Bifurcation from Periodic Solutions with Spatiotemporal Symmetry

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Abstract

In this paper, we discuss some recent developments in the understanding of generic bifurcation from periodic solutions with spatiotemporal symmetries. We focus mainly on the theory for bifurcation from *isolated* periodic solutions in dynamical systems with a compact symmetry group.

Moreover, we discuss how our theory justifies certain heuristic assumptions underlying previous approaches towards period preserving and period doubling bifurcation from periodic solutions.

1 Introduction

In dynamical systems of physical interest, the qualitative behavior of the dynamics may change as a function of external parameters. Such changes are referred to as *bifurcations.* A simple example of such a bifurcation can be observed when one studies the flow of a fluid past a cylinder. At low Reynolds number the flow is steady and two-dimensional (homogeneous in directions parallel to the axis of the cylinder). However, at higher Reynolds number the flow undergoes a supercritical Hopf bifurcation [9, 15] to a two-dimensional oscillatory flow, the so-called *Von Kármán vortex street.*

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In Figure 1, two snapshots of such a vortex street are depicted. We choose coordinates so that the x-axis is aligned with the direction of the flow and the z-axis is aligned with the axis of the cylinder. The figure displays a cross-section of the flow in the plane $z = 0$.

It is important to bear in mind that the fluid flows steadily from left to right. The vortex street begins immediately after the cylinder. The vortices grow in such a way that the fluid flow is periodic. That is, the vortex street at time $t = 0$ is identical to the vortex street at times $t = 1$, $t = 2$ and so on.

There are further regularity properties of the vortex street that can be described in terms of the underlying symmetries of the physical problem: namely, translations and reflections along the z-axis together with the reflection $y \mapsto -y$. Indeed the steady two-dimensional flow prior to the Hopf bifurcation is invariant under all of these symmetries. The bifurcation to Von Kármán vortices preserves the symmetries along the z-axis (so that the new solution remains two-dimensional) but breaks spontaneously the reflection $y \mapsto -y$. More precisely, the symmetries along the z-axis are *spatial symmetries* of the bifurcating periodic solution and preserve the form of the vortex street at all moments in time. The reflection $y \mapsto -y$ does not have this property and instead has the more subtle manifestation as a *spatiotemporal symmetry*. That is, at time $t = 1/2$ (after evolving for half a period) the vortex street is identical to the reflected image of the vortex street at time $t = 0$.

Aside from providing a language for describing the regularity of steady and oscillatory flows, knowledge of the symmetry of a physical problem provides a means of understanding and predicting the bifurcations that may take place. For example, it is evident following [9, 15] that the Hopf bifurcation takes place and breaks the $y \mapsto -y$ spatial symmetry. It is less evident from snapshots such as Figure 1 that the reflection symmetry reappears as a half-period spatiotemporal symmetry. The fact that this must be the case (in the absence of some highly degenerate events) is a consequence of the equivariant Hopf theorem [7].

There are numerous examples of periodic solutions with spatiotemporal symmetry that arise via Hopf bifurcation from symmetric steady-state solutions. The equivariant Hopf theorem [7] provides a mathematical framework for understanding such bifurcations. In this paper, we describe recent results on secondary bifurcation from periodic solutions with spatiotemporal symmetry.

Secondary bifurcations from the Von Kármán vortex street are noted in [1, 26]. These bifurcations break certain of the translation symmetries along the z-axis and hence lead to fully three-dimensional solutions. As described in [13], such bifurcations can be understood mathematically using existing techniques (as a fairly straightforward extension of methods of Fiedler [5]). However a mathematical framework for understanding secondary bifurcations from periodic solutions with arbitrary compact spatiotemporal symmetries has been formulated only recently in [13].

