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Bifurcation from Discrete Rotating Waves

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Abstract. Discrete rotating waves are periodic solutions that have discrete spatiotemporal symmetries in addition to their purely spatial symmetries. We present a systematic approach to the study of local bifurcation from discrete rotating waves. The approach centers around the analysis of diffeomorphisms that are equivariant with respect to distinct group actions in the domain and the range.

Our results are valid for dynamical systems with finite symmetry group, and more generally for bifurcations from *isolated* discrete rotating waves in dynamical systems with compact symmetry group.

1. Introduction

In systems of ordinary differential equations without symmetry, there is a complete theory of the generic local bifurcations that occur as a single bifurcation parameter is varied, see for example Guckenheimer and Holmes [8, Chapter 3]. Local bifurcations are by definition the bifurcations that occur in the neighborhood of a nonhyperbolic equilibrium or a nonhyperbolic periodic solution.

Equivariant bifurcation theory, Golubitsky, Stewart and Schaeffer [7], is concerned with the generalization of these results to the situation where the vector field is equivariant with respect to the action of a compact Lie group Γ. A systematic approach to bifurcation from equilibria is laid out in [7]. In contrast, such an approach to bifurcation from periodic solutions has previously proved elusive.

Suppose that P is a periodic solution of (minimal) period T, and let $x_0 \in P$. Let $x(t)$ be the trajectory with initial condition x_0, so $P = \{x(t) : 0 \leq t < T\}$. The symmetries that leave the periodic solution P invariant come in two forms. First, there is the group of *spatial symmetries*

$$\Delta = \{\gamma \in \Gamma : \gamma x_0 = x_0\}.$$

By definition, Δ is the isotropy subgroup of x_0. In fact, Δ is the isotropy subgroup of each point in P. Second, there is the group of *spatiotemporal symmetries*

$$\Sigma = \{\gamma \in \Gamma : \gamma P = P\}.$$

It is easy to see that for each $\sigma \in \Sigma$, there is a unique $T_\sigma \in [0, T)$ such that $\sigma x(t) = x(t + T_\sigma)$ for all t. Thus each spatiotemporal symmetry is the combination of a symmetry element σ composed with a time-shift by T_σ. The spatial symmetries are those spatiotemporal symmetries σ for which $T_\sigma = 0$. Moreover, Δ is a normal subgroup of Σ and either $\Sigma/\Delta \cong S^1$ or $\Sigma/\Delta \cong \mathbb{Z}_m$ for some $m \geq 1$.

When $\Sigma/\Delta \cong S^1$, the periodic solution P is called a *rotating wave*. Krupa [9] generalized the results of [7] to include local bifurcation from rotating waves (and more generally, from relative equilibria).

When $\Sigma/\Delta \cong \mathbb{Z}_m$, the periodic solution P is called a *discrete rotating wave* [5]. Special cases that have been studied include the case when $m = 1$, Chossat and Golubitsky [4], and the case when Σ is cyclic, Fiedler [5]. (As verified by Buono [3], many of Fiedler's results apply equally well when Σ is abelian.) For general discrete rotating waves, there are partial results due to Vanderbauwhede [23, 24], Nicolaisen and Werner [16] and Rucklidge and Silber [19]. However, these results rely explicitly on hypotheses that appear plausible but are not shown to be valid.

In this paper we systematically analyze generic local bifurcation from discrete rotating waves. The novelty of our approach lies in the use of the representation theory for compact Lie groups to develop a local bifurcation theory for *twisted equivariant* maps. The latter maps are the building blocks of return maps for discrete rotating waves. Incidentally, our results establish the genericity of the heuristic hypotheses in the papers [16,19,23,24]. We will discuss this in more detail in Section 4(d). Discrete rotating waves occur in a variety of situations, notably through Hopf bifurcation from equilibria. For example, Hopf bifurcation from an equilibrium with $\mathbf{O}(2)$ symmetry [7, Chapter XVII] leads to branches of rotating waves and standing waves (see [7, Chapter XI, Figure 1.4]). The rotating waves have no spatial symmetry, $\Delta = 1$, and rotational spatiotemporal symmetry $\Sigma = \mathbf{SO}(2)$ so that time evolution is equivalent to rigid rotation. The standing waves have spatial symmetry $\Delta = \mathbb{Z}_2$ and spatiotemporal symmetry $\Sigma = \mathbb{Z}_2 \times \mathbb{Z}_2$, corresponding to a state that is fixed at all times by a single reflection and for which evolution through half a period is the same as rotation through $180°$. These rotating waves and standing waves occur, for example, in the motion of a flexible hosepipe with circular cross-section [1]. In particular, the standing wave is an example of a discrete rotating wave. Hopf bifurcation with dihedral symmetry [7, Chapter XVIII] produces a particularly rich set of examples of discrete rotating waves. Examples of such discrete rotating waves are shown schematically in Figures 5.1(b) and 5.3(b) of Section 5.

The purpose of this paper is to understand secondary bifurcation from discrete rotating waves such as those that occur in Hopf bifurcation from equilibria. Throughout, we make the simplifying assumption that $\dim \Delta = \dim \Sigma = \dim \Gamma$. (It follows that generically the discrete rotating wave is isolated in the usual sense that there is a neighborhood U of the discrete rotating wave such that no further periodic solutions are contained entirely in U.) There are technical problems that arise in bifurcation from continuous group orbits of discrete rotating waves (or more generally from relative periodic solutions); these issues are addressed in Sandstede *et al.* [21]. In fact, the work of [21] in conjunction with the results in this paper, lead to a general theory for compact (and noncompact) symmetry groups, see Wulff *et al.* [26].

Chossat and Golubitsky [4] consider the case $m = 1$ ($\Sigma = \Delta$), where periodic solutions have spatial symmetry but no further spatiotemporal symmetry. They proceed by analyzing the Poincaré map G which in this case is a general Δ-equivariant diffeomorphism. The resulting theory is similar to the theory of [7] for bifurcation from equilibria.

When $m > 1$, the map G remains Δ-equivariant, but there are additional restrictions coming from the spatiotemporal symmetries in $\Sigma - \Delta$. In general, the restrictions on the Poincaré map are not of a type amenable to direct analysis. It is useful to consider the diffeomorphism f obtained by integrating for approximately $1/m$'th of the period and pulling back to the original cross-section by the appropriate symmetry element (see Figure 2.1). For example, when $\Sigma = \mathbb{Z}_2$, $\Delta = 1$, it can be shown that $G = f^2$, from which it follows that generically the linearized Poincaré map has no eigenvalues at -1 leading to the phenomenon known as *suppression of period-doubling*, see Swift and Wiesenfeld [22].

Fiedler [5] considers in detail the case when Σ is cyclic (with the emphasis on global, rather than local, bifurcation theory). For Σ cyclic, f is a general Δ-equivariant diffeomorphism. It is now possible to apply the methods in [4] to f and then to reinterpret the results in terms of the full problem. In Section 4(c), we describe a natural generalization of a local bifurcation-theoretic result of Fiedler [5, Theorem 5.11] that includes all cases when f is Δ-equivariant.

For general discrete rotating waves, the mapping f is not Δ-equivariant in the usual sense. Rather, there is an automorphism $\phi \in \mathrm{Aut}(\Delta)$ associated with the discrete rotating wave such that

$$f(\delta x) = \phi(\delta) f(x),$$

(cf. [17, 11]). The diffeomorphism f is completely general subject to this *twisted equivariance* condition. The structure of such diffeomorphisms (referred to as k-symmetric maps) has been studied in [13, 10]. (We note that these references consider also time-reversal symmetries, but such issues are not the subject of the present paper.)

In this paper, we systematically study bifurcation from discrete rotating waves by studying bifurcation from fixed points for a twisted equivariant diffeomorphism. In particular, we classify the local bifurcations in terms of the structure of the center subspace of the linearized twisted equivariant diffeomorphism. The nonlinear theory then follows the setup suggested in [11] (see also [19]).

In Section 2, we describe the basic setup and introduce the twisted equivariant map f that governs the local bifurcations. Our main results are presented in Sections 3, 4 and 6. There are two kinds of bifurcation, nonHopf and Hopf bifurcation, which are classified at the linear level in Section 3. Based on the linear theory, we show in Sections 4 and 6 that generically nonHopf and Hopf bifurcation reduce to bifurcations of the kind studied in [4]. Examples that illustrate our theory for nonHopf bifurcation can be found in Section 5. Example 6.5 illustrates the theory for Hopf bifurcation.

