EXISTENCE AND SMOOTHNESS OF THE STABLE FOLIATION FOR SECTIONAL HYPERBOLIC ATTRACTORS

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ABSTRACT. We prove the existence of a contracting invariant topological foliation in a full neighborhood for sectional hyperbolic attractors. Under certain bunching conditions it can then be shown that this stable foliation is smooth. In particular, we show that the stable foliation for the classical Lorenz equation (and nearby vector fields) is better than C^1 which is crucial for recent results on exponential decay of correlations. In fact the foliation is at least $C^{1.264}$.

1. INTRODUCTION

The goal of this paper is to establish existence and smoothness of the stable foliation for sectional hyperbolic flows. In particular, we treat the case of the classical Lorenz equations [11]

$$\begin{aligned}\dot{x}_1 &= 10(x_2 - x_1)\\ \dot{x}_2 &= 28x_1 - x_2 - x_1x_3\\ \dot{x}_3 &= x_1x_2 - \tfrac{8}{3}x_3\end{aligned}$$

showing that the stable foliation for the flow is at least $C^{1.264}$. This regularity ($C^{1+\varepsilon}$ for some $\varepsilon > 0$) is a crucial component of the analysis in [1, 2] where we prove exponential decay of correlations for the Lorenz attractor. An immediate consequence of our result is that the stable foliation for the associated Poincaré map is also at least $C^{1.264}$. The results are robust in the sense that we obtain smoothness of the stable foliations and exponential decay of correlations for smooth vector fields that are sufficiently C^1-close to the classical one.

As far as we know, this is the first complete proof that the stable foliation for the classical Lorenz equations (or even for the Poincaré map) exists and is better than Hölder continuous. By [12] and [18], the classical Lorenz attractor is a singular hyperbolic attractor. A consequence is the existence of smooth stable leaves through each point of the attractor. However, *a priori* it does not follow that these leaves form a topological foliation in a full neighborhood of the attractor; nor is there any information about smoothness of such a foliation. These issues are somewhat controversial, with various false claims in the literature.

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Careful analyses (see for example [9, 14, 15]) require additional conditions to establish smoothness and do not apply to the classical Lorenz attractor.

Recently [2] gave a verifiable criterion for smoothness of the stable foliation that is easily seen to hold for the classical Lorenz attractor. However, the argument in [2] presupposes that the stable leaves topologically foliate a full neighborhood of the attractor – a fact that is folklore but for which apparently there is no proof available in the literature.

In this paper, we consider the more general case of sectional hyperbolic attractors and give a complete proof of the existence of an topological foliation $\{W^s_x\}$ in a neighborhood of such attractors. The individual leaves W^s_x are smoothly embedded stable manifolds. In general, the leaves need not vary smoothly, but under a bunching condition [7] the foliation is smooth. The argument in [2] now applies, and we obtain existence and smoothness of the stable foliation for the classical Lorenz attractor. Our results hold for the flow, and hence also for the Poincaré map.

In addition, we extend the verifiable criterion of [2] to the sectional hyperbolic situation, and we give a lower bound for the smoothness for the classical Lorenz attractor. The condition is verifiable in the sense that it depends only on the linearised vector field and the location of the attractor and its equilibria.

In Section 2, we recall the notion of partially hyperbolic and sectional hyperbolic attractors. Section 3 contains general facts about cone fields for partially hyperbolic attractors, as well as the crucial step that the stable bundle extends continuously to a contracting invariant bundle over a neighborhood of the attractor. Section 4 contains general results about the stable foliation of partially hyperbolic attractors. In particular, the stable leaves define a topological foliation of a neighborhood of the attractor and is smooth under a bunching condition. In Section 5, we specialise to sectional hyperbolic attractors. Following [2], we give a verifiable condition for smoothness of the stable foliation and apply this to the classical Lorenz attractor.

2. Sectional hyperbolic attractors

In this section, we define what is understood as a sectional hyperbolic attractor; see e.g. [3] for an extended presentation of this theory,

Let M be a compact Riemannian manifold and $\mathfrak{X}^r(M)$, $r \geq 1$, be the set of C^r vector fields defined on M. Let X_t denote the flow generated by $G \in \mathfrak{X}^r(M)$. Given a compact invariant set Λ for $G \in \mathfrak{X}^r(M)$, we say that Λ is *isolated* if there exists an open set $U \supset \Lambda$ such that

$$\Lambda = \textstyle\bigcap_{t\in\mathbb{R}} X_t(U).$$

If U above can be chosen such that $X_t(U) \subset U$ for $t > 0$, then we say that Λ is an *attracting set*.