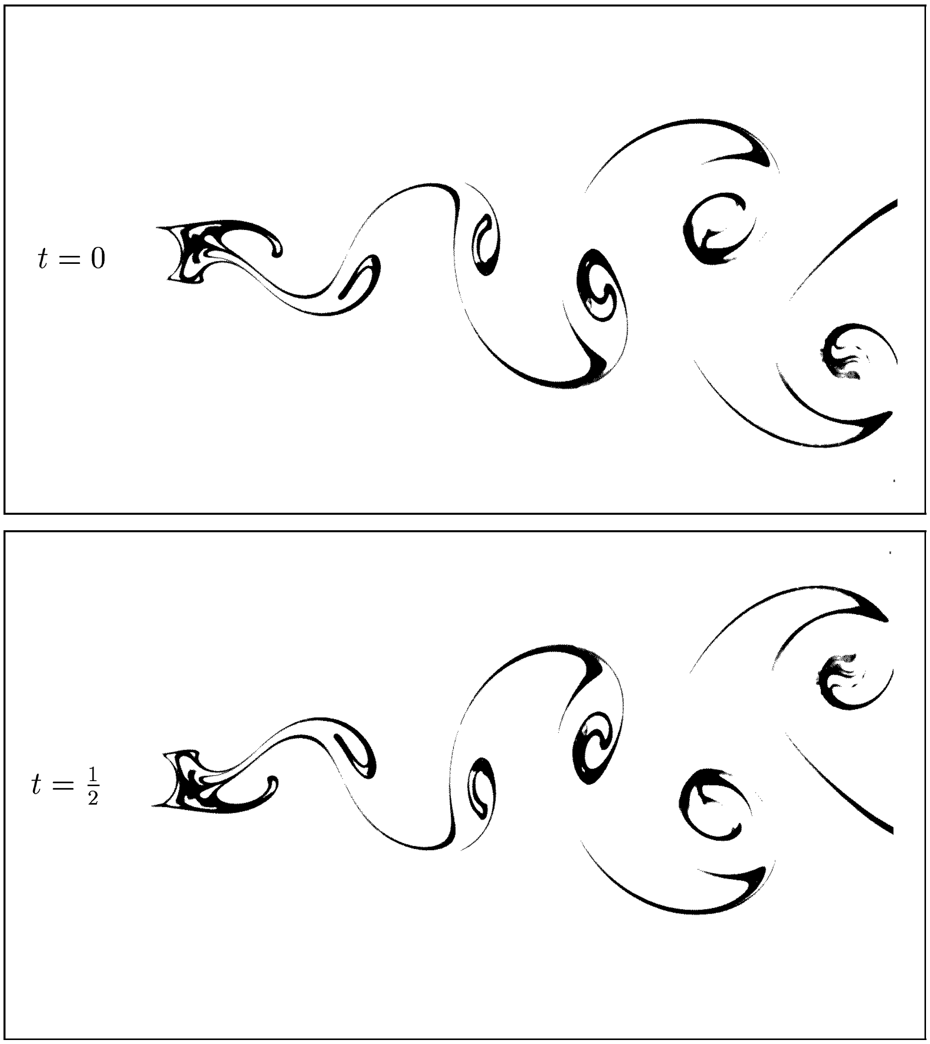

Figure 1: Two snapshots (at $t = 0$ and $t = \frac{1}{2}$) of a Von Kármán vortex street flow past a cylinder. The figure depicts a section of the flow in a plane orthogonal to the axis of the cylinder. The vortices move steadily from left to right and grow. The flow is periodic and after half a period ($t = \frac{1}{2}$) the vortex street is the reflection image of the original vortex street ($t = 0$). This is an example of a periodic solution with *spatiotemporal* symmetry. (Picture adapted from [4], with permission of Milton van Dyke. Photograph by S. Taneda.)

Bifurcation theory is at the heart of modern dynamical systems theory. In systems of differential equations without symmetry, there is a complete theory of the generic local bifurcations that occur as a single bifurcation parameter is varied, see for example Guckenheimer and Holmes [8, Chapter 3]. Local bifurcations are by definition the bifurcations that occur in the neighborhood of a nonhyperbolic steady-state or periodic solution.

Equivariant bifurcation theory [7] is concerned with the generalization of these results to differential equations that are equivariant with respect to the action of a (compact) Lie group Γ. A systematic approach to bifurcation from symmetric steady-state solutions is laid out in [7]. Importantly, bifurcations in equivariant systems are generally different from bifurcations one would expect in non-symmetric systems.

Until recently, a theory for bifurcation from symmetric periodic solutions was developed only to deal with certain special situations. The theory for periodic solutions with purely spatial symmetries was developed by Chossat and Golubitsky [3] building upon previous work of Ruelle [19]. Krupa [10] studied bifurcation from periodic solutions whose time evolution corresponds to a symmetry transformation for all time (rotating waves), see also Rand [20] and Renardy [21].

Fiedler [5] was one of the first to systematically study bifurcation from periodic solutions with discrete spatiotemporal symmetry using return map techniques. However, his study was confined to cyclic symmetry groups. The results generalize immediately to abelian symmetry groups, see Buono [2]. Previously, Swift and Wiesenfeld [23] made the observation that spatiotemporal symmetries may prevent a periodic solution form undergoing a period doubling bifurcation (even though such a bifurcation is typical for periodic solutions with purely spatial symmetry or no symmetry).

Vanderbauwhede [24, 25] set out to extend Fiedler's theory to study period preserving and period doubling bifurcations of periodic orbits with discrete spatiotemporal symmetry (though abandoning the use of return maps). However, the approach in [24, 25] (and similarly in Nicolaisen and Werner [16]) is based upon various heuristic assumptions.

The treatment of bifurcation from periodic orbits with spatiotemporal symmetries using return map techniques was taken up again recently by Lamb [12]. It turns out that an extension of Fiedler's approach involves consideration of *twisted equivariant* maps (called k-symmetric maps in [11, 12]), see also Nikolaev [17]. Rucklidge and Silber [18] recently used a similar approach in the study of certain examples of bifurcations from symmetric periodic solutions in convection problems. Finally, in [13], we developed a systematic theory for spatiotemporally symmetric periodic solutions (with compact symmetry groups) that are isolated in phase space, using the return map approach.

In this paper, we survey some of the main principles and results in studying bifurcation from periodic solutions with spatiotemporal symmetry. We summarize the main results of [13], focusing in particular on the linear theory that forms the foundation of the theory. It turns out that using only the linear theory, the nonlinear problems can be reduced to familiar (equivariant) bifurcation problems. Most proofs are omitted. For details we refer the reader to [13].

This paper is organized as follows. In Section 2, we discuss the different types of symmetry properties that periodic solutions of equivariant dynamical systems may possess. In Sections 3–5, we then focus on the theory for generic bifurcation from isolated discrete rotating waves based on the analysis of return maps. In particular, we discuss bifurcation from periodic solutions with no symmetry (Section 3), purely spatial symmetry (Section 4), and spatiotemporal symmetry (Section 5). Finally, in Section 6, we describe how heuristic assumptions in [24, 25, 16, 18] are shown to be justified by our results.

2 Symmetry properties of periodic solutions

Let $\Gamma \subset \mathbf{O}(n)$ be a compact Lie group acting orthogonally on $\mathbb{R}^n$. We consider dynamical systems (ODEs)

$$\frac{dx}{dt} = F(x) \tag{2.1}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth Γ-equivariant vector field, that is

$$\gamma F(x) = F(\gamma x),$$

for all $\gamma \in \Gamma$.