To apply the results of [4], it is necessary to compute the irreducible representations of certain cyclic extensions of the spatial symmetry group Δ. (We note that the appropriate cyclic extensions are related to the spatiotemporal symmetry group Σ but are not the group Σ itself.) In practice, it is often the case that the irreducible representations of Δ are 'known'. It is then a standard technique in representation theory, the theory of *induced representations*, to build up the irreducible representations of the cyclic extension of Δ from the irreducible representations of Δ. This is the subject of Section 7. We then reanalyze nonHopf bifurcation from this perspective in Section 8.

2. First hit maps and twisted equivariance

Let $\Gamma \subset \mathbf{O}(n)$ be a compact Lie group acting orthogonally on $\mathbb{R}^n$ and let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a Γ-equivariant vector field. Suppose that P_0 is a periodic solution with spatial symmetry group Δ and spatiotemporal symmetry group Σ. Suppose also that P_0 is an isolated discrete rotating wave so that $\Sigma/\Delta \cong \mathbb{Z}_m$, for some $m \geq 1$, and $\dim \Delta = \dim \Sigma = \dim \Gamma$.

Let T be the period of P_0 and choose $x_0 \in P_0$. Then $\psi(x_0, T/m) = \sigma x_0$ for some $\sigma \in \Sigma$ where ψ is the flow corresponding to the vector field F. It follows from the above considerations that Σ is generated by Δ and σ. In particular, $\sigma^m \in \Delta$. Note that σ is defined only up to multiplication by elements of Δ. The following result ensures that σ can be chosen to have finite order.

Lemma 2.1. *Suppose that Σ is a compact Lie group and that Δ is a closed normal subgroup such that $\Sigma/\Delta \cong \mathbb{Z}_m$. Then there is an element $\sigma \in \Sigma$ of finite order such that Σ is generated by Δ and σ.*

Proof. Choose $\sigma \in \Sigma$ so that Σ is generated by Δ and σ. We show that there is an element $\delta \in \Delta$ such that $\sigma' = \delta\sigma$ has finite order.

Let C be the closed subgroup of Δ generated by σ^m and let C^0 be the connected component of the identity in C. Then C^0 is a torus and $\sigma^{jm} \in C^0$ for some $j \geq 1$. Choose $\alpha \in C^0$ such that $\alpha^{jm} = \sigma^{jm}$, and set $\delta = \alpha^{-1} \in \Delta$, $\sigma' = \delta\sigma$. We claim that α commutes with σ. It then follows that $(\sigma')^{jm} = (\alpha^{-1}\sigma)^{jm} = \alpha^{-jm}\sigma^{jm} = 1$ as required.

It remains to prove the claim. Let D be the closed subgroup of Σ generated by σ. Then D is abelian and contains σ. Further, $\alpha \in C \subset D$ and hence α commutes with σ. $\square$

Let X be a Δ-invariant local section containing x_0 and define the first hit maps $g^{(j)} : X \to \sigma^j X$. Let $f = \sigma^{-1} g^{(1)}$, see Figure 2.1. The diffeomorphism $f : X \to X$ is equivariant with respect to distinct actions of Δ in the domain and the range. Indeed $f(\delta x) = (\sigma^{-1}\delta\sigma) f(x)$.

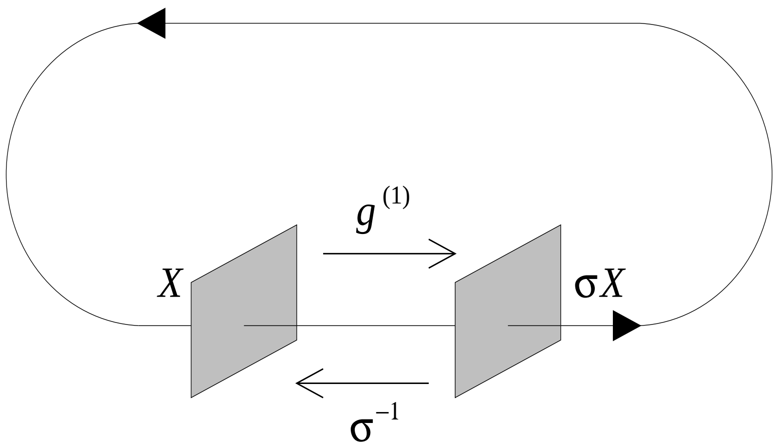

Fig. 2.1. The twisted equivariant diffeomorphism $f = \sigma^{-1}g^{(1)} : X \to X$

Note that σ does not act on the section X, but $\sigma^{-1}\delta\sigma \in \Delta$ and hence acts on X. We introduce the automorphism $\phi \in \mathrm{Aut}(\Delta)$ defined by $\phi(\delta) = \sigma^{-1}\delta\sigma$. More generally, ϕ^s denotes the automorphism induced by σ^s. The *twisted equivariance* condition satisfied by f now takes the form

$$f(\delta x) = (\sigma^{-1}\delta\sigma)f(x) = \phi(\delta)f(x). \tag{2.1}$$

Remark 2.2. (a) The automorphism ϕ depends on the choice of σ but is defined uniquely up to inner automorphism. In other words, there is a unique outer automorphism of Δ associated with the discrete rotating wave P_0.
(b) There is an integer k which depends on the specific choice of ϕ (or σ) and which plays a significant role in this paper. We define k to be the least positive integer such that σ^k lies in the centralizer of Δ. Equivalently, the automorphism ϕ has order k. (In [13], a twisted equivariant map f is called k-symmetric, reflecting the fact that the k'th iterate of f is equivariant.)

By Lemma 2.1, we can choose ϕ (or σ) so that k is finite. When Σ is a semidirect product of Δ and $\mathbb{Z}_m$, it is natural to choose σ to be a generator of $\mathbb{Z}_m$ so that $\sigma^m = 1$ and hence k divides m. When Σ is a direct product of Δ and $\mathbb{Z}_m$, this choice of σ gives $k = 1$.

When Σ is not a semidirect product of Δ and $\mathbb{Z}_m$, there are examples where $k > m$. Let $\mathbb{Z}_2^4$ be the group of diagonal 4×4 matrices with entries ± 1 on the diagonal and set $\Sigma = \mathbb{Z}_2^4 \rtimes \mathbb{Z}_4$ where $\mathbb{Z}_4$ is generated by a 4×4 cyclic permutation matrix σ. Define $\Delta = \mathbb{Z}_2^4 \rtimes \mathbb{Z}_2$ where $\mathbb{Z}_2$ is generated by σ^2. Clearly, Σ is a cyclic extension of Δ and $m = 2$. Moreover, σ^2 does not commute with elements of $\mathbb{Z}_2^4$ and it follows that the automorphism ϕ corresponding to σ has order $k = 4$. Replacing σ by another element of the coset $\sigma\Delta$ does not reduce k.

The first hit maps are recovered from f by the relation $g^{(j)} = \sigma^j f^j$. In particular, the Poincaré map for P_0 is given by

$$G = g^{(m)} = \sigma^m f^m. \tag{2.2}$$

Note that f and G define discrete dynamical systems on X, whereas in general, iteration of the maps $g^{(j)}$ is undefined.

Since σ^m has finite order and commutes with f, certain iterates of the Poincaré map G and the derived map f coincide. It follows that the dynamics for these maps are very closely related. In the following, we use the terminology 'periodic solution' specifically to refer to a periodic solution for the flow, whereas 'periodic orbit' refers to a periodic trajectory for the discrete dynamical system defined by f or G.

Proposition 2.3. *A point $x \in X$ (close to x_0) lies on a periodic orbit for f if and only if x lies on a periodic solution P lying in a small enough neighborhood of P_0. Moreover, x is hyperbolic, a periodic sink, a periodic saddle, etc for f if and only if P has the corresponding property for the underlying flow.*

Proof. Suppose that σ has order q. Then $G^q = f^{qm}$ and it follows that periodic points for f coincide with periodic points for G and hence with periodic solutions that lie in the neighborhood where the Poincaré maps are defined.

Specifically, suppose that $f^j(x) = x$. Then x is a fixed point for $f^{qmj} = G^{qj}$ and hence has the same stability type for each map. It follows that the periodic point x has the same stability type for f as for G. $\square$

Let $x \in X$ be a periodic point for f and let P be the corresponding periodic solution. Let $\Delta^{\rm bif} \subset \Delta$ denote the isotropy subgroup of x. The next result shows how to recover the symmetry group $\Sigma^{\rm bif} \subset \Sigma$ that fixes P as a set.