Given $x \in M$, we define $\omega_G(x)$ as the set of accumulation points of the set $\{X_t x; t \geq 0\}$ and define $\alpha_G(x) = \omega_{-G}(x)$. A subset $\Lambda \subset M$ is *transitive* if it has a full dense orbit, that is, there is $x \in \Lambda$ such that $\omega_G(x) = \Lambda = \alpha_G(x)$.

Definition 2.1. An *attractor* is a transitive attracting set, and a *repeller* is an attractor for the reversed vector field $-G$.

Definition 2.2. Let Λ be a compact invariant set for $G \in \mathfrak{X}^r(M)$. We say that Λ is *partially hyperbolic* if the tangent bundle over Λ can be written as a continuous DX_t-invariant sum

$$T_\Lambda M = E^s \oplus E^{cu},$$

where $d_s = \dim E^s \geq 1$ and $d_{cu} = \dim E^{cu} \geq 2$, and there exist constants $C > 0$, $\lambda \in (0,1)$ such that for every $t > 0$ and every $x \in \Lambda$, we have

- uniform contraction along E^s:

$$\|DX_t \mid E^s_x\| \leq C\lambda^t; \tag{2.1}$$

- domination of the splitting:

$$\|DX_t \mid E^s_x\| \cdot \|DX_{-t} \mid E^{cu}_{X_t x}\| \leq C\lambda^t. \tag{2.2}$$

We refer to E^s as the stable bundle and to E^{cu} as the center-unstable bundle.

Remark 2.3. By [6, Theorem 1], we may suppose without loss that $\|\cdot\|$ is an adapted metric so that $C = 1$.

Definition 2.4. The center-unstable bundle E^{cu} is *volume expanding* if there exists $K, \theta > 0$ such that $|\det(DX_t(x) \mid E^{cu}_x)| \geq K\, e^{\theta t}$ for all $x \in \Lambda$ and $t \geq 0$. More generally, E^{cu} is *sectional expanding* if for every two-dimensional subspace $P_x \subset E^{cu}_x$,

$$|\det(DX_t(x) \mid P_x)| \geq K e^{\theta t} \quad \text{for all } x \in \Lambda,\ t \geq 0.$$

An invariant set is *nontrivial* if it is neither a periodic orbit nor an equilibrium.

Definition 2.5. Let Λ be a compact nontrivial invariant set for $G \in \mathfrak{X}^r(M)$. We say that Λ is a *sectional hyperbolic set* if all the equilibria in Λ are hyperbolic, and Λ is partially hyperbolic with sectional expanding center-unstable bundle. A sectional hyperbolic set which is also an attractor is called a *sectional hyperbolic attractor*.

In the special case when E^{cu} is volume expanding, Λ is called a *singular* hyperbolic set/attractor.

An isolated set $\Lambda = \Lambda_G$ for a C^1 vector field G is *robustly transitive* if there is an open set $U \supset \Lambda$ such that $\Lambda_{\tilde G} = \bigcap_{t\in\mathbb{R}} \tilde X_t(U)$ is transitive and nontrivial for any vector field $\tilde G$ C^1-close to G.

Definition 2.6. An equilibrium σ for a 3-dimensional vector field G is *Lorenz-like* if the eigenvalues λ_j, $1 \leq j \leq 3$, of $DG(\sigma)$ are real and satisfy $\lambda_1 < \lambda_2 < 0 < -\lambda_2 < \lambda_3$.

For 3-dimensional vector fields, Morales, Pacifico, and Pujals proved in [12] that any robustly transitive invariant set Λ containing an equilibrium is a singular hyperbolic attractor or repeller. Moreover, every equilibrium in Λ is Lorenz-like for G or $-G$, and Λ is proper, i.e., $\Lambda \neq M$.

Tucker [18] gave a computer-assisted proof that the classical Lorenz attractor [11] is a robustly transitive invariant set containing an equilibrium. It then follows from [12] that the classical Lorenz attractor is singular hyperbolic.

3. Cone fields and the stable bundle for partially hyperbolic attractors

In this section, we analyse stable and center-unstable cone fields in a neighborhood of a partially hyperbolic attractor Λ, and we show that the stable bundle E^s extends to a continuous DX_t-invariant contracting bundle over a neighborhood of Λ.

Throughout, Λ is a partially hyperbolic attractor for a vector field $G \in \mathfrak{X}^r(M)$, $r \geq 1$, with invariant splitting $T_\Lambda M = E^s \oplus E^{cu}$ and contraction rate $\lambda \in (0,1)$. Sectional expansion is not assumed. Write $d = \dim M = d_s + d_{cu}$.