Suppose that P is a periodic solution of (2.1) of (minimal) period T, and let $x_0 \in P$. Let $x(t)$ be the trajectory with initial condition $x_0 = x(0)$, so $P = \{x(t) : 0 \leq t < T\}$. The symmetries that leave the periodic solution P invariant come in two forms. First, there is the group of *spatial symmetries*

$$\Delta = \{\gamma \in \Gamma : \gamma x_0 = x_0\}.$$

By definition, Δ is the isotropy subgroup of x_0. In fact, Δ is the isotropy subgroup of each point in P. Second, there is the group of *spatiotemporal symmetries*

$$\Sigma = \{\gamma \in \Gamma : \gamma P = P\}.$$

It is easy to see that for each $\sigma \in \Sigma$, there is a unique $T_\sigma \in [0, T)$ such that $\sigma x(t) = x(t + T_\sigma)$ for all t. Thus each spatiotemporal symmetry is the combination of a symmetry element σ composed with a time-shift by T_σ. The spatial symmetries are those spatiotemporal symmetries σ for which $T_\sigma = 0$. Moreover, Δ is a normal subgroup of Σ and either $\Sigma/\Delta \cong S^1$ or $\Sigma/\Delta \cong \mathbb{Z}_m$ for some $m \geq 1$.

When $\Sigma/\Delta \cong S^1$, the periodic solution P is called a *rotating wave*. When $\Sigma/\Delta \cong \mathbb{Z}_m$, the periodic solution P is called a *discrete rotating wave*. A brief overview of some key papers on bifurcation from rotating waves and discrete rotating waves is sketched in Table 1. In this paper, we confine ourselves to discussing bifurcation from isolated periodic solutions with compact spatiotemporal symmetry.

$\Sigma = \Delta$	: Purely Spatial Symmetry	Γ compact	[3, 19]
$\Sigma/\Delta \cong S^1$	: Rotating Wave	Γ compact	[10]
		Γ non-compact	[22]
$\Sigma/\Delta \cong \mathbb{Z}_m$	: Discrete Rotating Wave	Γ compact & P isolated	[13]
		remaining cases	[27]

Table 1: Overview of results on bifurcations from periodic solutions with spatiotemporal symmetry Σ and spatial symmetry Δ in Γ-equivariant dynamical systems

3 Periodic solutions with no symmetry

Bifurcation from a periodic solution P with no symmetry is conveniently studied as bifurcations from a fixed point of the associated Poincaré return map. This map is constructed as follows. For the periodic solution, a (local) *Poincaré section* is defined as a codimension one hyperplane X that transversally intersects the periodic solution P at some point x_0. In a neighborhood of the periodic solution P, the Poincaré map $G : X \to X$ keeps track of how solutions of the flow near the periodic solution return to the Poincaré section X. See Figure 2. Note that $G : X \to X$ is well-defined in a neighborhood of x_0 and is a diffeomorphism on this neighborhood. Moreover, $G(x_0) = x_0$ so that the periodic solution P for the flow is represented by the fixed point x_0 for the diffeomorphism G.

In the absence of symmetry (or other structure), it is well known [8] that in generic one-parameter families, fixed points of diffeomorphisms typically undergo (period preserving) saddle-node bifurcations, period doubling pitchfork bifurcations, or Hopf bifurcations. We refer to the first two types of bifurcations as *nonHopf* bifurcations.

NonHopf and Hopf bifurcation are characterized by the type of eigenvalue instabilities for the linearization $(dG)_{x_0}$ of the Poincaré map:

nonHopf bifurcation: an eigenvalue of $(dG)_{x_0}$ crosses the unit circle at ± 1 in the complex plane.

Hopf bifurcation: a pair of complex conjugate eigenvalues of $(dG)_{x_0}$ cross the unit circle in the complex plane at general position.

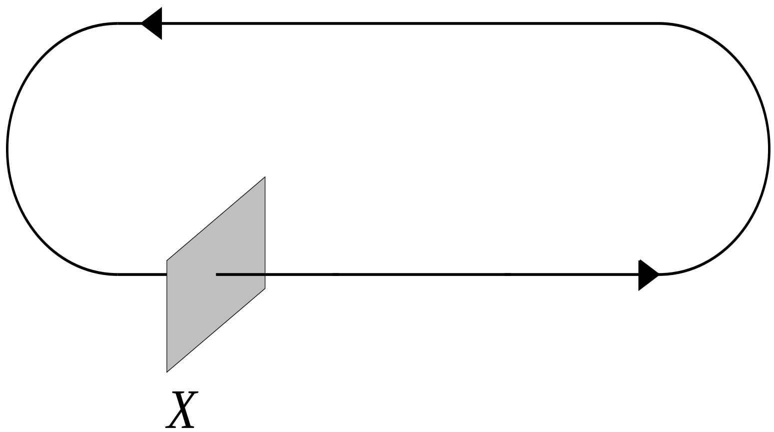

Figure 2: Poincaré section X for a periodic solution P.

To sketch the different types of bifurcations, let us suppose we have a stable periodic solution that is represented by a stable fixed point of the Poincaré map. Then at nonHopf bifurcation, generically one finds either (at eigenvalue $+1$) a saddle-node bifurcation of fixed points for G representing the merger of two periodic solutions of approximately the same period as P, or (at eigenvalue -1) a pitchfork bifurcation of period two points for G representing the birth of a periodic solution with approximately twice the period of the original solution. At Hopf bifurcation, generically the fixed point loses stability and an invariant circle is born, representing an invariant torus for the flow.

