Instantaneous Symmetry and Symmetry on Average in the Couette-Taylor and Faraday Experiments *

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Abstract

We describe some recent results on symmetry of attractors for dynamical systems with symmetry and consider the implications for the Couette-Taylor experiment and the Faraday surface wave experiment. In particular, we explore the relationship between symmetry of solutions at a fixed instant in time, and symmetry in the time-averaged solution. This leads to predictions that are somewhat surprising and which we believe require careful experimental exploration.

1 Introduction

Many physical interesting situations, including Rayleigh-Bénard convection and the Couette-Taylor experiment, are modeled by PDEs that have symme-

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tries, that is they are equivariant with respect to the action of a symmetry group Γ. Equilibrium and periodic solutions to these PDEs may be invariant as a subset of phase space under some of these symmetries. In this case, there is a well understood connection between the symmetries in phase space and symmetries in physical space [17].

Recently, there has been interest in interpreting the symmetry of a solution in physical space corresponding to a chaotic attractor in phase space, see [6, 8]. Thought of as a subset A of phase space, there are (at least) two ways in which A can be symmetric. There is a subgroup Σ_A of elements of Γ that fix A as a set, and a smaller subgroup T_A of elements of Γ that fix each point in A, see Section 2 below.

There are also two ways of interpreting the symmetry of the solution in physical space, the *instantaneous symmetry* which measures the symmetry at each moment in time, and the *symmetry on average* which is the symmetry in the time-averaged solution. Clearly, we may interpret T_A to be the instantaneous symmetry. The suggestion in Dellnitz et al [8] is that Σ_A should be identified with the symmetry on average. We shall make this identification, but note that this has been justified rigorously only under the assumption that the attractor A has an SBR measure.

The motivating example for the work of Chossat and Golubitsky [6] on symmetric attractors was provided by a chaotic state in the Couette-Taylor experiment that is known as turbulent Taylor vortices, see Brandstater and Swinney [5]. A picture of turbulent Taylor vortices at an instant in time is shown in Figure 1(a). The usual interpretation for this state is that there is no instantaneous symmetry, even though on average [22] there is the symmetry of the steady state Taylor vortices, Figure 1(b). More recently, experiments have been performed by Gollub and coworkers [16] on the Faraday surface wave experiment in square and circular geometries. Instantaneous and time-averaged pictures of states that they find are shown in Figures 2 and 3. Again there is evidence of symmetry that exists only on average.

The existence of symmetry only on average has been verified numerically [8] and is well-understood theoretically [3, 14]. One purpose of the present paper is to point out that the experimental evidence for symmetry only on average is not as conclusive as has been believed. In particular, we argue that the motivating example of turbulent Taylor vortices does not exhibit symmetry only on average, but that the symmetry in the time average is already present at each instant of time. In this paper we highlight those aspects of recent results about symmetric attractors in [14, 19] that have implications for PDEs, and then use these results to make predictions for experiments. As already indicated, these predictions are often contrary to intuition.

In Section 2 we define the instantaneous symmetry and symmetry on average for a subset A. Also, we state some elementary results including a normality condition that will prove crucial in the remainder of the paper. Then in Section 3 we describe some of the results in [14] concerning attractors for ODEs that have a finite group of symmetries Γ. To make the connection with PDEs, we introduce a class of 'high-dimensional' ODEs that includes discretizations of PDEs. As a preliminary application, we interpret these results for the Faraday experiment in a square container, taking $\Gamma = \mathbb{D}_4$ the symmetry group of the square.

In Section 4 we consider some results for continuous groups of symmetries and consider the implications for experiments. In particular, we consider the Faraday experiment in a circular container, and the Couette-Taylor experiment – the latter under the assumption of periodic boundary conditions.

Returning to the Faraday experiment in a square container, it is evident that the time-averaged state in Figure 2(b) has more structure than can be explained within the context of the symmetry group $\Gamma = \mathbb{D}_4$: there appear to be additional discrete translation symmetries parallel to the sides of the square container. Using the trick of embedding Neumann boundary conditions in periodic boundary conditions we are able to explain this structure but only if the structure is already present at any instant in time. A similar story is true for turbulent Taylor vortices if we assume Neumann boundary conditions rather than periodic boundary conditions. These issues are taken up in Section 5.