Lemma 2.4. *Suppose that $x \in X$ is a periodic point for f with corresponding periodic solution P. Let $p \geq 1$ be least such that $f^p(x) = \delta x$ for some $\delta \in \Delta$. Then $\Sigma^{\rm bif}$ is the group generated by $\Delta^{\rm bif}$ and $\sigma^p\delta$. The spatiotemporal symmetry $\sigma^p\delta$ corresponds to a time-shift by approximately the period of P_0 multiplied by p/m.*

Proof. Clearly $\Delta^{\rm bif} \subset \Sigma^{\rm bif}$. Furthermore, $g^{(p)}(x) = \sigma^p f^p(x) = \sigma^p\delta x$, so that $\sigma^p\delta x \in P$ and hence $\sigma^p\delta \in \Sigma^{\rm bif}$. Note also that $\psi(x,T) = \sigma^p\delta x$ for some $T > 0$ least (which is close to $\frac{p}{m}$ times the period of P_0). This T is also least such that $\psi(x,T) \in \Sigma x$, as otherwise p would not be least. Consequently, the symmetry group $\Sigma^{\rm bif}$ is generated by the isotropy subgroup $\Delta^{\rm bif}$ and $\sigma^p\delta$. $\square$

Remark 2.5. It is natural to measure time-shifts as a fraction of the period of the periodic solution P (as opposed to the old periodic solution P_0). Note that the period of P is approximately the period of P_0 multiplied by some positive integer q. Hence, the spatiotemporal symmetry $\sigma^p\delta$ in Lemma 2.4 corresponds to a phase-shift by (exactly) the period of P multiplied by $\frac{p}{qm}$. It is convenient to speak of the spatiotemporal symmetry $(\sigma^p\delta, \frac{p}{qm})$ signifying that $g^{(p)}(x) = \sigma^p\delta x$. In particular, the original periodic solution P_0 (for which $q = 1$) has the spatiotemporal symmetry $(\sigma, \frac{1}{m})$.

In this subsection, we have reduced the study of bifurcations from a discrete rotating wave to the study of bifurcation from a fixed point for the map f. Note that the Poincaré map G is Δ-equivariant in the usual sense, whereas f is equivariant only with respect to distinct actions in the domain and range. On the other hand, f is completely arbitrary up to the twisted equivariance condition (2.1), whereas the Poincaré map is constrained by the relation (2.2). It turns out to be simpler to apply genericity arguments to the twisted equivariant map f than to the constrained equivariant map G.

3. Classification of local bifurcations

Recall that $G = \sigma^m f^m$ is the Poincaré map associated with the discrete rotating wave P_0. We noted in the proof of Proposition 2.3 that certain powers of f and G coincide. It follows that $L = (df)_{x_0}$ and $(dG)_{x_0}$ share a common center subspace E^c. Since G is Δ-equivariant, the center subspace E^c is Δ-invariant.

Similarly, the nonlinear maps f and G share a common Δ-invariant center manifold. By the center manifold theorem [8,20], we can reduce to the center manifold associated with the periodic solution P_0. After center manifold reduction, the center subspace E^c is identified with the cross-section X. In the sequel, G denotes the restriction of G to the center manifold of x_0 and so on. The Δ-invariance of the center manifold guarantees that the structure of the original ODE is preserved by this reduction.

Now, define Δ_L to be the group generated by the actions of Δ and L on E^c. The twisted equivariance condition $L\delta = \phi(\delta)L$, $\delta \in \Delta$, implies that Δ is a normal subgroup of Δ_L. In general, Δ_L need not be compact since L need not be semisimple. It is a consequence of Theorem 3.2 below that twisted equivariant linear maps are generically semisimple.

In Remark 2.2, we introduced the integer k given by the order of the automorphism ϕ in (2.1). The integer k and the linear map L depend on the choice of σ. Of course, $(dG)_{x_0}$ is independent of the choice of σ.

Proposition 3.1. *The linear maps $(dG)_{x_0}$ and L^k commute with the action of Δ_L on E^c.*

Proof. Clearly, L^k commutes with L and $L^k\delta = \phi^k(\delta)L^k = \delta L^k$ by definition of k so that L^k is Δ-equivariant. The Poincaré map G is Δ-equivariant so that $(dG)_{x_0}$ commutes with Δ. Moreover, $L\sigma^m = \phi(\sigma^m)L = \sigma^m L$ and so $(dG)_{x_0} = \sigma^m L^m$ commutes with L. $\square$

Theorem 3.2. *Generically, the action of Δ_L on E^c is irreducible. Moreover, Δ_L is compact.*

Proof. Suppose that E^c is not Δ_L-irreducible. Let $V \subset E^c$ be a proper Δ_L-irreducible subspace. Such a subspace V exists since E^c is finite dimensional. Since Δ is compact, we have a splitting $E^c = V \oplus W$ where W is a Δ-invariant subspace.

Since Δ_L cannot be assumed to be compact, it is not necessarily the case that L leaves W invariant. By construction, $L(V) \subset V$ so that $L : V \oplus W \to V \oplus W$

has the block-matrix form

$$L = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.$$

Since Δ leaves V and W invariant, the twisted equivariance of L is given (in an obvious notation) by

$$A\delta_V = \phi(\delta)_V A, \qquad B\delta_W = \phi(\delta)_V B, \qquad D\delta_W = \phi(\delta)_W D. \tag{3.1}$$

Now define

$$L_\epsilon = \begin{pmatrix} A & B \\ 0 & (1+\epsilon)D \end{pmatrix}.$$

It is evident from the linearity of conditions (3.1) that L_ϵ is twisted equivariant for all ϵ. Moreover, the center subspace of E^c is given by the Δ_L-irreducible subspace V for all nonzero ϵ.

Next, we verify that Δ_L is compact. By Proposition 3.1, L^k commutes with the action of Δ_L, so that elements of Δ_L preserve the eigenspaces of L^k. In combination with the irreducibility of Δ_L, this implies that L^k must be a scalar (real or complex) multiple of the identity. Moreover, L has eigenvalues on the unit circle, so that L^k is a unit scalar multiple of the identity. In particular, the closed group $\langle L^k \rangle$ generated by L^k is compact.

Since Δ is compact, it follows immediately that $\Delta \times \langle L^k \rangle$ is compact. But Δ_L is a finite cyclic extension (by L) of $\Delta \times \langle L^k \rangle$ and hence is compact. □

Theorem 3.2 is the first step towards understanding generic bifurcation from discrete rotating waves. We note that this result, taken alone, is not so useful since *a priori* we do not know Δ_L. Part of the theory is to give a concrete description of Δ_L. This is done in Theorems 4.2 and 6.4 below. In addition, Theorem 3.4 below classifies the types of bifurcation that can occur in terms of the action of Δ_L.

A local bifurcation from P_0 is signaled by eigenvalues of $(dG)_{x_0}$ crossing the unit circle. There are three cases:

– **Period preserving bifurcation:** Eigenvalues at $+1$.
– **Period doubling bifurcation:** Eigenvalues at -1.
– **Hopf bifurcation:** Complex conjugate eigenvalues on the unit circle.

Remark 3.3. The terminology above is justified by the results in this paper (see for example, Proposition 4.5). It is important to note that these cases are distinguished by the eigenvalues of $(dG)_{x_0}$ and *not* by the eigenvalues of $L = (df)_{x_0}$. For many purposes, it is convenient to combine the period preserving and period doubling cases into a single bifurcation which we call *nonHopf* bifurcation.

Let V be an irreducible representation of a group Γ and let $\mathrm{Hom}_\Gamma(V)$ denote the space of linear maps $A : V \to V$ that commute with the action of Γ. Then it is well-known (see for example [7, Chapter XII,3]) that $\mathrm{Hom}_\Gamma(V)$ has the structure of a real division ring and hence is isomorphic to the reals $\mathbb{R}$, the complex numbers $\mathbb{C}$ or the quaternions $\mathbb{H}$. The representation is accordingly said to be irreducible of real, complex or quaternionic type. Irreducible representations of real type are also called absolutely irreducible and irreducible representations of complex or quaternionic type are said to be nonabsolutely irreducible.

Theorem 3.4. *Suppose that the discrete rotating wave P_0 undergoes local bifurcation. Generically, either*

(a) Δ_L *acts absolutely irreducibly on* E^c *and there is a nonHopf bifurcation (either period-preserving or period-doubling), or*
(b) The action of Δ_L *on* E^c *is irreducible of complex type and there is a Hopf bifurcation.*

Proof. By Theorem 3.2, generically Δ_L acts irreducibly on E^c. A priori, the action could be irreducible of real, complex or quaternionic type. We consider the three cases in turn.

If the action of Δ_L on E^c is absolutely irreducible, then it follows from Proposition 3.1 that $(dG)_{x_0} = \pm I$ and hence that there is nonHopf bifurcation.

Next suppose that the action of Δ_L on E^c is irreducible of complex type. By Proposition 3.1, L^k is a complex scalar multiple of the identity. Write $L^k = \alpha I$ where $\alpha \in \mathbb{C}$. Since E^c is the center subspace of L, and hence is the center subspace of L^k, it follows that $|\alpha| = 1$.