3.1. **Cone fields in a neighborhood of Λ.** Let $U_0 \subset M$ be a forward invariant neighborhood of Λ such that $\bigcap_{t \geq 0} X_t(U_0) = \Lambda$. Choose a continuous (not necessarily invariant) extension $T_{U_0}M = E^s \oplus E^{cu}$ of the splitting $T_\Lambda M = E^s \oplus E^{cu}$. Given $x \in U_0$ and $a > 0$ we define the cone fields

$$\mathcal{C}^s_x(a) = \{v = v^s + v^{cu} \in E^s_x \oplus E^{cu}_x : \|v^{cu}\| \leq a\|v^s\|\},$$
$$\mathcal{C}^{cu}_x(a) = \{v = v^s + v^{cu} \in E^s_x \oplus E^{cu}_x : \|v^s\| \leq a\|v^{cu}\|\}.$$

Proposition 3.1. *Fix T so that $\lambda^T = 1/150$. For any $a \in (0, \frac{1}{4}]$ there exists a positively invariant neighborhood U_0 of Λ, such that for all $x \in U_0$ the following hold:*

(a) *backward invariance of stable cones and forward invariance of center-unstable cones:*

$$DX_{-t}\big(\mathcal{C}^s_{X_t x}(b)\big) \subset \mathcal{C}^s_x(b), \tag{3.1}$$
$$DX_t\big(\mathcal{C}^{cu}_x(b)\big) \subset \mathcal{C}^{cu}_{X_t x}(b), \tag{3.2}$$

for all $b \geq a$, $t \geq T$.

(b) *backward expansion of stable cones and domination: there exist constants $c > 0$, $\tilde\lambda \in (0,1)$, such that for all $t > 0$,*

$$\|DX_{-t}(X_t x)v\| \geq c\tilde\lambda^{-t}\|v\| \quad \text{for all } v \in \mathcal{C}^s_{X_t x}(a), \tag{3.3}$$
$$\frac{\|DX_t(x)v\|}{\|v\|} \geq c\tilde\lambda^{-t}\frac{\|DX_t(x)u\|}{\|u\|} \quad \text{for all nonzero } v \in \mathcal{C}^{cu}_x(a),\ u \in DX_{-t}(\mathcal{C}^s_{X_t x}(a)). \tag{3.4}$$

Proof. If v lies in T_xM where $x \in U_0$, then we write $v = v^s + v^{cu} \in E^s_x \oplus E^{cu}_x$. If $v \in \mathcal{C}^*_x(a)$, then $(1-a)\|v^*\| \leq \|v\| \leq (1+a)\|v^*\|$ where throughout $* \in \{s, cu\}$.

For $x \in \Lambda$, it follows from invariance of the splitting $E^s \oplus E^{cu}$ that $(DX_t(x)v)^* = DX_t(x)v^*$ for all $v \in T_xM$ and $t \in \mathbb{R}$.

We fix the neighborhood U_0 as follows. For each $x \in \Lambda$, we choose a neighborhood $U_x \subset M$ of x such that U_x is diffeomorphic to $\mathbb{R}^d$ where $d = \dim M$. Then $T_{U_x}M$ is identified with $U_x \times \mathbb{R}^d$. Given $y_1, y_2 \in U_x$, a vector $v \in \mathbb{R}^d$ corresponds to vectors $v_{y_j} \in T_{y_j}M$ via this identification. Using the smoothness of the flow, we can choose U_x so small that $\|DX_t(y_1)v_{y_1}\| \leq 2\|DX_t(y_2)v_{y_2}\|$ for all $x \in \Lambda$, $y_1, y_2 \in U_x$, $v \in \mathbb{R}^d$, $t \in [-T, T]$. Using moreover the continuity of the splitting $E^s \oplus E^{cu}$, for $a > 0$ fixed we can ensure for all $b \geq a/8$, $t \in [-T, T]$, that if $DX_t(y_1)v_{y_1} \in \mathcal{C}^*_{X_t y_1}(b)$, then $DX_t(y_2)v_{y_2} \in \mathcal{C}^*_{X_t y_2}(2b)$.