We end in Section 6 with some conclusions. In particular, we discuss at some length the likelihood that turbulent Taylor vortices have no symmetry on average except for that symmetry that is already present at any instant in time.

2 Symmetry groups of sets

In this section, we define the symmetry of a set. At this point, we work quite generally, and do not require mention of any dynamics. Suppose that X is a set, and that Γ is a group acting on X. If $A \subset X$, we define

$$T_A = \{\gamma \in \Gamma;\ \gamma x = x \text{ for all } x \in A\},$$

and

$$\Sigma_A = \{\gamma \in \Gamma;\, \gamma A = A\}.$$

For reasons explained in the introduction, we call T_A the *instantaneous symmetry* and Σ_A the *symmetry on average*. Note that A is contained in the fixed-point set of T_A: $A \subset \mathrm{Fix}(T_A)$. The following result is elementary, but crucial.

Proposition 2.1 *T_A is a normal subgroup of Σ_A.*

Proof Suppose that $t \in T_A$ and $\sigma \in \Sigma_A$. If $x \in A$, $\sigma x \in A$ and hence $t\sigma x = \sigma x$. It follows that $\sigma^{-1} t \sigma x = x$ and $\sigma^{-1} t \sigma \in T_A$ as required. ∎

If T is a subgroup of Γ, then $N(T)$ denotes the normalizer of T in Γ, namely the largest subgroup of Γ that contains T as a normal subgroup. An equivalent formulation of the normality condition in Proposition 2.1 is that

$$T_A \subset \Sigma_A \subset N(T_A).$$

For example, suppose that $\Gamma = \mathbf{O}(2)$. Up to conjugacy, the subgroups of $\mathbf{O}(2)$ are

$$\mathbb{Z}_k,\ k \geq 1, \quad \mathbb{D}_k,\ k \geq 1, \quad \mathbf{SO}(2), \quad \mathbf{O}(2).$$

(In particular, $\mathbb{Z}_1 = \mathbf{1}$ is the trivial group, and $\mathbb{D}_1$ is the two element group generated by any element $\kappa \in \mathbf{O}(2) - \mathbf{SO}(2)$.) The subgroups $\mathbb{Z}_k$, $\mathbf{SO}(2)$ and $\mathbf{O}(2)$ are normal in $\mathbf{O}(2)$. Hence the normality condition in Proposition 2.1 comes into effect only when $T_A = \mathbb{D}_k$ which has normalizer $\mathbb{D}_{2k}$. It then follows that either $\Sigma_A = \mathbb{D}_k$ or $\Sigma_A = \mathbb{D}_{2k}$. Proposition 2.1 can also be applied in the opposite direction. For example if $\Sigma_A = \mathbf{O}(2)$ then we deduce that $T_A \neq \mathbb{D}_k$ for any k.

3 Finite symmetry groups

In this section we recall results of Field et al [14] which classify the possible symmetry groups of attractors for flows that are equivariant with respect to the action of a finite group of symmetries Γ. The points that we emphasize here are rather different from those in [14], in particular we focus on the implications for applications modeled by PDEs. A consequence is that generally the representation-theoretic restrictions obtained in [14] (see also [20]) are not applicable.

Since the results in [14] are stated within the context of finite-dimensional flows, it is necessary to make the transition between PDEs and 'high-dimensional' ODEs. Note that if we were working with a PDE, we would expect every subgroup $T \subset \Gamma$ to be an isotropy subgroup with fixed-point subspace $\mathrm{Fix}(T)$ of infinite dimension. Moreover if I is another subgroup and $I \not\subset T$, then the intersection of $\mathrm{Fix}(I)$ with $\mathrm{Fix}(T)$ should be of infinite codimension in $\mathrm{Fix}(T)$. Passing to a discretization of such a PDE leads to a high-dimensional ODE and the dimensions and codimensions mentioned above should also be very high. In particular, we expect that the following conditions are satisfied.

(i) T is an isotropy subgroup,

(ii) $\dim(\mathrm{Fix}(T)) \geq 5$, and

(iii) $\dim(\mathrm{Fix}(T)) - \dim(\mathrm{Fix}(I) \cap \mathrm{Fix}(T)) \geq 2$ for all $I \subset \Gamma$, $I \not\subset T$.