We claim that α can be any complex number with absolute value one. To see this, let $\beta \in \mathbb{C}$ be a general complex number with absolute value one. Since the action of Δ_L is of complex type, βI commutes with L. It follows that $(\beta L)^k = \beta^k L^k = \beta^k \alpha I$. This proves the claim since $\beta^k \alpha$ is general.

It follows from the claim that generically the eigenvalues of L are not roots of unity. Since certain powers of L and $(dG)_{x_0} = \sigma^m L^m$ coincide, it must be the case that the eigenvalues of $(dG)_{x_0}$ are not real.

Finally, we show that the remaining possibility, namely that the action of Δ_L is irreducible of quaternionic type, is nongeneric. Suppose for contradiction that E^c is irreducible of quaternionic type. By Proposition 3.1, we can write $L^k = \alpha I$ where $\alpha \in \mathbb{H}$. Proceeding just as in the complex case, we can perturb L so that α is a general unit quaternion. In particular, by the noncommutativity of $\mathbb{H}$, there is a quaternion $\beta \in \mathbb{H}$ that does not commute with α. But βI commutes with the action of Δ_L and hence commutes with L^k. This leads to the equation $\alpha\beta I = L^k \beta I = \beta L^k = \beta\alpha I$ contradicting the fact that α and β do not commute. □

Remark 3.5. Theorem 3.4 is well-known for periodic solutions with purely spatial symmetries ($m = 1$), and is implicit in the work of Chossat and Golubitsky [4] where the representation of Δ on E^c is stressed (absolutely irreducible in the case of nonHopf bifurcation and Δ-*simple* in the case of Hopf bifurcation), instead of the representation of Δ_L on E^c.

4. Analysis of NonHopf bifurcation

In this section, we analyze nonHopf bifurcation from discrete rotating waves. There is a systematic approach to bifurcation from periodic solutions with only spatial symmetry due to Chossat and Golubitsky [4]. We obtain a similarly systematic approach by reducing our problem to the one studied in [4].

There are four subsections. In Subsection (a), we obtain an explicit description of the representation of the group Δ_L from Theorem 3.4(a). The nonlinear analysis is given in Subsection (b) and relies on Birkhoff normal form theory for twisted equivariant maps [10]. Generalized versions of *suppression of period-doubling* are given in Remark 4.6. In Subsection (c), we consider the simplest case ($k = 1$) where twisted equivariance reduces to equivariance in the usual sense. In Subsection (d), we compare our results with previous approaches.

(a). Structure of the group Δ_L

In the nonHopf case, Δ_L acts absolutely irreducibly on E^c. It follows from Proposition 3.1 that

$$(dG)_{x_0} = \pm I, \qquad L^k = \pm I.$$

We stress that the signs of $(dG)_{x_0}$ and L^k are independent and play different roles. The sign of $(dG)_{x_0}$ determines whether there is a period preserving or period doubling bifurcation, see Proposition 4.5. The sign of L^k determines the group Δ_L, see Theorem 4.2.

Define the abstract group $\Delta \rtimes \mathbb{Z}_k$ to be the cyclic extension of Δ by an element τ of order k such that $\tau^{-1}\delta\tau = \phi(\delta) = \sigma^{-1}\delta\sigma$. We stress that this group depends only on Δ and the choice of ϕ.

A second cyclic extension $\Delta \rtimes \mathbb{Z}_{2k}$ is defined similarly, except that τ now has order $2k$. In this case, τ^k is a nonidentity element that commutes with the elements of Δ.

Remark 4.1. (a) When k is odd, $\Delta\rtimes\mathbb{Z}_{2k} = (\Delta\rtimes\mathbb{Z}_k)\times\mathbb{Z}_2$. No such decomposition exists when k is even, as demonstrated below in Example 5.3. There, we have $\Sigma = \mathbb{D}_4$, $\Delta = \mathbb{D}_2$. It turns out that $k = 2$ and $\Delta\rtimes\mathbb{Z}_2 \cong \mathbb{D}_4$. However $\Delta\rtimes\mathbb{Z}_4 \ncong \mathbb{D}_4\times\mathbb{Z}_2$.
(b) In the case that Σ is a semidirect product of Δ and $\mathbb{Z}_m$, it is natural to take σ to be a generator of $\mathbb{Z}_m$. In particular, σ has order m and k divides m. Moreover, we have the group isomorphism

$$\Delta \rtimes \mathbb{Z}_k \cong \Sigma/\mathbb{Z}_{m/k},$$

where $\mathbb{Z}_{m/k}$ is generated by σ^k. In the extreme cases $k = 1$ and $k = m$, we have $\Delta \rtimes \mathbb{Z}_k = \Delta$ and $\Delta \rtimes \mathbb{Z}_k \cong \Sigma$ respectively.

The group isomorphism is obtained as followed. Elements of Σ can be represented uniquely as $\delta\sigma^j$ where j is computed $\bmod\, m$. Similarly, elements of $\Delta \rtimes \mathbb{Z}_k$ can be represented uniquely as $\delta\tau^j$ where j is computed $\bmod\, k$. Since k divides m, the map $\chi : \Sigma \to \Delta \rtimes \mathbb{Z}_k$ given by $\chi(\delta\sigma^j) = \delta\tau^j$ is well-defined. Since σ and τ induce the same automorphism ϕ of Δ, the map χ is a group homomorphism. Moreover, it is clear that χ is onto and $\ker\chi = \mathbb{Z}_{m/k}$. Hence χ induces the required isomorphism.

Theorem 4.2. *Suppose that Δ_L acts absolutely irreducibly on E^c. Then either*

(i) $\Delta_L = \Delta \rtimes \mathbb{Z}_k$ *(if* $L^k = I$*), or*
(ii) $\Delta_L = \Delta \rtimes \mathbb{Z}_{2k}$ *(if* $L^k = -I$*).*

Moreover, the action of $\mathbb{Z}_k$ *(resp.* $\mathbb{Z}_{2k}$*) is generated by* L*.*

Proof. By Proposition 3.1, L^k commutes with the action of Δ_L. Hence $L^k = \pm I$. Moreover $L\delta = \phi(\delta)L$. It follows from the definitions of $\Delta \rtimes \mathbb{Z}_k$ and $\Delta \rtimes \mathbb{Z}_{2k}$ that Δ_L is isomorphic to one of these groups with L^{-1} playing the role of τ. □

Remark 4.3. To apply Theorem 4.2, it is necessary only to consider the absolutely irreducible representations of $\Delta \rtimes \mathbb{Z}_{2k}$, since these representations incorporate the absolutely irreducible representations of $\Delta \rtimes \mathbb{Z}_k$, but with L^k acting trivially. (It should be noted that in general the order of L may be any integer that divides $2k$. For example, Δ may act trivially on the center subspace in which case the center subspace is one-dimensional and $L = \pm I$ regardless of the size of k.)

In the nonequivariant context, we have $\Delta = \Sigma = \mathbf{1}$ and $\Delta_L = \mathbf{1}$ or $\Delta_L = \mathbb{Z}_2$. Although it is usual practice to consider these cases separately, one could equally say that $\Delta_L = \Delta \rtimes \mathbb{Z}_{2k} = \mathbb{Z}_2$ acts absolutely irreducibly in each case — either by the one-dimensional trivial representation ($L = I$), or by the one-dimensional nontrivial representation ($L = -I$).

In Section 8 we work throughout with the group $\Delta \rtimes \mathbb{Z}_{2k}$. Using methods developed in Section 7, it is possible to derive the absolutely irreducible representations of $\Delta \rtimes \mathbb{Z}_{2k}$ from the irreducible representations of Δ, see Theorem 8.1.

(b). Nonlinear analysis of nonHopf bifurcation

Following Lamb [10], we write $f = Lh$. Since f and L are twisted equivariant, it follows that h is Δ-equivariant in the usual sense. Moreover, h is a general Δ-equivariant diffeomorphism satisfying $h(x_0) = x_0$ and $(dh)_{x_0} = I$.

Lemma 4.4 [10]. *Up to arbitrarily high order, coordinates can be chosen so that* h *is* Δ_L*-equivariant.*

Proof. We have already observed that h is Δ-equivariant (to all orders). Moreover, L is semisimple (indeed $L^k = \pm I$). Hence it follows from Birkhoff normal form theory for twisted equivariant maps [10] that f can be transformed (by changes of coordinates that preserve twisted equivariance) so that f is L-equivariant up to any specified order in its Taylor expansion. In particular, $h = L^{-1}f$ is Δ_L-equivariant up to any specified order. □

We proceed by supposing that h is Δ_L-equivariant to *all* orders. This simplification is lifted at the end of the subsection. Thus $h : E^c \to E^c$ is a general diffeomorphism, satisfying $h(x_0) = x_0$ and $(dh)_{x_0} = I$, equivariant under an absolutely irreducible representation of the compact Lie group Δ_L. This is precisely the setting discussed in Chossat and Golubitsky [4] — bifurcation from a fixed point with eigenvalue passing through 1 for a diffeomorphism with Δ_L symmetry.