Two key components of the proofs of these results are 'center manifold reduction' and 'Birkhoff normal form theory'. Recall that the *center subspace* $E^c \subset X$ of $(dG)_{x_0}$ is defined to be the sum of the generalized eigenspaces corresponding to eigenvalues on the unit circle in the complex plane. Generically, $\dim E^c = 1$ at nonHopf bifurcation and $\dim E^c = 2$ at Hopf bifurcation. Center manifold reduction allows us to reduce to a low-dimensional Poincaré map $G : E^c \to E^c$. Birkhoff normal form theory then states that there exist changes of coordinates under which G commutes with $(dG)_{x_0}$ to any specified order in its Taylor expansion. Thus in the case of period doubling, the Birkhoff normal form of G is odd. Said differently, G commutes with the $\mathbb{Z}_2$ action $x \mapsto -x$. In the case of Hopf bifurcation, under the 'nonresonance' assumption that the eigenvalues of $(dG)_{x_0}$ lie at irrational angles on the unit circle, the Birkhoff normal form of G commutes with an action of the circle group S^1. To summarize:

Theorem 3.1 (nonHopf without symmetry) *Suppose that a periodic solution undergoes nonHopf bifurcation. Then generically* $\dim E^c = 1$ *and* $G : E^c \to E^c$ *is a general diffeomorphism satisfying* $g(x_0) = x_0$ *and* $(dG)_{x_0} = \pm 1$*. In the case of period* *doubling* $((dG)_{x_0} = -1)$*, the Birkhoff normal form of* G *is* $\mathbb{Z}_2$*-equivariant.*

Theorem 3.2 (Hopf without symmetry) *Suppose that a periodic solution undergoes Hopf bifurcation. Then generically* $\dim E^c = 2$ *and* $G : E^c \to E^c$ *is a general diffeomorphism satisfying* $G(x_0) = x_0$ *and* $(dG)_{x_0} = \alpha$ *where* α *is a complex number in general position on the unit circle. Moreover, generically the Birkhoff normal form of* G *is* S^1*-equivariant.*

Under the assumptions of nonresonance and Birkhoff normal form symmetry, it is straightforward to study the dynamics associated generically with nonHopf and Hopf bifurcation, see [8]. In addition, it is not difficult to see that few resonances are harmful, again see [8]. The issues associated with the terms in the tail of the Poincaré map are more complicated and are dealt with in Ruelle [19] and Field [6]. We will not mention these issues again in this paper but refer to [13] for further details.

4 Periodic solutions with purely spatial symmetry

In this section, we consider a periodic solution P with purely spatial symmetry. In other words, we have $\Sigma = \Delta$. The Poincaré section X can be chosen to be invariant under Δ. It then follows that the Poincaré map $G : X \to X$ is Δ-equivariant.

In fact, G is a general Δ-equivariant diffeomorphism with Δ-symmetric fixed point x_0. Hence, the bifurcation theory for the periodic solution now follows from the bifurcation theory for fixed points of equivariant diffeomorphisms. Generic nonHopf and Hopf bifurcations for equivariant diffeomorphisms have been discussed by Chossat and Golubitsky [3] and by Ruelle [19].

In order to describe their main results, we need to introduce some notions from the representation theory of compact Lie groups. A (real) representation of a group Δ is a linear action of the group, or in other words a homomorphism from Δ into $\mathbf{GL}(X)$. A given representation is *reducible* if its action on X can be written as the action on two disjoint non-empty invariant linear subspaces (so that the matrix representation block-diagonalizes). A representation is called *irreducible* if it is not reducible.

Representation theory states that the irreducible representations of compact Lie groups are finite dimensional, and that the linear maps that commute with an irreducible representation are scalar multiples of the identity, where the scalars lie either in $\mathbb{R}$ (the real numbers), in $\mathbb{C}$ (the complex numbers), or in $\mathbb{H}$ (the quaternions). A representation is called *absolutely irreducible* if it is irreducible of type $\mathbb{R}$. A representation is called *nonabsolutely irreducible* if it is irreducible of types $\mathbb{C}$ or $\mathbb{H}$. Finally, a representation is called Δ*-simple* when the representation is either nonabsolutely irreducible, or the direct sum of two isomorphic absolutely irreducible representations.

The following theorems describe the generic action of Δ on the center subspace E^c at nonHopf and Hopf bifurcation.

Theorem 4.1 (spatial nonHopf [3]) *Suppose that a periodic solution with purely spatial symmetry Δ undergoes nonHopf bifurcation. Then generically Δ acts absolutely irreducibly on E^c and $G : E^c \to E^c$ is a general Δ-equivariant diffeomorphism satisfying $g(x_0) = x_0$, $(dG)_{x_0} = \pm I$. If $(dG)_{x_0} = -I$, then the Birkhoff normal form of G is $\Delta \times \mathbb{Z}_2$-equivariant.*

Theorem 4.2 (spatial Hopf [3, 19]) *Suppose that a periodic solution with purely spatial symmetry Δ undergoes Hopf bifurcation. Then generically E^c is Δ-simple, equivalently E^c is an irreducible representation of type $\mathbb{C}$ for $\Delta \times S^1$.*

In Birkhoff normal form, $G : E^c \to E^c$ is a general $\Delta \times S^1$-equivariant diffeomorphism satisfying $G(x_0) = x_0$, $(dG)_{x_0} = \alpha I$, where α is a complex number in general position on the unit circle.

Note that the trivial group $\Delta = \mathbf{1}$ has a unique irreducible representation, namely the trivial one-dimensional representation, and this representation is absolutely irreducible. Similarly, there is a unique $\mathbf{1}$-simple representation, and this is two-dimensional. It follows that Theorems 4.1 and 4.2 reduce to Theorems 3.1 and 3.2 when $\Delta = \mathbf{1}$.