We shall say that a representation of Γ on $\mathbb{R}^n$ is *high-dimensional* if each subgroup $T \subset \Gamma$ satisfies conditions (i)–(iii). Also a Γ-equivariant ODE or vector field is *high-dimensional* if the representation of Γ is high-dimensional. Under this assumption of high-dimensionality, the normality condition in Proposition 2.1 is necessary and sufficient.

Theorem 3.1 ([14]) *Suppose that Γ is a finite subgroup of $\mathbf{O}(n)$ and that the representation of Γ on $\mathbb{R}^n$ is high-dimensional. Let T and Σ be subgroups of Γ. Then there exists a C^∞ Γ-equivariant vector field on $\mathbb{R}^n$ possessing an Axiom A attractor A with $T_A = T$ and $\Sigma_A = \Sigma$ if and only if T is a normal subgroup of Σ.*

Remark 3.2 Axiom A attractors are structurally stable, so it follows from the theorem that whenever T is a normal subgroup of Σ, attractors with instantaneous symmetry T and symmetry on average Σ are unavoidable.

Application: the Faraday experiment in a square geometry Theorem 3.1 can be applied to the Faraday surface wave experiment performed in a square container [16], see Figure 2. For the time being we work only with the naive group of symmetries which is the symmetry group of the square $\Gamma = \mathbb{D}_4$. Later, in Section 5, we shall consider the issues that arise when trying to understand the additional structure that is evident in Figure 2(b).

It appears from the figure that we have a state with instantaneous symmetry $T = \mathbf{1}$ and symmetry on average $\Sigma = \mathbb{D}_4$. Note that this scenario is entirely consistent with Theorem 3.1 but is only one of several possibilities that is equally consistent with the theorem.

The subgroups of $\mathbb{D}_4$ are

$$\mathbf{1},\ \mathbb{Z}_2,\ \mathbb{Z}_4,\ \mathbb{D}_1,\ \mathbb{D}_2,\ \mathbb{D}_4.$$

Each of these subgroups is normal in $\mathbb{D}_4$ with the exception of the subgroup $\mathbb{D}_1$ which has normalizer $\mathbb{D}_2$. Hence if $T \neq \mathbb{D}_1$, Σ can be any subgroup of $\mathbb{D}_4$ that contains T. In particular, if $T = \mathbf{1}$ then there are no restrictions on Σ. Thus we should not be surprised by symmetry on average, but neither should we expect it.

As a result of later considerations in this paper, it will become apparent that the instantaneous symmetry T is not so easy to deduce from Figure 2(a). However it is clear from Figure 2(b) that $\Sigma = \mathbb{D}_4$. Applying Theorem 3.1, we can deduce at least that $T \neq \mathbb{D}_1$.

4 Continuous groups

In this section, we consider ω-limit sets A for flows in $\mathbb{R}^n$ that are equivariant with respect to a continuous (nonfinite but compact) group of symmetries $\Gamma \subset \mathbf{O}(n)$.

Theorem 3.1 states that for high-dimensional representations of a finite group Γ, given T_A we can expect Σ_A to be any subgroup of Γ that satisfies $T_A \subset \Sigma_A \subset N(T_A)$. Moreover, all such possibilities for Σ_A occur in a structurally stable manner (since A can be chosen to be an Axiom A attractor).

As we indicate in this section, the situation is completely different for continuous groups. Typically there are larger lower bounds for the continuous symmetries in Σ_A. These bounds depend on T_A and also in a subtle way on the dynamics in $\omega(x)$.

Recall that if G is a group, we denote by G^0 the connected component of the identity in G. Suppose that $A = \omega(x)$ is an ω-limit set for a Γ-equivariant vector field on $\mathbb{R}^n$. Then results of [2, 10, 19, 21] indicate that, provided the dynamics in A is 'sufficiently chaotic', typically

$$(N(T_A)/T_A)^0 \subset \Sigma_A/T_A \subset N(T_A)/T_A. \tag{4.1}$$

We do not wish to define here precisely what is meant by 'sufficiently chaotic'. It follows from [19] that at least when Γ^0 is abelian (which is the case for the applications considered in this paper) it is enough that A satisfies conditions that can be thought of as a equivariant generalization of 'Devaney's definition of chaos' [11]. However these hypotheses are unnecessarily stringent and are undergoing constant revision at present.

