Correction to: A note on diffusion limits of chaotic skew-product flows

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Abstract

This fixes a gap in the averaging argument in our paper: A note on diffusion limits of chaotic skew product flows. *Nonlinearity* (2011) 1361–1367, and moreover shows that the large deviation estimate assumed there is redundant.

Recall from [2] that $Z^{(\epsilon)}(t) = \int_0^t g(x^{(\epsilon)}(s), y^{(\epsilon)}(s))\, ds$ where $g(x,y) = f(x,y) - F(x)$. In [2, Section 3], it is argued that $Z^{(\epsilon)} \to 0$ in $L^1(C([0,T], \mathbb{R}^d); \mu)$, but the proof is incorrect. Specifically, the proof introduces a random variable J_n (see below) that depends on $x^{(\epsilon)}(n\epsilon^{3/2})$ and $y^{(1)}(s)$, and derives an estimate for $\mathbb{E}|J_n|$. This estimate takes into account the randomness of $y^{(1)}(s)$ but overlooks the randomness of $x^{(\epsilon)}(n\epsilon^{3/2})$.

In this note, we correct the argument in [2]. Moreover, in contrast to [2], our proof does not require any large deviation estimates. Hence the weak invariance principle is a sufficient (as well as necessary) condition for the main result in [2].

Lemma 1 *$Z^{(\epsilon)} \to 0$ in $L^1(C[0,T], \mathbb{R}^d); \mu)$ as $\epsilon \to 0$ for each $T > 0$.*

Proof Following the calculation in [2, Section 3] with $\delta = \epsilon^{3/2}$, we obtain

$$\max_{[0,T]} |Z^{(\epsilon)}| = I_1 + I_2 + O(\epsilon^{3/2}) = I_2 + O(\epsilon^{1/2}) = \epsilon^{3/2} \sum_{n=0}^{[T\epsilon^{-3/2}]-1} |J_n| + O(\epsilon^{1/2}), \tag{1}$$

where

$$J_n = \epsilon^{1/2} \int_{n\epsilon^{-1/2}}^{(n+1)\epsilon^{-1/2}} g(x^{(\epsilon)}(n\epsilon^{3/2}), y^{(1)}(s))\, ds.$$

(The intermediate expressions I_1 and I_2 are defined in [2] but the formulas are not required here.)

For $u \in \mathbb{R}^d$ fixed, we define

$$\tilde{J}_n(u) = \epsilon^{1/2} \int_{n\epsilon^{-1/2}}^{(n+1)\epsilon^{-1/2}} g(u, y^{(1)}(s))\, ds = \epsilon^{1/2} \int_{n\epsilon^{-1/2}}^{(n+1)\epsilon^{-1/2}} A_u \circ \phi_s\, ds, \qquad A_u(y) = g(u,y).$$

Note that $\tilde{J}_n(u) = \tilde{J}_0(u) \circ \phi_{n\epsilon^{-1/2}}$, and so $\mathbb{E}|\tilde{J}_n(u)| = \mathbb{E}|\tilde{J}_0(u)|$. By the ergodic theorem, $\mathbb{E}|\tilde{J}_0(u)| \to 0$ as $\epsilon \to 0$ for each u.

Let $Q > 0$ and write $Z^{(\epsilon)} = Z^{(\epsilon)}_{Q,1} + Z^{(\epsilon)}_{Q,2}$ where

$$Z^{(\epsilon)}_{Q,1}(t) = Z^{(\epsilon)}(t)1_{B_\epsilon(Q)}, \quad Z^{(\epsilon)}_{Q,2}(t) = Z^{(\epsilon)}(t)1_{B_\epsilon(Q)^c}, \quad B_\epsilon(Q) = \big\{\max_{[0,T]} |x^{(\epsilon)}| \le Q\big\}.$$

For any $a > 0$, there exists a finite subset $S \subset \mathbb{R}^d$ such that $\operatorname{dist}(x, S) \le a/(2 \operatorname{Lip} f)$ for any x with $|x| \le Q$. Then for all $n \ge 0$, $\epsilon > 0$,

$$1_{B_\epsilon(Q)}|J_n| \le \sum_{u \in S} |\tilde{J}_n(u)| + a.$$

Hence by (1),

$$\begin{aligned}\mathbb{E} \max_{[0,T]} |Z^{(\epsilon)}_{Q,1}| &\le \epsilon^{3/2} \sum_{n=0}^{[T\epsilon^{-3/2}]-1} \sum_{u \in S} \mathbb{E}|\tilde{J}_n(u)| + Ta + O(\epsilon^{1/2}) \\ &= \epsilon^{3/2} \sum_{n=0}^{[T\epsilon^{-3/2}]-1} \sum_{u \in S} \mathbb{E}|\tilde{J}_0(u)| + Ta + O(\epsilon^{1/2}) \le T \sum_{u \in S} \mathbb{E}|\tilde{J}_0(u)| + Ta + O(\epsilon^{1/2}).\end{aligned}$$

Since $a > 0$ is arbitrary, we obtain for each fixed Q that $\max_{[0,T]} |Z^{(\epsilon)}_{Q,1}| \to 0$ in L^1, and hence in probability, as $\epsilon \to 0$.

Next, since $x^{(\epsilon)} - W^{(\epsilon)}$ is bounded on $[0, T]$, for Q sufficiently large

$$\mu\big\{\max_{[0,T]} |Z^{(\epsilon)}_{Q,2}| > 0\big\} \le \mu\big\{\max_{[0,T]} |x^{(\epsilon)}| \ge Q\big\} \le \mu\big\{\max_{[0,T]} |W^{(\epsilon)}| \ge Q/2\big\}.$$

Fix $c > 0$. Increasing Q if necessary, we can arrange that $\mu\{\max_{[0,T]} |\sqrt{\Sigma}W| \ge Q/2\} < c/4$. By the continuous mapping theorem, $\max_{[0,T]} |W^{(\epsilon)}| \to_d \max_{[0,T]} |\sqrt{\Sigma}W|$. Hence there exists $\epsilon_0 > 0$ such that $\mu\{\max_{[0,T]} |W^{(\epsilon)}| \ge Q/2\} < c/2$ for all $\epsilon \in (0, \epsilon_0)$. For such ϵ,

$$\mu\big\{\max_{[0,T]} |Z^{(\epsilon)}_{Q,2}| > 0\big\} < c/2.$$

Shrinking ϵ_0 if necessary, we also have that $\mu\{\max_{[0,T]} |Z^{(\epsilon)}_{Q,1}| > c/2\} < c/2$. Hence $\mu\{\max_{[0,T]} |Z^{(\epsilon)}| > c\} < c$, and so $\max_{[0,T]} |Z^{(\epsilon)}| \to 0$ in probability. Finally, since $|\max_{[0,T]} |Z^{(\epsilon)}||_\infty \le 2|f|_\infty T$, it follows from the bounded convergence theorem that $\lim_{\epsilon \to 0} \mathbb{E} \max_{t \in [0,T]} |Z^{(\epsilon)}(t)| = 0$ as required. ∎

Remark 2 The subsequent paper [1] contains the same error (see [1, Appendix A]). The gap is fixed in identical manner to above, and the large deviation assumptions throughout [1] are again unnecessary.

References

[1] G. A. Gottwald and I. Melbourne. Homogenization for deterministic maps and multiplicative noise. *Proc. R. Soc. London A* (2013) 20130201.

[2] I. Melbourne and A. Stuart. A note on diffusion limits of chaotic skew product flows. *Nonlinearity* (2011) 1361–1367.