We now fix U_0 to be a positively invariant neighborhood of Λ contained in $\bigcup_{x \in \Lambda} U_x$. By construction, for every $y \in U_0$, there exists $x \in \Lambda$ such that

(i) $DX_t(x)v_x \in \mathcal{C}^*_{X_t x}(b)$ implies that $DX_t(y)v_y \in \mathcal{C}^*_{X_t y}(2b)$,

(ii) $DX_t(y)v_y \in \mathcal{C}^*_{X_t y}(b)$ implies that $DX_t(x)v_x \in \mathcal{C}^*_{X_t x}(2b)$, and

(iii) $\frac{1}{2}\|DX_t(x)v_x\| \le \|DX_t(y)v_y\| \le 2\|DX_t(x)v_x\|$,

for all $v \in \mathbb{R}^d$, $b \ge a/8$, $t \in [-T, T]$.

We now proceed with the proof of part (a). By (2.2),

$$\begin{aligned}\|(DX_t(x)v)^s\| &= \|DX_t(x)v^s\| \le \|DX_t|E^s_x\|\|v^s\| \le \lambda^t\|DX_{-t}|E^{cu}_{X_tx}\|^{-1}\|v^s\| \\ &= \lambda^t\|(DX_t|E^{cu}_x)^{-1}\|^{-1}\|v^s\| \le \lambda^t\|(DX_t(x)v)^{cu}\|\|v^{cu}\|^{-1}\|v^s\|,\end{aligned}$$

for all $x \in \Lambda$, $v \in T_xM$, $t \ge 0$. In particular, $DX_t(\mathcal{C}^{cu}_x(b)) \subset \mathcal{C}^{cu}_{X_tx}(b\lambda^t)$ for all $x \in \Lambda$, $b > 0$, $t \ge 0$.

Now let $y \in U_0$, $b \ge a$, $v \in \mathcal{C}^{cu}_y(b)$. We can pass to a nearby point $x \in \Lambda$ with corresponding vector $v_x \in \mathcal{C}^{cu}_x(2b)$ by (ii). Then $DX_t(x)v_x \in \mathcal{C}^{cu}_{X_tx}(2b\lambda^t)$ for all $t \ge 0$. In particular, since $\lambda^T = 1/150 \le 1/16$,

$$DX_T(x)v_x \in \mathcal{C}^{cu}_{X_Tx}(b/8) \quad \text{and} \quad DX_t(x)v_x \in \mathcal{C}^{cu}_{X_tx}(2b) \text{ for all } t \ge 0.$$

By (i),

$$DX_T(\mathcal{C}^{cu}_y(b)) \subset \mathcal{C}^{cu}_{X_Ty}(b/4) \subset \mathcal{C}^{cu}_{X_Ty}(b) \quad \text{and} \quad DX_r(\mathcal{C}^{cu}_y(b)) \subset \mathcal{C}^{cu}_{X_ry}(4b), \tag{3.5}$$

for all $r \in [0, T]$, $y \in U_0$.

By positive invariance of U_0, it follows inductively from (3.5) that $DX_{kT}(\mathcal{C}^{cu}_y(b)) \subset \mathcal{C}^{cu}_{X_{kT}y}(b/4) \subset \mathcal{C}^{cu}_{X_{kT}y}(b)$ for all $y \in U_0$, $k \in \mathbb{N}$.

For general $t \ge T$, write $t = kT + r$ where $k \ge 1$ and $r \in [0, T)$. Again using positive invariance of U_0 together with (3.5),

$$DX_t(\mathcal{C}^{cu}_y(b)) = DX_{kT} \cdot DX_r(\mathcal{C}^{cu}_y(b)) \subset DX_{kT}(\mathcal{C}^{cu}_{X_ry}(4b)) \subset \mathcal{C}^{cu}_{X_ty}(b).$$

This completes the proof of (3.2).

The proof of (3.1) is similar, so we only sketch the details. Using (2.2) as before, we obtain that $DX_{-t}(\mathcal{C}^s_{X_tx}(b)) \subset \mathcal{C}^s_x(b\lambda^t)$ for all $x \in \Lambda$, $b > 0$, $t \ge 0$. Let $y \in U_0$, $b \ge a$, $v \in \mathcal{C}^s_{X_ty}(b)$ where $t \ge 0$, and pass to a nearby point $x_t \in \Lambda$ such that $v_{X_tx_t} \in \mathcal{C}^s_{X_tx_t}(2b)$. (The only difference here is the dependence of x_t on t.) As before, we obtain that

$$DX_{-T}(X_Tx_T)v_{X_Tx_t} \in \mathcal{C}^s_{x_T}(b/8) \quad \text{and} \quad DX_{-t}(X_tx_t)v_{X_tx_t} \in \mathcal{C}^s_{x_t}(2b) \text{ for all } t \ge 0,$$

from which it follows that

$$DX_{-T}(\mathcal{C}^s_{X_Ty}(b)) \subset \mathcal{C}^s_y(b/4) \subset \mathcal{C}^s_y(b) \quad \text{and} \quad DX_{-r}(\mathcal{C}^s_{X_ry}(b)) \subset \mathcal{C}^s_y(4b),$$

for all $r \in [0, T]$, $y \in U_0$. The last formulas are the direct analogy to those in (3.5), and the remainder of the proof of (3.1) is identical to the proof of (3.2).