Since $f = Lh$ where h and L commute and $L^k = \pm I$, we have $h^{2k} = f^{2k}$. Therefore, periodic points for h coincide with periodic points for f and their stabilities are identical. It follows from Proposition 2.3 that locally the existence and stability of bifurcating periodic solutions for the underlying flow is reduced to the existence and stability of periodic points for the diffeomorphism h.

The methods of [4] lead to a partial determination of the local dynamics of h. In particular, bifurcation of fixed points is equivalent to bifurcation of steady-states for Δ_L-equivariant vector fields. Hence the branching and stability of fixed points for h can be read off from the corresponding results for steady-state bifurcations [7].

In the remainder of this subsection, we discuss how properties of periodic solutions for the flow can be read off from properties of the fixed points for h.

First we determine the periods of the periodic solutions corresponding to fixed points for h. In the process, we justify the terminology 'period preserving/doubling bifurcation' introduced in Section 3.

Proposition 4.5. *In the period preserving case ($(dG)_{x_0} = I$), branches of fixed points x_λ for h correspond to branches of periodic solutions P_λ for the flow with period approximately equal to the period of P_0. In the period doubling case ($(dG)_{x_0} = -I$), branches of fixed points x_λ for h correspond to branches of periodic solutions P_λ for the flow with period approximately twice the period of P_0.*

Proof. By our assumption of Birkhoff normal form symmetry to all orders, h is L-equivariant. The Poincaré map satisfies

$$G = \sigma^m f^m = \sigma^m (Lh)^m = \sigma^m L^m h^m = (dG)_{x_0} h^m = \pm h^m.$$

Hence, a fixed point of h corresponds to a fixed point of G in the period preserving case and to a period two point of G in the period doubling case. $\square$

Remark 4.6. It is well-known that in certain cases, the spatiotemporal symmetry of a discrete rotating wave *suppresses* the possibility of period-doubling. The simplest case is when $\Sigma = \mathbb{Z}_2$ and $\Delta = \mathbf{1}$, see [22].

We describe a generalized version of this phenomenon. Suppose that k divides m. Suppose further that m/k is even and that σ^m has odd order. Then we have suppression of period-doubling (independent of the action of Δ_L).

To see this, recall that $(dG)_{x_0} = \sigma^m L^m$. Since $L^k = \pm I$ and m/k is even, it follows that $L^m = I$. But then $\sigma^m = (dG)_{x_0} = \pm I$. Since σ^m has odd order, we must be in the case $(dG)_{x_0} = +I$.

An important special case is when Σ is a semidirect product of Δ and $\mathbb{Z}_m$ (so that $\sigma^m = 1$). Suppression of period-doubling occurs when m/k is even.

Next, we discuss the symmetries of the periodic solutions obtained in Proposition 4.5. Suppose that x is a fixed point for h with isotropy subgroup $J \subset \Delta_L$. Then the spatial symmetry group of the corresponding periodic solution P is given by $\Delta^{\rm bif} = J \cap \Delta$. The spatiotemporal symmetries $\Sigma^{\rm bif}$ can be read off from J as follows [11].

Lemma 4.7. *Suppose that $x \in E^c$ is a fixed point for h with isotropy subgroup $J \subset \Delta_L$. Let P be the corresponding periodic solution. Let $p \geq 1$ be least such that $L^{-p}\delta \in J$ for some $\delta \in \Delta$. Then $\Sigma^{\rm bif}$ is the group generated by $\Delta^{\rm bif}$ and $(\sigma^p \delta, \frac{p}{qm})$, where $q = 1$ in the period-preserving case and $q = 2$ in the period-doubling case.*

Proof. We apply Lemma 2.4, searching for the least p such that $f^p(x) = \delta x$ where $\delta \in \Delta$. Since x is a fixed point for h, we compute that $f^p(x) = L^p h^p(x) = L^p x$. Hence $f^p(x) = \delta x$ if and only if $L^p x = \delta x$. Equivalently, $L^{-p}\delta \in J$. □

When computing the quantity $\frac{p}{qm}$, it should be understood that m is known from the outset and that q takes the values 1 or 2 depending on whether $(dG)_{x_0} = \sigma^m L^m = \pm I$.

We note that many fixed points for h (and hence periodic solutions for the flow) can be obtained from the equivariant branching lemma [7]. Let J be an axial isotropy subgroup of Δ_L (that is, J is an isotropy subgroup with one-dimensional fixed-point subspace). Then the equivariant branching lemma predicts the existence generically of a branch of fixed points with isotropy J for the mapping h.

As promised, we end this subsection by dropping the assumption that h is Δ_L-equivariant to all orders. A fundamental determinacy theorem of Field [6] states that branches of fixed points for Δ_L-equivariant diffeomorphisms are generically determined and hyperbolic for the truncation of h at some fixed finite order (dependent only on the particular absolutely irreducible representation of the compact Lie group Δ_L). Moreover, Field [6] shows that there is a (higher) finite order at which perturbations that break the Δ_L-equivariance do not affect the branches of fixed points for h. Since we consider only perturbations that preserve Δ-equivariance of h, it follows easily that our statements about spatiotemporal symmetries are unchanged by these perturbations.

(c). The untwisted case ($k = 1$)

In this subsection, we obtain a natural generalization of the results of Fiedler [5] who studied the case when the spatiotemporal symmetry group Σ is cyclic. Indeed, we consider the case when the automorphism $\phi \in \mathrm{Aut}(\Delta)$ in equation (2.1) is the trivial automorphism. In other words, σ lies in the centralizer of Δ and $k = 1$. This includes the case when Σ is abelian [3] and the case $\Sigma \cong \Delta \times \mathbb{Z}_m$.

The condition that ϕ is the trivial automorphism implies that f is Δ-equivariant. We analyze nonHopf bifurcations, applying the results of Sections 3 and 4.

Proposition 4.8. *Let P_0 be a discrete rotating wave with spatial symmetry group Δ and spatiotemporal symmetry group Σ and suppose that $k = 1$. Suppose also that P_0 undergoes a nonHopf bifurcation. Then the center subspace is generically Δ-absolutely irreducible and either $\Delta_L = \Delta$ (if $L = I$) or $\Delta_L = \Delta \times \mathbb{Z}_2$ (if $L = -I$).*

Proof. By Theorem 3.4, the center subspace of $L = (df)_{x_0}$ (equivalently of $(dG)_{x_0}$) is generically Δ_L-absolutely irreducible. It follows from Proposition 3.1 that $L = L^k = \pm I$. Since the automorphism ϕ is trivial, Theorem 4.2 implies that $\Delta_L = \Delta$ if $L = I$ and that $\Delta_L = \Delta \times \mathbb{Z}_2$ if $L = -I$. In particular, Δ_L acts absolutely irreducibly if and only if Δ acts absolutely irreducibly. □

It follows also from Proposition 3.1 that $(dG)_{x_0} = \pm I$. Once again, we stress that the distinction between period preserving or period doubling is unrelated to the distinction between normal form symmetry Δ or $\Delta \times \mathbb{Z}_2$.

Finally, we note that σ^m lies in the center of Δ and hence acts as $\pm I$ on E^c. In the following result, the notation $\sigma^m = \pm I$ refers to the action of σ^m on E^c.

Theorem 4.9. *Let P_0 be a discrete rotating wave with spatial symmetry group Δ and spatiotemporal symmetry group Σ where $\Sigma/\Delta \cong \mathbb{Z}_m$ and suppose that $k = 1$. Suppose also that P_0 undergoes a generic nonHopf bifurcation. Each branch of fixed points for h corresponds to a branch of periodic solutions for the flow, and the periods of the periodic solutions are determined as follows:*

$\sigma^m = I$, *m even. Period preserving.*
$\sigma^m = I$, *m odd. Period preserving if $L = I$, period doubling if $L = -I$.*
$\sigma^m = -I$, *m even. Period doubling.*
$\sigma^m = -I$, *m odd. Period doubling if $L = I$, period preserving if $L = -I$.*

Proof. By Proposition 4.5, it is sufficient to determine whether $(dG)_{x_0} = I$ or $(dG)_{x_0} = -I$. But $(dG)_{x_0} = \sigma^m L^m = \sigma^m(\pm I)^m$ from which the result is immediate. □

We note that the sign of σ^m depends on the absolutely irreducible representation of Δ, whereas the parity of m is purely group theoretic.