We note that it is certainly necessary to rule out dynamics that is too regular. For example, if A is a subset of a single group orbit (A is a relative equilibrium) then Σ_A/T_A is easily seen to be abelian, typically it is a maximal torus in $N(T_A)/T_A$, see [12, 18]. The situation is more complicated for relative periodic orbits, see [18, 13], but Σ_A/T_A remains abelian. This is not the case for general ω-limit sets. For example, the existing theory indicates that typically equation (4.1) is valid under very weak hypotheses on the irregularity of the dynamics in A. From now on we shall assume that equation (4.1) is valid at least for the states shown in Figures 1, 2 and 3.

Application: the Faraday experiment in a circular geometry We consider the Faraday surface wave experiment performed in a circular geometry [16], see Figure 3. The solution shown in the figure would appear to have no instantaneous symmetry but to have full symmetry on average. In our notation, this is $\Sigma = \Gamma = \mathbf{O}(2)$ and $T = \mathbf{1}$.

Actually, we cannot tell from the figure whether there is $\mathbf{SO}(2)$ or $\mathbf{O}(2)$ symmetry on average. This is due to the fact that all subsets $A \subset \mathbb{R}^2$ with $\mathbf{SO}(2) \subset \Sigma_A$ automatically satisfy $\Sigma_A = \mathbf{O}(2)$. In the language of [4], the observation that is being averaged is not a *detective*; it cannot distinguish between $\Sigma = \mathbf{SO}(2)$ and $\Sigma = \mathbf{O}(2)$.

It turns out that the information on symmetry on average contained in Figure 3(b) can be predicted from knowledge of the instantaneous symmetry. This follows from the entries in Table 1 where we enumerate the subgroups T of $\mathbf{O}(2)$ and then list the possibilities for Σ that are typical according to equation (4.1). In particular, if we know that there is no instantaneous symmetry, then we can predict that there is at least $\mathbf{SO}(2)$ symmetry on average. Hence with hindsight we can say that Figure 3(b) gives no further information on the symmetry on average. However, averaging the correct detective should determine whether the symmetry on average is $\mathbf{SO}(2)$ or $\mathbf{O}(2)$.

Finally, applying Table 1 in the reverse direction, we can deduce that if Σ_A contains $\mathbf{SO}(2)$ then either there are no reflection symmetries present instantaneously or there is full symmetry instantaneously. (This is an application of the normality condition in Proposition 2.1.)

T	$N(T)$	$N(T)/T$	Σ/T	Σ
$\mathbb{Z}_k$, $k \geq 1$	$\mathbf{O}(2)$	$\mathbf{O}(2)$	$\mathbf{SO}(2)$, $\mathbf{O}(2)$	$\mathbf{SO}(2)$, $\mathbf{O}(2)$
$\mathbb{D}_k$, $k \geq 1$	$\mathbb{D}_{2k}$	$\mathbb{Z}_2$	$\mathbf{1}$, $\mathbb{Z}_2$	$\mathbb{D}_k$, $\mathbb{D}_{2k}$
$\mathbf{SO}(2)$	$\mathbf{O}(2)$	$\mathbb{Z}_2$	$\mathbf{1}$, $\mathbb{Z}_2$	$\mathbf{SO}(2)$, $\mathbf{O}(2)$
$\mathbf{O}(2)$	$\mathbf{O}(2)$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{O}(2)$

Table 1: This table shows the interaction between instantaneous symmetry T and symmetry on average Σ for ω-limit sets in ODEs with $\mathbf{O}(2)$ symmetry. For each subgroup $T \subset \mathbf{O}(2)$ we list the typical possibilities for Σ as guaranteed by equation (4.1).

Application: the Couette-Taylor experiment In the Couette-Taylor experiment, the underlying symmetry group is $\Gamma = \mathbf{O}(2) \times \mathbf{SO}(2)$ where $\mathbf{SO}(2)$ corresponds to the azimuthal rotations and $\mathbf{O}(2)$ corresponds to the axial translations together with the mid-cylinder flip. Here we are assuming periodic boundary conditions at the ends of the cylinder. In Section 5 we shall consider what happens if we make the more reasonable assumption of Neumann boundary conditions at the ends of the cylinder. Our predictions are then unchanged though the analysis is completely different.