Again, nonHopf bifurcation is period preserving ($(dG)_{x_0} = I$) or period doubling ($(dG)_{x_0} = -I$), and Hopf bifurcation gives rise to invariant tori. Further details depend on the analysis of the remaining singularity theoretical problem. Note in this respect that period preserving nonHopf bifurcation no longer need to be of saddle-node type, but can also be of pitchfork or transcritical type. Many bifurcating solutions (but not always all solutions!) can be found by application of the equivariant branching lemma [7].

Remark 4.3 An unusual but unified reformulation of these results is the following. Observe that $(dG)_{x_0}$ acts on E^c and hence we can form a closed group $\widetilde{\Delta}$ acting on E^c generated by the actions of Δ and $(dG)_{x_0}$ on E^c. Since $(dG)_{x_0}$ commutes with the action of Δ, it follows that $\widetilde{\Delta}$ is a direct product of Δ and the closed group generated by $(dG)_{x_0}$. In general, $\widetilde{\Delta}$ need not be compact. However, it turns out generically that $(dG)_{x_0}$ is semisimple, that $\widetilde{\Delta}$ is compact, and that $\widetilde{\Delta}$ acts irreducibly on E^c. The action is either absolutely irreducible (nonHopf) or irreducible of complex type (Hopf). Moreover, $\widetilde{\Delta}$ is given by either Δ (period-preserving bifurcation), $\Delta \times \mathbb{Z}_2$ (period-doubling bifurcation), or $\Delta \times S^1$ (Hopf bifurcation). In all cases, the Birkhoff normal form of G is $\widetilde{\Delta}$-equivariant.

This approach of forming the group $\widetilde{\Delta}$ is crucial in the approach of [13] to bifurcation from periodic solutions with spatiotemporal symmetry, as outlined in the next section. However, we caution that in general it is not $(dG)_{x_0}$ but a different linear map that must be adjoined to Δ.

5 Periodic solutions with spatiotemporal symmetries

In case we have only spatial symmetries, a key step in doing the bifurcation analysis is to realize that generic bifurcation of a fixed point of an equivariant diffeomorphism corresponds to generic bifurcation of a spatially symmetric periodic solution. This connection can be made because there are *no* constraints on the Poincaré map, other than the fact that it is equivariant.

When we want to proceed in the same way with periodic solutions with spatiotemporal symmetries, we encounter a problem. As before, the spatial symmetries Δ of the periodic solution P imply that the Poincaré map $G : X \to X$ is a Δ-equivariant diffeomorphism with Δ-invariant fixed point x_0. However, the presence of spatiotemporal symmetries Σ for the periodic solution P may impose additional structure on G that is less easily codified.

Let $x(t)$ denote the periodic solution with initial condition x_0, and suppose that $s > 0$ is least such that $x(s) \in \Sigma x_0$. Write $x(s) = \sigma x_0$ where $\sigma \in \Sigma$. Then Σ is the closed subgroup of Γ generated by Δ and σ. Because the spatiotemporal symmetry σ does not fix the Poincaré section X, it does not give rise to equivariance. Instead, it relates the flow between the sections X and σX and the rest of the flow (between σX and X).

Let us denote the first hit map from X to σX by $g^{(1)}$. Then $g^{(1)} : X \to \sigma X$ is a Δ-equivariant map and has the advantage (in comparison with G) that $g^{(1)}$ is a general Δ-equivariant diffeomorphism. However, $g^{(1)}$ cannot be iterated because its domain and range are not the same. This can be repaired by using σ^{-1} to transport σX back to the section X. In this way, we thus construct a map [5, 17, 12]

$$f = \sigma^{-1} g^{(1)}.$$

See Figure 3.

Clearly, $f : X \to X$ is a diffeomorphism and can be iterated and interpreted as a dynamical system. It remains now to keep track of the symmetry properties of f induced by the spatiotemporal symmetry properties of the periodic solution P.

We noticed before that $g^{(1)}$ is Δ-equivariant. However, σ need not be Δ-equivariant and hence neither need f. We may systematically keep track of the possible non-

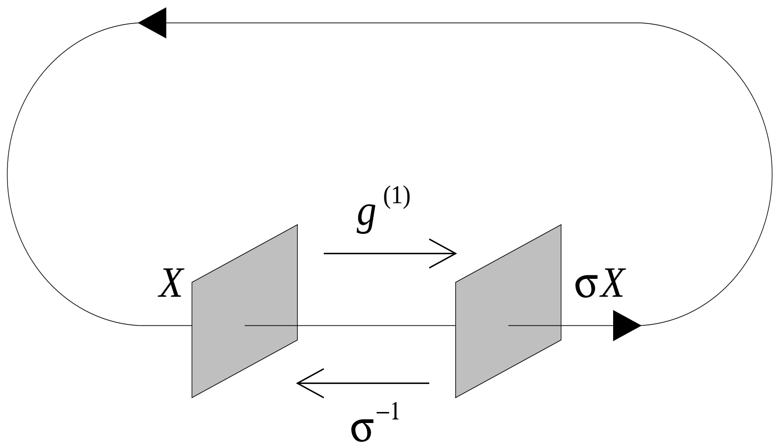

Figure 3: Construction of the first hit map $f := \sigma^{-1} g^{(1)}$.

commutativity of σ with Δ by introducing a map ϕ that satisfies

$$f\delta = \phi(\delta) f, \tag{5.1}$$

for all $\delta \in \Delta$. In fact, $\phi(\delta) = \sigma^{-1}\delta\sigma$ so that ϕ is an automorphism of Δ, that is $\phi : \Delta \to \Delta$ is a group isomorphism.