Next we turn to part (b). The choices of T and U_0 are unchanged. Recall that $a \in (0, \frac{1}{4}]$ is fixed. First we prove (3.3). Suppose that $x \in \Lambda$ and $v \in \mathcal{C}^s_{X_Tx}(2a)$. By (3.1), $DX_{-T}(X_Tx)v \in \mathcal{C}^s_x(2a)$, so using (2.1),

$$\begin{aligned}\|DX_{-T}(X_Tx)v\| &\ge (1-2a)\|(DX_{-T}(X_Tx)v)^s\| = (1-2a)\|(DX_T(x))^{-1}v^s\| \\ &\ge (1-2a)\lambda^{-T}\|v^s\| \ge (1+2a)^{-1}(1-2a)\lambda^{-T}\|v\| \ge 50\|v\| \ge 8\|v\|.\end{aligned}$$

Now let $y \in U_0$, $v \in \mathcal{C}^s_{X_Ty}(a)$. As in part (a), we can pass to a nearby point $x \in \Lambda$ with corresponding vector $v_x \in \mathcal{C}^s_{X_Tx}(2a)$ and so $\|DX_{-T}(X_Tx)v_x\| \ge 8\|v_x\|$. Using (iii) together with positive invariance of U_0, we have that $\|DX_{-T}(X_Ty)v\| \ge 2\|v\|$ for all $v \in \mathcal{C}^s_{X_Ty}(a)$.

By positive invariance of U_0 and (3.1), it follows inductively that

$$\|DX_{-kT}(X_{kT}y)v\| \geq 2^k\|v\| \quad \text{for } y \in U_0,\ v \in \mathcal{C}^s_{X_{kT}y}(a),\ k \geq 0. \tag{3.6}$$

Finally, we consider the case of general $t = kT + r$ where $k \in \mathbb{N}$, $r \in [0,T)$. Let $v \in \mathcal{C}^s_{X_ty}(a)$. Then $DX_{-t}(X_ty)v = DX_{-r}(X_ry)\, DX_{-kT}(X_ty)v$ so it follows from positive invariance and (3.6) that

$$\|DX_{-t}(X_ty)v\| \geq c\|DX_{-kT}(X_{kT}(X_ry))v\| \geq c2^k\|v\|,$$

where $c = \inf_{r\in[0,T]} \inf_{y\in U_0} \inf_{v\in T_yM,\, v\neq 0} \|DX_{-r}(y)v\|/\|v\| > 0$. This completes the proof of (3.3).

To prove (3.4), we start from (2.2) so for $x \in \Lambda$, $u, v \in T_xM$,

$$\frac{\|DX_T(x)u^s\|}{\|u^s\|} \leq \|DX_T|E^s_x\| \leq \lambda^T\|(DX_T|E^{cu}_x)^{-1}\|^{-1} \leq \lambda^T \frac{\|DX_T(x)v^{cu}\|}{\|v^{cu}\|}.$$

Let $u \in DX_{-T}(\mathcal{C}^s_{X_Tx}(2a))$, $v \in \mathcal{C}^{cu}_x(2a)$. By (3.1) and (3.2),

$$\frac{\|DX_T(x)v^{cu}\|}{\|v^{cu}\|} \leq \frac{(1+2a)\|DX_T(x)v\|}{(1-2a)\|v\|}, \quad \text{and} \quad \frac{\|DX_T(x)u\|}{\|u\|} \leq \frac{(1+2a)\|DX_T(x)u^s\|}{(1-2a)\|u^s\|},$$

and so

$$\frac{\|DX_T(x)u\|}{\|u\|} \leq 9\lambda^T \frac{\|DX_T(x)v\|}{\|v\|} \leq \frac{3}{50} \frac{\|DX_T(x)v\|}{\|v\|}$$

for all $v \in \mathcal{C}^{cu}_x(2a)$, $u \in DX_{-T}(\mathcal{C}^s_{X_Tx}(2a))$. Using (iii), it follows that

$$\frac{\|DX_T(y)u\|}{\|u\|} \leq \frac{24}{25} \frac{\|DX_T(y)v\|}{\|v\|}$$

for all $v \in \mathcal{C}^{cu}_y(a)$, $u \in DX_{-T}(\mathcal{C}^s_{X_Ty}(a))$. For general $t \geq 0$, we write $t = kT + r$, $k \geq 0$, $r \in [0,T)$ and proceed as in the proof of (3.3). □