First observe that the center of Γ is $\mathbf{SO}(2)$, and it follows that $N(T)$ contains $\mathbf{SO}(2)$ for all subgroups T. Hence we can predict that on average all solutions will have full azimuthal symmetry, independent of the instantaneous symmetry.

Our second prediction is more interesting and concerns the solution known as turbulent Taylor vortices which is shown in Figure 1. Thanks to the previous prediction, we may as well factor out the azimuthal symmetry so that $\Gamma = \mathbf{O}(2)$. The current trend is to believe that turbulent Taylor vortices have no instantaneous symmetry but have the symmetry of Taylor vortices on average, namely $\Sigma = \mathbb{D}_k$ (or $\mathbb{D}_k \times \mathbf{SO}(2)$ if we reinstate the azimuthal symmetry). However, by Table 1, if we take $T = \mathbf{1}$ then we expect $\Sigma = \mathbf{SO}(2)$ or $\mathbf{O}(2)$. But then the averaged solution is totally homogeneous (Couette flow), which is clearly not the case. On the other hand, if we have $T_A = \mathbb{D}_k$, then $N(T_A) = \mathbb{D}_{2k}$ and we would expect either $\Sigma_A = \mathbb{D}_k$ or $\Sigma_A = \mathbb{D}_{2k}$. Based on this calculation, we propose that turbulent Taylor vortices have $T = \Sigma = \mathbb{D}_k$.

5 Embedding Neumann boundary conditions in periodic boundary conditions

In Section 3 we considered the Faraday surface wave experiment in a square domain (Figure 2) but only taking into account the naive symmetry group $\Gamma = \mathbb{D}_4$. On the other hand, the time-averaged solution shown in Figure 2(b) clearly has additional structure that cannot be explained in terms of elements of $\mathbb{D}_4$.

Recently there has been much interest in problems satisfying certain boundary conditions, in particular Neumann boundary conditions (NBC), that admit the possibility of embedding the problem in a larger problem (on a larger domain) where the solutions satisfy periodic boundary conditions (PBC). The extra symmetry that arises in the PBC problem imposes constraints on the original NBC problem and changes the generic behavior of the associated dynamical system as well as increasing the available range of symmetries for the solutions. See Crawford et al [7] which expands upon work of Fujii, Mimura and Nishiura [15] and Armbruster and Dangelmayr [1].

In this section, we review the construction and then reconsider the Couette-Taylor and Faraday experiments in this light. Since we are interested in global dynamics, the implications for genericity do not concern us. However the additional symmetries will be of great importance.

Suppose that we have a Euclidean-equivariant PDE on the unit interval $[0, 1]$ and that we impose NBC at the ends of the interval. Then the only Euclidean transformation that preserves the domain is the flip $x \to 1 - x$ which generates the symmetry group $\mathbb{D}_1^F$. By reflecting a solution across 0 we obtain a solution that satisfies the PDE on $[-1, 1]$. Moreover the solution satisfies PBC on this larger domain. (There is a technical problem concerning regularity of the solution obtained in this way but in many cases, including the ones that we shall consider, regular solutions to the NBC problem on $[0,1]$ extend to regular solutions to the PBC problem on $[-1,1]$.)

Conversely, solutions satisfying PBC on $[-1,1]$ restrict to solutions satisfying NBC on $[0,1]$ if and only if they are invariant under the reflection κ that sends x to $-x$. Define $\mathbb{D}_1^N$ to be the group generated by κ. Now we consider the enlarged problem of PBC on $[-1,1]$ but then restrict to $\mathrm{Fix}(\mathbb{D}_1^N)$ in order to recover the NBC problem. The idea is that the PBC problem is $\mathbf{O}(2)$-equivariant and solutions in $\mathrm{Fix}(\mathbb{D}_1^N)$ may pick up symmetries in $\mathbf{O}(2)$ that do not lie in the original group $\mathbb{D}_1^F$.

Now, if we have a solution to the NBC problem, we may compute the instantaneous symmetry T and the symmetry on average Σ as subgroups of $\mathbf{O}(2)$ instead of $\mathbb{D}_1^F$. An important observation is that T must contain $\mathbb{D}_1^N$ (since solutions are assumed to satisfy Neumann boundary conditions at all times). In particular, if there is no discernible structure at an instant in time, then this should be interpreted as $T = \mathbb{D}_1^N$.