Remark 4.10. (a) Fiedler [5, Theorem 5.11] obtained the results described in Theorem 4.9 in the case when Σ is cyclic. His terminology is based on whether $\sigma^m = -I$, referred to as *flop*, and $L = (df)_{x_0} = -I$, referred to as *flip*. For example, *flip-flop doubling* corresponds to the case when $\sigma^m = -I$, $L = -I$ and m is even. (It should be noted however that this terminology is dependent on the choice of σ and does not correspond to intrinsic properties of the bifurcation.) Buono [3] extended Fiedler's results to the case when Σ is abelian.
(b) Suppose that x is a fixed point for h with isotropy subgroup $J \subset \Delta_L$. Then the corresponding periodic solutions consist of points with isotropy $\Delta^{\rm bif} = J \cap \Delta$. The spatiotemporal group $\Sigma^{\rm bif}$ can be determined using Lemma 4.7. There are two cases to consider: $L = \pm I$ (note that $L^{-1} = L$ in all cases). When $L = I$, trivially $L \in J$ so that $\Sigma^{\rm bif} = \langle \Delta^{\rm bif}, \sigma\rangle$. When $L = -I$, there are two subcases. If there exists an element $\delta \in \Delta$ such that $L\delta \in J$, then $\Sigma^{\rm bif} = \langle \Delta^{\rm bif}, \sigma\delta\rangle$. Otherwise, $\Sigma^{\rm bif} = \langle \Delta^{\rm bif}, \sigma^2\rangle$.

Next we make explicit the interpretation of $\Sigma^{\rm bif}$ as spatiotemporal symmetries. First in the period-preserving case ($(dG)_{x_0} = \sigma^m L^m = I$), we have $\Sigma^{\rm bif} = \langle \Delta^{\rm bif}, (\sigma, \frac{1}{m})\rangle$ if $L = I$, and either $\Sigma^{\rm bif} = \langle \Delta^{\rm bif}, (\sigma\delta, \frac{1}{m})\rangle$ or $\Sigma^{\rm bif} = \langle \Delta^{\rm bif}, (\sigma^2, \frac{2}{m})\rangle$ if $L = -I$.

Similarly, in the period-doubling case ($(dG)_{x_0} = \sigma^m L^m = -I$) we have $\Sigma^{\rm bif} = \langle \Delta^{\rm bif}, (\sigma, \frac{1}{2m})\rangle$ if $L = I$, and either $\Sigma^{\rm bif} = \langle \Delta^{\rm bif}, (\sigma\delta, \frac{1}{2m})\rangle$ or $\Sigma^{\rm bif} = \langle \Delta^{\rm bif}, (\sigma^2, \frac{1}{m})\rangle$ if $L = -I$.

(d). Comparison with alternative approaches

In this subsection we compare our results to some previous attempts to study nonHopf bifurcation from periodic solutions with spatiotemporal symmetry in the twisted case $k \geq 2$. In particular, we mention the papers of Vanderbauwhede [23, 24], Nicolaisen and Werner [16] and Rucklidge and Silber [19].

A central problem in the development of the theory has been that the spatial symmetry group Δ is insufficient to characterize the bifurcations when $k \geq 2$ and yet the spatiotemporal symmetry group Σ does not act on the cross-section X and hence does not *a priori* act on the center subspace E^c. More precisely, the action of Δ on E^c need not *a priori* extend to an action of Σ on E^c.

To counteract this problem, Vanderbauwhede [23,24] observed that a certain group Σ_0 related to Σ acts on the domain of the Floquet matrix and assumed as a hypothesis in the case of nonHopf bifurcation that the center subspace of the Floquet matrix is an absolutely irreducible representation of Σ_0. See also Nicolaisen and Werner [16]. We know of no direct proof that this hypothesis holds generically, but the first indirect proof (as a corollary of our results) is presented in Theorem 4.11 below.

More recently, Rucklidge and Silber [19] restricted to the period preserving case and, following [11], considered twisted equivariant maps. Even though Σ does not act on X, it was assumed as a hypothesis in [19] that Σ acts on E^c as a symmetry of the normal form equations. Again, this hypothesis is justified by Theorem 4.11.

Theorem 4.11. *Let P_0 be a discrete rotating wave with spatiotemporal symmetry Σ and spatial symmetry Δ, with $\Sigma/\Delta \cong \mathbb{Z}_m$.*

If P_0 undergoes a nonHopf bifurcation, then generically the center subspace E^c is an absolutely irreducible representation of the group

$$\Sigma_0 = \langle \Delta, (\sigma, \tfrac{1}{2m}) \rangle \subset \Sigma \times \mathbb{Z}_{2m}.$$

Moreover, there is a one-to-one correspondence between the absolutely irreducible representations of Δ_L and Σ_0. In particular,

Period preserving nonHopf, *cf [23,19]: The generic nonHopf bifurcation is period preserving if and only if E^c is an absolutely irreducible representation of Σ, that is, an absolutely irreducible representation of Σ_0 with $(\mathrm{id}, \frac{1}{2}) \in \Sigma_0$ acting as I. Moreover, there is a one-to-one correspondence between the absolutely irreducible representations of Δ_L and Σ, given by $L^{-1} \sim \sigma$.*

Period doubling nonHopf, *cf [24]: The generic nonHopf bifurcation is period doubling if and only if E^c is an absolutely irreducible representation of Σ_0, with $(\mathrm{id}, \frac{1}{2}) \in \Sigma_0$ acting as $-I$. Moreover, there is a one-to-one correspondence between the absolutely irreducible representations of Δ_L and the absolutely irreducible representations of Σ_0 with $(\mathrm{id}, \frac{1}{2})$ acting as $-I$, given by $L^{-1} \sim (\sigma, \frac{1}{2m})$.*

Proof. At nonHopf bifurcation we have

$$(dG)_{x_0} = \sigma^m L^m = \pm I.$$

In the period preserving case, we have $\sigma^m L^m = +I$ and hence $L^{-m} = \sigma^m$. Under the identification of L^{-1} with σ, it follows that every absolutely irreducible representation of Δ_L is at the same time an absolutely irreducible representation of Σ.

In the period doubling case, we have $\sigma^m L^m = -I$ and hence $L^{-m} = -\sigma^m$. Under the identification of L^{-1} with $(\sigma, \frac{1}{2m})$, we have $L^{-m} = (\sigma^m, \frac{1}{2}) = \sigma^m(\mathrm{id}, \frac{1}{2})$ and it follows that every absolutely irreducible representation of Δ_L is at the same time an absolutely irreducible representation of Σ_0 that has $(\mathrm{id}, \frac{1}{2})$ acting as $-I$. □

Remark 4.12. The identifications $L^{-1} \sim (\sigma, \frac{1}{m})$ and $L^{-1} \sim (\sigma, \frac{1}{2m})$ can be used directly in the identification of the spatiotemporal symmetries of periodic solutions represented by the fixed points arising in the steady state bifurcation of the Δ_L-equivariant diffeomorphism h.

The groups Σ and Σ_0 are precisely the groups that one would expect to act geometrically on the center bundle of a periodic solution at period preserving and period doubling nonHopf bifurcation. This appears to be the main idea underlying the approach by Vanderbauwhede [23,24].

Despite the natural geometrical interpretation of the group Σ_0, we prefer using the group $\Delta \rtimes \mathbb{Z}_{2k}$ in the analysis of nonHopf bifurcation. In particular, the group $\Delta \rtimes \mathbb{Z}_{2k}$ often has a simpler structure (and hence a simpler representation theory) than the group Σ_0.

Example 4.13. As a simple illustration of complications in the alternative approach that are avoided by our method, let us consider a discrete rotating wave for which $k = 1$. In that case $\Delta \rtimes \mathbb{Z}_{2k}$ is isomorphic to $\Delta \times \mathbb{Z}_2$, whereas Σ_0 is isomorphic to $\Delta \times \mathbb{Z}_{2m}$. For instance, when $(\Sigma, \Delta) = (\mathbb{D}_6, \mathbb{D}_3)$, then $k = 1$ and hence $\Delta \rtimes \mathbb{Z}_{2k} \cong \mathbb{D}_3 \times \mathbb{Z}_2 \cong \mathbb{D}_6$, whereas $\Sigma_0 \cong \mathbb{D}_3 \times \mathbb{Z}_4$.