3.2. **Stable bundle over a neighborhood of Λ.** Whereas the original splitting $T_\Lambda M = E^s \oplus E^{cu}$ is DX_t-invariant, in general the extension E^{cu} of the center-unstable bundle cannot be assumed to be invariant. However the extension E^s of the stable bundle may be chosen to be DX_t-invariant:

Proposition 3.2. *The continuous bundle E^s over U_0 can be chosen to be DX_t-invariant and uniformly contracting: $\|DX_t \mid E^s_x\| \leq c^{-1}\tilde{\lambda}^t$ for all $t \geq 0$, $x \in U_0$, where $c > 0$, $\tilde{\lambda} \in (0,1)$ are the constants in Proposition 3.1.*

Proof. We begin with the original choice of continuous splitting $T_{U_0}M = E^s \oplus E^{cu}$. Let $a \in (0, \frac{1}{4}]$ and choose T and U_0 as in Proposition 3.1. For $x \in U_0$, define

$$F_x = \textstyle\bigcap_{t\geq 0} DX_{-t}\big(\mathcal{C}^s_{X_tx}(a)\big).$$

We show that $\{F_x\}$ is the desired stable bundle. That is, we show that for all $t \geq 0$,

(i) $x \mapsto F_x$ is a continuous map from U_0 to the Grassmannian bundle $\mathcal{G} = \{\mathcal{G}_x,\, x \in U_0\}$ where $\mathcal{G}_x$ is the space of d_s-dimensional subspaces of T_xM,
(ii) $F_x = E^s_x$ for $x \in \Lambda$,
(iii) $\{F_x,\, x \in U_0\}$ is DX_t-invariant and uniformly contracting.

Now $\{DX_{-t}(\mathcal{C}^s_{X_tx}(a)),\, t \ge 0\}$ is a nested family of closed cones, and by (3.1) the cones are contained in $\mathcal{C}^s_x(a)$ for $t \ge T$. In particular, $F_x \subset \mathcal{C}^s_x(a)$.

We can also regard $\{DX_{-t}(\mathcal{C}^s_{X_tx}(a)),\, t \ge 0\}$ as a nested family of closed subsets of $\mathcal{G}_x$, so F_x is a closed subset of $\mathcal{G}_x$. By compactness of $\mathcal{G}_x$, the elements $DX_{-t}E^s_{X_tx} \in \mathcal{G}_x$ have a convergent subsequence $DX_{-t_n}E^s_{X_{t_n}x}$ with limit $\tilde{F}_x \in \mathcal{G}_x$. Since $DX_{-t}E^s_{X_tx} \in DX_{-t}(\mathcal{C}^s_{X_tx}(a))$ and F_x is closed, it follows that $\tilde{F}_x \in F_x$.

To summarise, we have shown that there exists a d_s-dimensional subspace $\tilde{F}_x$ such that $\tilde{F}_x \subset F_x$ and $\tilde{F}_x = \lim_{n\to\infty} DX_{-t_n}E^s_{X_{t_n}x}$ (in $\mathcal{G}_x$). Without loss we may suppose that $t_n \ge T$ for all n.

Next we show that $F_x = \tilde{F}_x$. Choose vectors $u_n \in E^s_{X_{t_n}x}$ such that $\|DX_{-t_n}(X_{t_n}x)u_n\| = 1$. Suppose for contradiction that $F_x \ne \tilde{F}_x$. Then F_x is a nontrivial cone containing $\tilde{F}_x$, and so there exists $v \in E^{cu}_x$ nonzero such that $w_n = DX_{-t_n}(X_{t_n}x)u_n + v \in F_x$ for n sufficiently large. It follows from the definition of F_x that $DX_{t_n}(x)w_n = u_n + DX_{t_n}(x)v \in \mathcal{C}^s_{X_{t_n}x}(a)$. Hence