Application: the Couette-Taylor experiment revisited In Section 4 we considered the Couette-Taylor experiment and made two predictions based upon the assumption of periodic boundary conditions. Now we show that these predictions are unchanged if we assume Neumann boundary conditions.

The prediction concerning full azimuthal symmetry on average goes through easily no matter what boundary conditions are assumed. Hence we turn to the second prediction concerning turbulent Taylor vortices. As before, we may factor out the azimuthal symmetry. Also, the radial direction plays no important role and so we may reduce to the situation of a NBC problem on the unit interval.

We now have an $\mathbf{O}(2)$-equivariant problem and we restrict attention to those solutions with instantaneous symmetry at least $\mathbb{D}_1^N$ (that is, those solutions satisfying NBC). If there is no further structure, then we have instantaneous symmetry $T = \mathbb{D}_1^N$. But then the normality condition in Proposition 2.1 guarantees that the symmetry on average Σ is isomorphic to $\mathbb{D}_1$ or $\mathbb{D}_2$, see Table 1. This is inconsistent with Figure 1(b). On the other hand if $T \cong \mathbb{D}_k$ then we have $\Sigma \cong \mathbb{D}_k$ or $\Sigma \cong \mathbb{D}_{2k}$. We are again led to our prediction that $T = \Sigma \cong \mathbb{D}_k$.

Application: the Faraday experiment revisited Here we consider the Faraday experiment in a square container, but in such a way that we can account for additional structure in the time-averaged solution that is not accounted for by the symmetries of the square alone. The analysis here is almost identical to that for the Couette-Taylor experiment under the assumption of NBC. The corresponding questions raised by the normality condition in Proposition 2.1 were first pointed out to me by M. Golubitsky.

Solutions to the NBC problem on the square extend to solutions satisfying PBC on a square of four times the size. This problem is equivariant under the semidirect product of the naive symmetries $\mathbb{D}_4$ with translation symmetries in the directions parallel to the sides of the square. So we have $\Gamma = \mathbb{D}_4 \dot{+} (\mathbf{SO}(2) \times \mathbf{SO}(2))$. Equivalently, we can write $\Gamma = \mathbb{D}_1 \dot{+} (\mathbf{O}(2) \times \mathbf{O}(2))$ which is more convenient for our purposes. Again we restrict attention to solutions that satisfy NBC on the original square: these solutions have instantaneous symmetry at least $T = \mathbb{D}_1^N \times \mathbb{D}_1^N$.

The time-averaged solution in Figure 2(b) would seem to have symmetry on average $\Sigma = \mathbb{D}_1 \dot{+} (\mathbb{D}_k \times \mathbb{D}_k)$ for a fairly large value of k. The normality condition implies that if there is no instantaneous symmetry ($T = \mathbb{D}_1^N \times \mathbb{D}_1^N$) then there is an upper bound on the symmetry on average ($\Sigma \cong \mathbb{D}_1 \dot{+} (\mathbb{D}_2 \times \mathbb{D}_2)$) which is incompatible with Figure 2(b). We propose that $T = \Sigma \cong \mathbb{D}_1 \dot{+} (\mathbb{D}_k \times \mathbb{D}_k)$.

6 Conclusions

In this paper we have made the somewhat contentious proposal that turbulent Taylor vortices have no more symmetry in the time-average than they have at any instant in time. This is surprising since much of the work on symmetry of attractors and on the existence of symmetry only in the time-average has been motivated by turbulent Taylor vortices.

The obvious objection to our proposal is that this symmetry is not very evident in Figure 1(a), and certainly not as evident as in Figure 1(b). (This is also the case for the state observed in the Faraday experiment in a square container, Figure 2.) A possible explanation is that the underlying symmetry in our model is not exact in the experiment and that the instantaneous symmetry is necessarily approximate, hence difficult to deduce from the snapshot at an instant in time. However on average the discrepancies (which are essentially random) may be expected to cancel out so that the symmetry is much clearer.

There is a related state in the Couette-Taylor experiment called turbulent wavy vortices which has a time-periodic counterpart called wavy vortices, see Figure 4. Wavy vortices have not full but discrete azimuthal symmetry (in fact the symmetry is the product of an azimuthal symmetry with the mid-cylinder flip). In addition, there is the discrete axial symmetry that is present in Taylor vortices. An intriguing question for some time has been how to distinguish on grounds of symmetry between turbulent Taylor vortices and turbulent wavy vortices.