5. Examples of nonHopf bifurcation

In this section, we illustrate the theory obtained in this paper with examples of nonHopf bifurcation from a discrete rotating wave. In Subsection (a), we give a complete analysis of examples in the untwisted case ($k = 1$). Examples to illustrate the genuinely twisted equivariant case ($k \geq 2$) are given in Subsection (b).

A complete treatment of the examples for $k \geq 2$ relies on the computation of the absolutely irreducible representations of Δ_L. The required representation theory is given in Section 7 and a complete analysis of the examples for $k \geq 2$ is deferred to Section 8.

(a). Examples in the untwisted case ($k = 1$)

Example 5.1. We consider nonHopf bifurcation from a discrete rotating wave with spatial symmetry $\Delta = \mathbb{D}_n$ and spatiotemporal symmetry $\Sigma = \mathbb{D}_{2n}$ where n is odd.

Let R_θ denote counterclockwise rotation through angle θ and let κ denote a specific reflection in $\mathbb{D}_n$ (in the x-axis say). Then $\mathbb{D}_n$ is generated by $R_{2\pi/n}$ and κ. Moreover $\mathbb{D}_{2n} = \mathbb{D}_n \times \mathbb{Z}_2$ where $\mathbb{Z}_2$ is generated by R_π. Evolution on the discrete rotating wave through half a period is thus the same as rotation through π (equivalently, by any other element of $\mathbb{D}_{2n}$ that does not lie in $\mathbb{D}_n$).

Choosing $\sigma = R_\pi$, we have $k = 1$ and it follows from Proposition 4.8 that $\Delta_L = \mathbb{D}_n$ (if $L = I$) or $\Delta_L = \mathbb{D}_{2n} = \mathbb{D}_n \times \mathbb{Z}_2$ (if $L = -I$). We note that there exist 'bad' choices of σ for which $k \geq 2$.

Since $k = 1$, we can apply the methods of Section 4(c). We note that the decomposition $\mathbb{D}_{2n} = \mathbb{D}_n \times \mathbb{Z}_2$ is no longer valid when n is even. In that case, we have $k = 2$ and require the full strength of the theory presented in this paper. The case n even is analyzed partially in Example 5.4 and the complete analysis is given in Example 8.3.

Note that Σ is a direct product of Δ and $\mathbb{Z}_m$ with $m = 2$ and hence $m/k = 2$ is even. It follows from Remark 4.6 that we have suppression of period doubling.

The first step is to enumerate the absolutely irreducible representations of Δ_L. By Proposition 4.8, it suffices to enumerate the absolutely irreducible representations of $\Delta = \mathbb{D}_n$. The irreducible representations of the group $\mathbb{D}_n$, n odd, are all absolutely irreducible and can be enumerated as follows. There are 2 one-dimensional representations $V_{0,\pm}$ and $(n-1)/2$ two-dimensional representations V_j, where the action of the generators $R_{2\pi/n}$ and κ is given by

	$V_{0,+}$	$V_{0,-}$	$V_j,\ 1 \leq j < n/2$
$R_{2\pi/n}$	1	1	$\begin{pmatrix} \cos 2\pi j/n & -\sin 2\pi j/n \\ \sin 2\pi j/n & \cos 2\pi j/n \end{pmatrix}$
κ	1	-1	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(The integer subscript j indicates the action of $R_{2\pi/n}$. There is a unique irreducible representation of $\mathbb{D}_n$ for each $j \geq 1$ and two irreducible representations when $j = 0$ which are distinguished by the action of κ, hence the subscript $\pm$.)

The branches of periodic solutions (and their symmetries) corresponding to the absolutely irreducible representations of $\mathbb{D}_n$ are summarized in Table 5.1. We now describe the details behind these results for the representations V_j, $1 \leq j < n/2$. (The cases $V_{0,\pm}$ are easier and we omit the details.)

First suppose that $L = I$. Then $\Delta_L = \mathbb{D}_n$. In particular, when $j = 1$ we have the standard action of $\mathbb{D}_n$ on the plane. It follows from [7, Chapter XIII,5] that (up to conjugacy) there is a unique axial isotropy subgroup $J = \mathbb{D}_1(\kappa) \subset \Delta_L$. By Remark 4.10(b), the corresponding periodic solutions have spatial symmetry $\Delta^{\text{bif}} = J \cap \Delta = \mathbb{D}_1$ and spatiotemporal symmetry $\Sigma^{\text{bif}} = \langle \Delta^{\text{bif}}, (\sigma, 1/2) \rangle = \langle \kappa, (R_\pi, 1/2) \rangle \cong \mathbb{D}_2$. In general, for $j \geq 1$ the action has a kernel $\mathbb{Z}_{j'}$ where $j' = \gcd(j, n)$ so that $J = \mathbb{D}_{j'}(R_{2\pi/j'}, \kappa)$. By Remark 4.10(b), the corresponding periodic solutions have spatial symmetry $\Delta^{\text{bif}} = J \cap \Delta = \mathbb{D}_{j'}$ and spatiotemporal symmetry $\Sigma^{\text{bif}} = \langle \Delta^{\text{bif}}, (\sigma, 1/2) \rangle = \langle R_{2\pi/j'}\kappa, (R_\pi, 1/2) \rangle = \langle \kappa, (R_{\pi/j'}, 1/2) \rangle \cong \mathbb{D}_{2j'}$.

Next, suppose that $L = -I$. Then $\Delta_L = \mathbb{D}_{2n}$. Suppose for simplicity that $j = 1$ (this restriction is lifted just as in the case $L = I$ by incorporating the kernel $\mathbb{Z}_{j'}$). By [7], there are now two conjugacy classes of axial isotropy subgroups $J = \mathbb{D}_1(\kappa)$ and $J = \mathbb{D}_1(L\kappa)$. The respective spatial symmetry groups are given

Table 5.1. Spatiotemporal symmetry of bifurcating solutions in nonHopf bifurcation from a discrete rotating wave with symmetry $(\Sigma, \Delta) = (\mathbb{D}_{2n}, \mathbb{D}_n)$, n odd. All bifurcations are period-preserving. All bifurcations are pitchforks unless stated otherwise. Notation: $j' = \gcd(j, n)$.

Space	L	Δ^{bif}	Σ^{bif}	Remarks
$V_{0,+}$	1	$\mathbb{D}_n$	$\langle \kappa, (R_{\pi/n}, 1/2)\rangle \cong \mathbb{D}_{2n}$	saddle-node
$V_{0,+}$	-1	$\mathbb{D}_n$	$\mathbb{D}_n$	
$V_{0,-}$	1	$\mathbb{Z}_n(R_{2\pi/n})$	$\langle (R_{\pi/n}, 1/2)\rangle \cong \mathbb{Z}_{2n}$	
$V_{0,-}$	-1	$\mathbb{Z}_n(R_{2\pi/n})$	$\langle R_{2\pi/n}, (R_\pi\kappa, 1/2)\rangle \cong \mathbb{D}_n$	
V_j $1 \le j < n/2$	I	$\mathbb{D}_{j'}(R_{2\pi/j'}, \kappa)$	$\langle \kappa, (R_{\pi/j'}, 1/2)\rangle \cong \mathbb{D}_{2j'}$	transcritical if $n/j' = 3$
V_j $1 \le j < n/2$	$-I$	$\mathbb{D}_{j'}(R_{2\pi/j'}, \kappa)$ $\mathbb{Z}_{j'}(R_{2\pi/j'})$	$\mathbb{D}_{j'}$ $\langle R_{2\pi/j'}, (R_{\pi/j'}\kappa, 1/2)\rangle \cong \mathbb{D}_{j'}$	

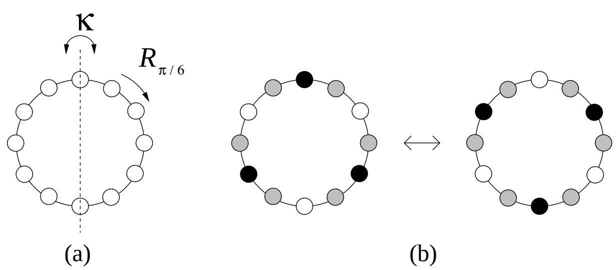

Fig. 5.1. Example of a discrete rotating wave on a system of twelve coupled cells. (a) The action of the symmetry group $\mathbb{D}_{12}$ generated by a reflection κ and a rotation $R_{\pi/6}$. (b) Schematic stroboscopic picture of a discrete rotating wave with symmetry $(\Sigma, \Delta) = (\mathbb{D}_6, \mathbb{D}_3)$ on the twelve coupled cells. Cells with identical coloring denote cells with identical states. The cells are depicted at two instants in time, half a period apart. Note that the spatiotemporal symmetry forces the grey cells to oscillate with twice the frequency of the other (black/white) cells.

by $\Delta^{\mathrm{bif}} = \mathbb{D}_1(\kappa)$ and $\Delta^{\mathrm{bif}} = \mathbf{1}$. The respective spatiotemporal symmetry groups are $\Sigma^{\mathrm{bif}} = \langle\kappa\rangle \cong \mathbb{D}_1$ and $\Sigma^{\mathrm{bif}} = \langle (R_\pi\kappa, 1/2)\rangle \cong \mathbb{D}_1$.