$$\|(DX_{t_n}(x)v)^{cu}\| \le a\|u_n + (DX_{t_n}(x)v)^s\|. \tag{3.7}$$

Since $v \in E^{cu}_x$, it follows from (3.2) that $DX_{t_n}(x)v \in \mathcal{C}^{cu}_x(a)$ and hence $\|(DX_{t_n}(x)v)^s\| \le a\|(DX_{t_n}(x)v)^{cu}\|$ and $\|DX_{t_n}(x)v\| \le (1+a)\|(DX_{t_n}(x)v)^{cu}\|$. Substituting into (3.7) yields $(1-a^2)\|(DX_{t_n}(x)v)^{cu}\| \le a\|u_n\|$ and then

$$\|DX_{t_n}(x)v\| \le (1+a)(1-a^2)^{-1}a\|u_n\|.$$

On the other hand, $u_n \in E^s_{X_{t_n}x}$, $v \in E^{cu}_x$, so by (3.4),

$$\frac{\|DX_{t_n}(x)v\|}{\|v\|} \ge c\tilde{\lambda}^{-t_n}\frac{\|u_n\|}{\|DX_{-t_n}(X_{t_n}x)u_n\|} = c\tilde{\lambda}^{-t_n}\|u_n\|.$$

Letting $n \to \infty$ yields the desired contradiction, and so F_x and $\tilde{F}_x$ coincide. In particular, $F_x \in \mathcal{G}_x$ for all $x \in U_0$.

To prove continuity of the map $x \mapsto F_x$, fix $x \in U_0$ and let $\mathcal{U} \subset \mathcal{G}$ be a neighborhood of F_x. There exists $t_0 \ge 0$ such that $\bigcap_{t\le t_0} DX_{-t}(\mathcal{C}^s_{X_tx}(a)) \subset \mathcal{U}$. By smoothness of the flow, $F_y \subset \bigcap_{t\le t_0} DX_{-t}(\mathcal{C}^s_{X_ty}(a)) \subset \mathcal{U}$ for y sufficiently close to x. This completes the proof of (i).

It is immediate from invariance of the bundle $E^s|\Lambda$ that $E^s_x \subset F_x$ for all $x \in \Lambda$. Hence $E^s_x = F_x$ for all $x \in \Lambda$ establishing (ii).

For $r \ge 0$,

$$\begin{aligned} DX_rF_x &= \textstyle\bigcap_{t\ge0} DX_{r-t}(\mathcal{C}^s_{X_{t-r}(X_rx)}(a)) = \bigcap_{t\ge r} DX_{r-t}(\mathcal{C}^s_{X_{t-r}(X_rx)}(a)) \\ &= \textstyle\bigcap_{t\ge0} DX_{-t}(\mathcal{C}^s_{X_t(X_rx)}(a)) = F_{X_rx}, \end{aligned}$$

so the bundle $\{F_x\}$ is DX_t-invariant. Finally, if $v \in F_x$, $t \ge 0$, then $DX_t(x)v \in \mathcal{C}^s_{X_tx}(a)$ so by (3.3), $\|v\| \ge c\tilde{\lambda}^{-t}\|DX_t(x)v\|$. Hence $\|DX_t \mid F_x\| \le c^{-1}\tilde{\lambda}^t$ so (iii) holds. □

From now on, we suppose that the continuous extension $T_{U_0}M = E^s \oplus E^{cu}$ of $T_\Lambda M = E^s \oplus E^{cu}$ is chosen so that E^s is invariant and uniformly contracted.

Remark 3.3. In the definition of partial hyperbolicity, we assumed uniform contraction along E^s and dominated splitting. However the uniform contraction assumption (2.1) was used only to ensure that the extended stable bundle is uniformly contracting.

For attracting sets satisfying just the dominated splitting assumption (2.2), it still follows from the arguments above that the bundle E^s over Λ extends to a continuous invariant bundle over a neighborhood of Λ.

4. The stable foliation for partially hyperbolic attractors

In this section, we discuss the existence and regularity properties of the stable foliation associated with a partially hyperbolic attractor Λ satisfying the conditions in Definition 2.2. Sectional expansion is not assumed. In Subsection 4.1, we prove that the stable bundle E^s integrates to a contracting invariant topological foliation of a neighborhood of Λ with smooth leaves. In Subsection 4.2, we obtain smoothness of the foliation under a suitable bunching condition.

Remark 4.1. The results in this section follow entirely from standard arguments. However the proof that the extended stable bundle E^s in Section 3.2 integrates to a topological foliation is complicated by the noninvariance of the complementary bundle E^{cu}. Since we have been unable to find a formulation in the literature that does not assume invariance of both E^s and E^{cu}, we present below the details of the standard arguments suitably modified.