We say that a map $f : X \to X$ satisfying the condition (5.1) is *twisted equivariant*. Since $g^{(1)}$ is a general Δ-equivariant diffeomorphism satisfying $g^{(1)}(x_0) = \sigma x_0$, it follows that f is a general twisted equivariant diffeomorphism satisfying $f(x_0) = x_0$.

It is now interesting to note that there is a simple relationship between the Poincaré map G and the map f

$$G = \sigma^m f^m. \tag{5.2}$$

Since σ can always be chosen to be of finite order [13], it follows that there exist positive integers p, q so that $G^p = f^q$. Hence, periodic points of f and G are in one-to-one correspondence and have the same stability properties.

The spatiotemporal symmetry properties of a solution of the flow can be determined by using the map f (rather than the Poincaré map G). Let $x(t)$ be a solution of the flow, $x(0) = x_0 \in X$, $\delta \in \Delta$. Then,

$$f^j(x_0) = \delta(x_0) \quad \Leftrightarrow \quad x(t + j/m) = \sigma^j \delta x(t).$$

The main philosophy now is to study generic bifurcation from a fixed point for f, and then interpret the result in terms of the underlying flow.

We introduce the integer k to denote the *order* of the automorphism ϕ, that is, k is the smallest positive integer for which ϕ^k is trivial. (In [14, 11, 12], twisted equivariant diffeomorphisms were called k-symmetric, with reference to the notion of k as introduced here.) There are basically two cases to consider: either ϕ is trivial ($k=1$) and f is Δ-equivariant, or ϕ is nontrivial ($k \geq 2$) and f is not Δ-equivariant. We discuss these cases below.

(a) The untwisted case, $k = 1$

When f is Δ-equivariant, generic bifurcation for f follows the discussion in section 4. Indeed f is a general Δ-equivariant diffeomorphism with Δ-invariant fixed point x_0. It follows that generically Δ acts either absolutely irreducibly (nonHopf) or Δ-simply (Hopf) on the center subspace of $L = (df)_{x_0}$. In the nonHopf case, $L = \pm I$ so that f has Birkhoff normal form symmetry Δ or $\Delta \times \mathbb{Z}_2$. In the Hopf case, f has Birkhoff normal form symmetry $\Delta \times S^1$.

It follows from (5.2) that

$$(dG)_{x_0} = \sigma^m L^m. \tag{5.3}$$

Again, we have that certain powers of $(dG)_{x_0}$ and L coincide so that both linear maps have the same center subspace E^c. Since G is also Δ-equivariant, it follows in the nonHopf case that $(dG)_{x_0} = \pm I$.

To summarize, in the case of nonHopf bifurcation, we have shown the following result. This result can be viewed as a straightforward extension of the work of Fiedler [5] who treated cyclic spatiotemporal symmetry groups (see also Buono [2] for the abelian case).

Theorem 5.1 (nonHopf, $k = 1$) *Suppose that a periodic solution with spatiotemporal symmetry Σ and spatial symmetry Δ satisfying $k = 1$ undergoes a nonHopf bifurcation. Then generically Δ acts absolutely irreducibly on E^c and $f : E^c \to E^c$ is a a general Δ-equivariant diffeomorphism satisfying $f(x_0) = x_0$, $L = \pm I$. If $L = -I$, then the Birkhoff normal form of f is $\Delta \times \mathbb{Z}_2$-equivariant.*

It is important to clear up possible confusion regarding the parities $\pm I$ of the linear maps $(dG)_{x_0}$ and L. These parities are independent and have different consequences. The parity of $(dG)_{x_0}$ determines the type of nonHopf bifurcation: $(dG)_{x_0} = 1$ corresponds to period preserving bifurcation, $(dG)_{x_0} = -1$ corresponds to period doubling bifurcation. The parity of L determines the Birkhoff normal form symmetry of the Δ-equivariant diffeomorphism f that we analyze.

For example, suppose that $(dG)_{x_0} = I$ and $L = -I$. Since $L = -I$, our analysis of f proceeds as in the period-doubling case in Section 4 leading to period two points for f. However, since $(dG)_{x_0} = I$, the corresponding periodic solutions for the flow turn out to have period close to that of the underlying periodic solution P.

It often happens that condition (5.3) precludes the possibility that $(dG)_{x_0} = -I$ in the nonHopf case. For example, suppose that $\Sigma = \Delta \times \mathbb{Z}_m$. Then σ can be chosen to be a generator of $\mathbb{Z}_m$ so that $\sigma^m = \mathrm{id}$. If in addition, m is even, then we have $(dG)_{x_0} = L^m = (\pm I)^m = I$. This phenomenon is referred to as *suppression of period doubling* [23].

The situation for Hopf bifurcation in the untwisted case is analogous.

Theorem 5.2 (Hopf, $k = 1$) *Suppose that a periodic solution with spatiotemporal symmetry Σ and spatial symmetry Δ satisfying $k = 1$ undergoes Hopf bifurcation. Then generically E^c is Δ-simple, equivalently E^c is an irreducible representation of type $\mathbb{C}$ for $\Delta \times S^1$.*

In Birkhoff normal form $f : E^c \to E^c$ is a general $\Delta \times S^1$-equivariant diffeomorphism satisfying $f(x_0) = x_0$, $L = \alpha I$, where α is a complex number in general position on the unit circle.

(b) The twisted case, $k \geq 2$

Recall that E^c is the center subspace of both the linear map $(dG)_{x_0}$ (which is Δ-equivariant) and the linear map L. In particular, E^c is L-invariant as well as Δ-invariant and we can define an action of the closed group Δ_L on E^c generated by the actions of Δ and L. We note that in Section 4, the linear maps $(dG)_{x_0}$ and L coincide so that the group $\widetilde{\Delta}$ defined in Remark 4.3 is the same as the group Δ_L. In this more general situation, Δ_L is a semidirect product with normal subgroup Δ and the product structure is defined by the automorphism ϕ.

