

Proof of Theorem 1.1 Fix $T > 0$, and define

$$M_T(t) = m_T - m_{T-t} = m_t \circ F_{T-t}, \quad t \in [0,T].$$

Also, define the filtration $\mathcal{G}_{T,t} = F_{T-t}^{-1}\mathcal{B}$. It is immediate that $M_T(t) = m_t \circ F_{T-t}$ is $\mathcal{G}_{T,t}$-measurable. Also, for $s < t$ we have $M_T(t) - M_T(s) = m_{T-s} - m_{T-t} = m_{t-s} \circ F_{T-t}$, so

$$\mathbb{E}(M_T(t) - M_T(s)|\mathcal{G}_{T,s}) = \mathbb{E}(m_{t-s} \circ F_{T-t}|F_{T-s}^{-1}\mathcal{B}) = \mathbb{E}(m_{t-s}|F_{t-s}^{-1}\mathcal{B}) \circ F_{T-t} = 0$$

by Proposition 2.1(ii). Hence M_T is a martingale for each $T > 0$. Next,

$$|M_T(t)|_\infty = |m_t \circ F_{T-t}|_\infty \le |m_t|_\infty \le |v_t|_\infty + 2|\chi|_\infty \le T|v|_\infty + 2|\chi|_\infty.$$

Hence $M_T(t)$, $t \in [0,T]$, is a bounded martingale.

By Proposition 2.1(i), M_T has C^η sample paths. Since $\eta > \frac{1}{2}$, it follows from general martingale theory that $M_T \equiv 0$ a.e. Taking $t = T$, we obtain $m_T = 0$ a.e. Hence $v_T = \chi \circ F_T - \chi$ a.e. for all $T > 0$ as required.

For completeness, we include the argument that $M_T \equiv 0$ a.e. We require two standard properties of the quadratic variation process $t \mapsto [M_T](t)$; a reference for these is [4, Theorem 4.1]. First, $[M_T](t)$ is the limit in probability as $n \to \infty$ of

$$S_n(t) = \sum_{j=1}^n (M_T(jt/n) - M_T((j-1)t/n))^2.$$

Second (noting that $M_T(0) = 0$),

$$[M_T](t) = M_T(t)^2 - 2\int_0^t M_T\,dM_T,$$

where the stochastic integral has expectation zero. In particular, $\mathbb{E}([M_T]) \equiv \mathbb{E}(M_T^2)$.

Since M_T has Hölder sample paths with exponent $\eta > \frac{1}{2}$, we have a.e. that

$$|S_n(t)| = O(t^\eta n^{-(2\eta-1)}) \to 0 \quad \text{as } n \to \infty.$$

Hence $[M_T] \equiv 0$ a.e. It follows that $\mathbb{E}(M_T^2) \equiv 0$ and so $M_T \equiv 0$ a.e. ∎

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