The stability of the bifurcating periodic solutions, and the form of the branches, can be read off from [7, Chapter XIII,5]. To reduce the number of possibilities, we assume that the underlying discrete rotating wave is asymptotically stable subcritically and unstable supercritically. When $L = I$ and $n/j' = 3$, the branches are transcritical and unstable. When $L = I$ and $n/j' \ge 5$, the branches are asymmetric pitchforks (asymmetric in the sense that the two states on the pitchfork are unrelated by symmetry). If the pitchfork bifurcation is subcritical, then the periodic solutions are unstable. If the pitchfork bifurcation is supercritical, then each pitchfork consists of a periodic sink and a periodic saddle.

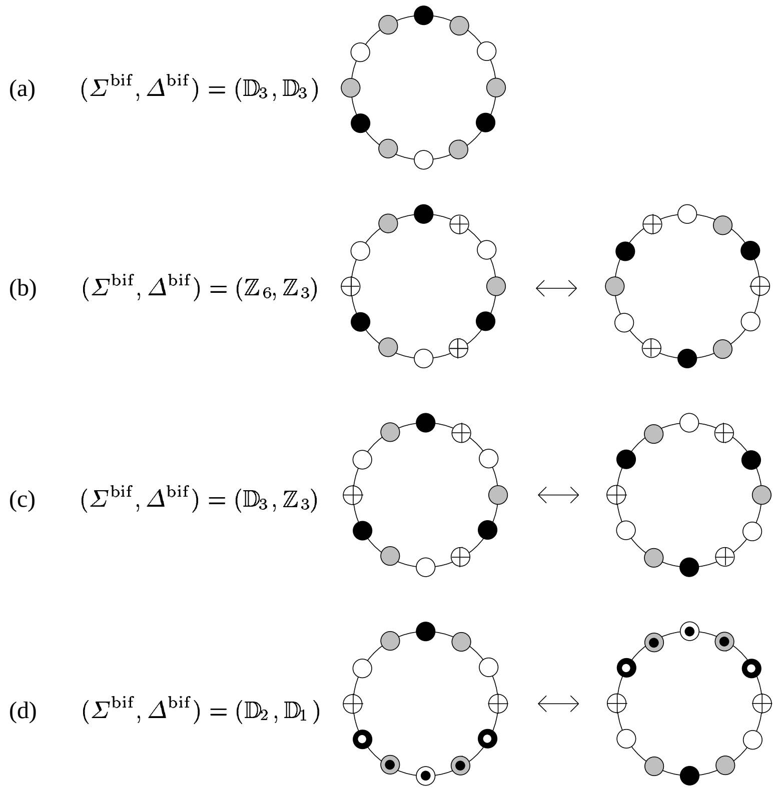

Fig. 5.2. Discrete rotating waves that generically bifurcate from the discrete rotating wave with spatiotemporal symmetry $(\Sigma, \Delta) = (\mathbb{D}_6, \mathbb{D}_3)$ in Figure 5.1, with reference to Table 5.1. (a) $[V_{0,+},\ L = -1]$ $(\Sigma^{\text{bif}}, \Delta^{\text{bif}}) = (\mathbb{D}_3, \mathbb{D}_3)$. The spatiotemporal symmetry is broken, and all cells oscillate with the same frequency. (b) $[V_{0,-},\ L = 1]$ $(\Sigma^{\text{bif}}, \Delta^{\text{bif}}) = (\mathbb{Z}_6, \mathbb{Z}_3)$. All spatial and spatiotemporal reflection symmetries are broken, and all cells oscillate with the same frequency. (c) $[V_{0,-},\ L = -1]$ $(\Sigma^{\text{bif}}, \Delta^{\text{bif}}) = (\mathbb{D}_3, \mathbb{Z}_3)$. The spatial reflection symmetries are broken. The grey and crossed cells oscillate with twice the frequency of the other (black/white) cells. But the grey and crossed cells no longer oscillate in phase with each other. (d) $[V_1,\ L = I]$ $(\Sigma, \Delta) = (\mathbb{D}_2, \mathbb{D}_1)$. All symmetries are broken except for one spatial and one spatiotemporal reflection. The crossed cells oscillate with twice the frequency of the other cells.

When $L = -I$, both branches of periodic solutions are symmetric pitchforks. Provided both solutions bifurcate supercritically, precisely one of the solutions is stable. Otherwise, both solutions are unstable.

Finally, in all cases in Table 5.1, the entire local dynamics consists of the enumerated periodic solutions together with their stabilities.

As an illustration, in Figure 5.1, we schematically depict a system of 12 symmetrically coupled cells ($\Gamma = \mathbb{D}_{12}$), with a discrete rotating wave with spatiotemporal symmetry $\Sigma = \mathbb{D}_6$ and spatial symmetry $\Delta = \mathbb{D}_3$ (so $n = 3$). Such a discrete rotating wave may occur through Hopf bifurcation from a $\mathbb{D}_{12}$ invariant steady state. In Figure 5.2 we schematically depict some of the discrete rotating waves that bifurcate from such a solution by nonHopf bifurcation. See the figure captions for more details.

Example 5.2. We consider nonHopf bifurcation from a discrete rotating wave with spatial symmetry $\Delta = \mathbf{O}(2)$ and spatiotemporal symmetry $\Sigma = \mathbf{O}(2) \times \mathbb{Z}_2$. Evolution on the discrete rotating wave through half a period is the same as applying the nontrivial element of $\mathbb{Z}_2$.

Denote the elements of $\mathbf{O}(2)$ by R_θ and κ and let ρ be the generator of $\mathbb{Z}_2$. Choosing $\sigma = \rho$, we ensure that $k = 1$. (Observe that in this example there exist 'bad' choices of σ for which k is arbitrarily large or for which $k = \infty$.) As in Example 5.1, we have suppression of period doubling.

The irreducible representations of $\mathbf{O}(2)$ (all of which are absolutely irreducible) are as follows:

	$V_{0,+}$	$V_{0,-}$	$V_j, j \geq 1$
R_θ	1	1	$\begin{pmatrix} \cos j\theta & -\sin j\theta \\ \sin j\theta & \cos j\theta \end{pmatrix}$
κ	1	-1	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

The corresponding branches of periodic solutions are summarized in Table 5.2. The details are similar to, but easier than, the arguments in Example 5.1. The main point is to recall from [7] that in the standard irreducible representation V_1 of $\mathbf{O}(2)$, there is up to conjugacy a unique axial isotropy subgroup $J = \mathbb{D}_1$.

Table 5.2. Spatiotemporal symmetry of bifurcating solutions in nonHopf bifurcation from a discrete rotating wave with $(\Sigma, \Delta) = (\mathbf{O}(2) \times \mathbb{Z}_2, \mathbf{O}(2))$ symmetry. All bifurcations are period-preserving. All bifurcations are pitchforks unless stated otherwise.

Space	L	Δ^{bif}	Σ^{bif}	Remarks
$V_{0,+}$	1	$\mathbf{O}(2)$	$\langle \mathbf{O}(2), (\rho, 1/2) \rangle \cong \mathbf{O}(2) \times \mathbb{Z}_2$	saddle-node
$V_{0,+}$	-1	$\mathbf{O}(2)$	$\mathbf{O}(2)$	
$V_{0,-}$	1	$\mathbf{SO}(2)$	$\langle \mathbf{SO}(2), (\rho, 1/2) \rangle \cong \mathbf{SO}(2) \times \mathbb{Z}_2$	
$V_{0,-}$	-1	$\mathbf{SO}(2)$	$\langle \mathbf{SO}(2), (\rho\kappa, 1/2) \rangle \cong \mathbf{O}(2)$	
$V_j,\ j \geq 1$	I	$\mathbb{D}_j$	$\langle \mathbb{D}_j, (\rho, 1/2) \rangle \cong \mathbb{D}_j \times \mathbb{Z}_2$	
$V_j,\ j \geq 1$	$-I$	$\mathbb{D}_j$	$\langle \mathbb{D}_j, (\rho R_{\pi/j}, 1/2) \rangle \cong \mathbb{D}_{2j}$	

This particular nonHopf bifurcation has been observed in experiments, numerical simulations and numerical linear stability calculations associated with three-dimensional wake flows behind cylinders. See [2,25] and references therein, see