The next theorem is a generalization of some of the results described in Remark 4.3.

Theorem 5.3 ([13]) *Suppose that E^c is nontrivial (so that there is a bifurcation). Generically, L is semisimple, Δ_L is compact and Δ_L acts irreducibly on E^c.*

Moreover, either Δ_L acts absolutely irreducibly and $(dG)_{x_0} = \pm I$ (nonHopf bifurcation), or the action of Δ_L is irreducible of type $\mathbb{C}$ and $(dG)_{x_0} = \alpha I$ where α is a complex number in general position on the unit circle (Hopf bifurcation).

Taken alone, this result is not so useful. First, we require an explicit description of Δ_L and of its representations. Second, the diffeomorphism $f : E^c \to E^c$ is not Δ-equivariant and hence is certainly not Δ_L-equivariant.

The second question is answered by Birkhoff normal form theory for twisted equivariant maps [11]. Write $f = Lh$. Since f and L are twisted equivariant, it follows that $h : E^c \to E^c$ is Δ-equivariant in the usual sense. Moreover, since L is semisimple it follows that in Birkhoff normal form f and hence h are L-equivariant. By [11], the transformations into Birkhoff normal form can be done so as to preserve twisted equivariance of f, with the result that h is Δ_L-equivariant as required.

Theorem 5.4 (nonHopf, $k \geq 2$) *Suppose that Δ_L acts absolutely irreducibly on E^c. Then $L^k = \pm I$ and hence Δ_L is a cyclic extension of Δ of order k or $2k$. In Birkhoff normal form, h is a general Δ_L-equivariant diffeomorphism satisfying $h(x_0) = x_0$, $(dh)_{x_0} = I$.*

Thus, nonHopf bifurcation from periodic solutions with spatiotemporal symmetry reduces to nonHopf bifurcation from periodic solutions with purely spatial symmetry, but with G replaced by h and Δ enlarged to Δ_L. Moreover, we always reduce to the period preserving case.

The terminology 'period preserving' and 'period doubling' is justified in the general situation of spatiotemporal symmetry by the following calculation. Suppose that x is a fixed point for h. We compute that

$$G(x) = \sigma^m f^m(x) = \sigma^m (Lh)^m(x) = \sigma^m L^m h^m(x) = \sigma^m L^m x = (dG)_{x_0} x = \pm x.$$

Note that this calculation relies on the fact that h commutes with L when in Birkhoff normal form.

Symmetry breaking from Δ_L in the bifurcation problem can be related to symmetry breaking from the spatiotemporal symmetries of the periodic solutions. In particular, L^{-1} can be identified with the spatiotemporal symmetry $(\sigma, \frac{1}{m})$ (the spatial transformation σ combined with a $\frac{1}{m}$th period time-shift) in the period preserving case. In the period doubling case, L^{-1} is identified with $(\sigma, \frac{1}{2m})$.

More precisely, a fixed point of h with isotropy containing $L^{-j}\delta$ represents a periodic solution with symmetry $(\sigma^j\delta, \frac{j}{m})$ (resp. $(\sigma^j\delta, \frac{j}{2m})$).

Theorem 5.5 (Hopf, $k \geq 2$) *Suppose that E^c is an irreducible representation of type $\mathbb{C}$ for Δ_L. Then $L^k = \alpha I$ where $\alpha \in \mathbb{C}$ is in general position on the unit circle. Define $L_0 = \alpha^{-1/k} L$ and set $f = L_0 h$. Generically, $\Delta_L = \Delta_{L_0} \times S^1$ where Δ_{L_0} is a cyclic extension of Δ of order k, and E^c is Δ_{L_0}-simple. Moreover, in Birkhoff normal form, h is a general $\Delta_{L_0} \times S^1$-equivariant diffeomorphism satisfying $h(x_0) = x_0$ and $(dh)_{x_0} = \alpha^{1/k} I$.*

In the Hopf case, L can be identified with σ^{-1} in the sense that when j is the smallest positive integer for which for some $\delta \in \Delta$, $L^j\delta \in S^1$ (and so maps the bifurcating invariant circle of h to itself), then $\sigma^{-j}\delta$ fixes the bifurcating invariant torus setwise.

In principle, Theorems 5.4 and 5.5 allow us to carry out the bifurcation analysis for general spatiotemporal symmetry groups. It should be noted that an important part in the further analysis consists of determining the irreducible representations of the group Δ_L in the nonHopf case and Δ_{L_0} in the Hopf case. These groups are cyclic extensions of order k or $2k$ of the spatial symmetry group Δ. In [13], this issue is discussed in the context of *induced representation theory* (building the irreducible representations of Δ_L and Δ_{L_0} given the irreducible representations of Δ). We also refer to [13] for further examples and details of this bifurcation theory.

6 Comparison with alternative approaches

In the last section of this paper we would like to point out how our results compare to some previous attempts to study bifurcation from periodic solutions with spatiotemporal symmetry in the twisted case $k \geq 2$. In particular, we mention the papers [24, 25, 16, 18].

A central problem in the development of the theory has been that the spatial symmetry group Δ is insufficient to characterize the bifurcations when $k \geq 2$ and yet the spatiotemporal symmetry group Σ does not act on the cross-section X and hence does not *a priori* act on the center subspace E^c. More precisely, the action of Δ on E^c need not extend to an action of Σ on E^c.