Again, it is often claimed that turbulent wavy vortices have no instantaneous symmetry. We propose that turbulent wavy vortices have wavy vortex symmetry instantaneously. Then our prediction that there is always full azimuthal symmetry in the time average leads to the expectation that on average there is Taylor vortex symmetry. In particular, we have that symmetry on average does not distinguish between turbulent Taylor vortices and turbulent wavy vortices. However, the instantaneous symmetry does distinguish between the two states.

In principle it is not difficult to devise an experimental test of our proposal. The primary bone of contention lies in the existence or nonexistence of the mid-cylinder flip as a instantaneous symmetry. Consider the detective v consisting of the absolute value of the difference of observations taken at two reflection related points (perhaps averaged over time). For turbulent Taylor vortices, we expect the value of v to be close to zero.

The numerical difficulty of what constitutes a value close to zero can be overcome as in [4] by computing v during a transition from turbulent wavy vortices to turbulent Taylor vortices. If our predictions are correct, v should jump from a value far from zero to a value close to zero. At the same time, if we average the difference in the observations without taking the absolute value, we should obtain a value w near zero throughout the transition, indicating the presence of the mid-cylinder flip in the time-average. Note that because the symmetry on average should be much cleaner than the instantaneous symmetry, we expect w to be much closer to zero than v, even for turbulent Taylor vortices.

A second and more difficult objection to our proposal has been raised by M. Field. In experiments, there is typically a natural direction in the variation of parameters. In the case of the Couette-Taylor experiment, it is usual to consider the transitions as the Reynolds number is increased.

Then one prechaotic scenario is that solutions lie in a low-dimensional fixed-point subspace (Couette flow) from which there are bifurcations to solutions lying in higher-dimensional fixed-point subspaces (such as Taylor vortices and wavy vortices) and even to solutions with no symmetry. It is rather difficult to imagine why there should then be a transition back to a (now chaotic) solution in a lower-dimensional fixed-point subspace. However, this is what we are arguing to be the case with turbulent Taylor vortices. At present, we have no answer to this line of argument except to say that this is all the more reason to obtain a better understanding of the states that occur and only then to consider the transitions as parameters are varied.

We end by giving an argument in support of our proposal that we feel is particularly compelling. In phase space, the instantaneous symmetry corresponds to the symmetry of a single point in an attractor, whereas the symmetry on average corresponds to the symmetry of the whole attractor. It is clear in this setting that the symmetry of one point yields little or no information about additional symmetry that fixes the attractor but not the point itself. Hence in physical space, the symmetries (or structure) visible in the picture of a state at an instant in time should not give any clue to additional symmetry (structure) that may be present in the time average.

Evidence for this argument is provided by Figure 3(a) where there is no sign of the symmetry that appears on average 3(b) (but note that equation (4.1) predicts the time average symmetry in Figure 3(b) as a consequence of the lack of structure in Figure 3(a)). Contrast this with the picture of turbulent Taylor vortices in Figure 1(a) where it is already possible to 'see' the putative symmetry on average.

In conclusion, we suggest that whenever it is possible to guess the existence of symmetry on average by looking at the instantaneous picture, then that symmetry is probably there even instantaneously. To compute additional symmetry on average, it is necessary to appeal to theory (such as equation (4.1) and/or to use detectives [4, 9]. By the same token, even the instantaneous symmetry may not be transparent by simply observing the solution. This suggests that it is necessary in general to use detectives to compute the instantaneous symmetry as well as the symmetry on average.

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Figure 1: Turbulent Taylor vortices (a) and Taylor vortices (b) in the Couette-Taylor experiment. Pictures supplied by H.L. Swinney. The state in (a), when time-averaged, looks like the state in (b) [22].

(a) (b)

Figure 2: Instantaneous symmetry (a) and symmetry on average (b) of a state in the Faraday experiment in a square geometry. Pictures supplied by J.P. Gollub.

(a)

(b)

Figure 3: Instantaneous symmetry (a) and symmetry on average (b) of a state in the Faraday experiment in a circular geometry. Pictures supplied by J.P. Gollub.

(a)

(b)

Figure 4: Wavy vortices in the Couette-Taylor experiment. Picture supplied by H.L Swinney.