To counteract this problem, Vanderbauwhede [24, 25] observed that a certain group Σ_0 related to Σ acts on the domain of the Floquet matrix and assumed as a hypothesis in the case of nonHopf bifurcation that the center subspace of the Floquet matrix is an absolutely irreducible representation of E^c. See also Nicolaisen and Werner [16]. We know of no direct proof that this hypothesis holds generically, but the first (indirect) proof is presented in Theorem 6.1 below.

More recently, Rucklidge and Silber [18] restricted to the period preserving case and, following [12], considered the map f described in Section 5. Even though Σ does not act on X, it was assumed as a hypothesis in [18] that Σ acts on E^c. Again, this hypothesis is justified by Theorem 6.1.

Theorem 6.1 (NonHopf bifurcation) *Let P be a discrete rotating wave with spatiotemporal symmetry Σ and spatial symmetry Δ, with $\Sigma/\Delta \cong \mathbb{Z}_m$.*

Period preserving, cf [24, 18] *If P undergoes a period preserving nonHopf bifurcation, then generically Δ_L acts on the center subspace as an absolutely irreducible representation of Σ. Moreover, there is a one-to-one correspondence between the absolutely irreducible representations of Δ_L and Σ, with $L^{-1} \sim \sigma$.*

Period doubling, cf [25] *If P undergoes a period doubling nonHopf bifurcation, then generically Δ_L acts on the center subspace as an absolutely irreducible representation of the group*

$$\Sigma_0 := \langle \Delta, (\sigma, \tfrac{1}{2m}) \rangle \subset \Sigma \times \mathbb{Z}_{2m},$$

with $(\mathrm{id}, \frac{1}{2}) \in \Sigma_0$ *acting as* $-I$*. Moreover, there is a one-to-one correspondence between the absolutely irreducible representations of* Δ_L *and the absolutely irreducible representations of* Σ_0 *with* $(\mathrm{id}, \frac{1}{2})$ *acting as* $-I$*; the correspondence being given by* $L^{-1} \sim (\sigma, \frac{1}{2m})$*.*

Proof At nonHopf bifurcation we have

$$(dG)_{x_0} = \sigma^m L^m = \pm I.$$

In the period preserving case, we have $\sigma^m L^m = +I$ and hence $L^{-m} = \sigma^m$. Under the identification of L^{-1} with σ, it follows that every absolutely irreducible representation of Δ_L is at the same time also an absolutely irreducible representation of Σ.

In the period doubling case, we have $\sigma^m L^m = -I$ and hence $L^{-m} = -\sigma^m$. Under the identification of L^{-1} with $(\sigma, \frac{1}{2m})$, we have $L^{-m} = (\sigma^m, \frac{1}{2}) = \sigma^m(\mathrm{id}, \frac{1}{2})$ and it follows that every absolutely irreducible representation of Δ_L is at the same time also an absolutely irreducible representation of Σ_0 that has $(\mathrm{id}, \frac{1}{2})$ acting as $-I$. ∎

Remark 6.2

1. The groups Σ and Σ_0 are precisely the groups that one would expect to act geometrically on the center bundle of a periodic solution at period preserving and period doubling nonHopf bifurcation. This appears to be the main idea underlying the approach by Vanderbauwhede [24, 25].

2. A general nonHopf theorem could be formulated with reference to Σ_0 only. Namely, in the period preserving case, the absolutely irreducible representations of Σ are precisely those absolutely irreducible representations of Σ_0 in which $(\mathrm{id}, \frac{1}{2})$ acts as $+I$.

Theorem (NonHopf bifurcation) *If* P *undergoes a nonHopf bifurcation, then generically* Δ_L *acts on the center subspace as an absolutely irreducible representation of* Σ_0*. Moreover, there is a one-to-one correspondence between the absolutely irreducible representations of* Δ_L *and* Σ_0*.*

In particular, suppression of period doubling arises precisely when $(\mathrm{id}, \frac{1}{2})$ acts as $+I$ in all of the absolutely irreducible representations of Σ_0.

3. The identifications $L^{-1} \sim (\sigma, \frac{1}{m})$ and $L^{-1} \sim (\sigma, \frac{1}{2m})$ can be used directly in the identification of the spatiotemporal symmetries of periodic solutions represented by the fixed points arising in the steady state bifurcation of the Δ_L-equivariant diffeomorphism h.

In the case of Hopf bifurcation we can rephrase our results as follows.

Theorem 6.3 (Hopf bifurcation) *If P undergoes a Hopf bifurcation, then generically $\Delta_L \cong \Delta_{L_0} \times S^1$ acts on the center subspace as an irreducible representation of type $\mathbb{C}$ of $\Sigma \times S^1$.*

Proof At Hopf bifurcation we have

$$(dG)_{x_0} = \sigma^m L^m = \beta I \qquad \beta \in S^1.$$

Hence, under the identification of L with $\beta^{1/m}\sigma^{-1}$, it follows that every irreducible representation of type $\mathbb{C}$ of Δ_L is at the same time an irreducible representation of type $\mathbb{C}$ of $\Sigma \times S^1$. ∎

Remark 6.4 We note that there is not necessarily a one-to-one correspondence between the irreducible representations of type $\mathbb{C}$ for the groups $\Delta_{L_0} \times S^1$ and $\Sigma \times S^1$. Indeed, the second group may have many irreducible representations for each irreducible representation of the first group. For example, suppose that $\Delta = \mathbf{1}$ and $\Sigma = \mathbb{Z}_{24}$ say. In this case, $k = 1$ so that $\Delta_{L_0} = \mathbf{1}$. Clearly it is more efficient to work with the group S^1 than with the group $\mathbb{Z}_{24} \times S^1$ even though working with the second group does not lead to error.

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