4.1. **Stable foliation in a neighborhood of Λ.** Let $\mathbb{D}^k$ denote the k-dimensional open unit disk and let $\mathrm{Emb}^r(\mathbb{D}^k, M)$ denote the set of C^r embeddings $\phi : \mathbb{D}^k \to M$ endowed with the C^r distance.

Theorem 4.2. *There exists a positively invariant neighborhood U_0 of Λ, and a constant $\nu \in (0,1)$, such that the following are true.*

(a) For every point $x \in U_0$ there is a C^r embedded d_s-dimensional disk $W^s_x \subset M$, with $x \in W^s_x$, such that

(1) $T_x W^s_x = E^s_x$.
(2) $X_t(W^s_x) \subset W^s_{X_t x}$ *for all* $t \geq 0$.
(3) $d(X_t x, X_t y) \leq \nu^t d(x,y)$ *for all* $y \in W^s_x$, $t \geq 0$.

(b) The disks W^s_x depend continuously on x in the C^0 topology: there is a continuous map $\gamma : U_0 \to \mathrm{Emb}^0(\mathbb{D}^{d_s}, M)$ such that $\gamma(x)(0) = x$ and $\gamma(x)(\mathbb{D}^{d_s}) = W^s_x$.

(c) The family of disks $\{W^s_x : x \in U_0\}$ defines a topological foliation of U_0.

To prove Theorem 4.2, we begin by following the exposition in [10, Section 6.4(b)].

Let $T > 0$, $c > 0$, $\tilde\lambda \in (0,1)$ be the constants in Propositions 3.1 and 3.2. Increase $T > 0$ if necessary so that $\hat\lambda = c^{-1}\tilde\lambda^T \in (0,1)$. Define the diffeomorphism $f = X_T : U_0 \to U_0$.

For each $x \in U_0$, we consider the exponential map $\exp_x : T_x M \to M$. This transforms a small enough neighborhood of 0 diffeomorphically onto a neighborhood of x, and $D\exp_x(0) = I$.

Choose orthonormal bases on $\mathbb{R}^{d_s}$, $\mathbb{R}^{d_{cu}}$. Also for each $x \in U_0$, choose orthonormal bases on E^s_x and E^{cu}_x. Let $P^s_x : \mathbb{R}^{d_s} \to E^s_x$, $P^{cu}_x : \mathbb{R}^{d_{cu}} \to E^{cu}_x$ be the corresponding isometric isomorphisms. The splitting $E^s \oplus E^{cu}$ is continuous so we can arrange that $x \mapsto P^s_x$ and $x \mapsto P^{cu}_x$ are continuous families of isomorphisms.

Define $P_{x,n} = P^s_{f^n x} + Df^n(x)P^{cu}_x : \mathbb{R}^d \to T_{f^n x}M$. Note that $x \mapsto P_{x,n}$ is a continuous family of isomorphisms for each n. In general $P_{x,n}$ is not an isometric isomorphism since $Df^n E^{cu}_x$ is not necessarily orthogonal to $E^s_{f^n x}$. However, it follows from (3.2) that $Df^n E^{cu}_x \subset \mathcal{C}^{cu}_{f^n x}(a)$ for some $a \in (0, \frac{1}{4}]$, and so the angle between the subspaces $E^s_{f^n x}$ and $Df^n E^{cu}_x$ is bounded away from zero. Hence there is a constant $C_1 \geq 1$ such that $C_1^{-1} \leq \|P_{x,n}\| \leq C_1$ for all $x \in U_0$, $n \geq 0$.

Next, $Q_{x,n} = \exp_{f^n x} \circ P_{x,n} : \mathbb{R}^d \to M$ maps a neighborhood of 0 in $\mathbb{R}^d$ diffeomorphically onto a neighborhood of $f^n x$. Again, $x \mapsto Q_{x,n}$ is a continuous family of diffeomorphisms for each n.

Let $D_\rho \subset \mathbb{R}^d$ denote the ρ-neighborhood of 0. Using boundedness of $\|P_n\|$ and compactness of Λ, and shrinking U_0 if necessary, we can choose $\rho > 0$ so that $Q_{x,n} : D_\rho \to M$ is a diffeomorphism onto its range for all n. Moreover, there is a constant $C_2 \geq 1$ such that

$$C_2^{-1}\|p\| \leq d(f^n x, Q_{x,n}(p)) \leq C_2\|p\| \text{ for all } x \in U_0,\ n \geq 0,\ p \in D_\rho.$$

Now define the family of maps $f_{x,n} = Q^{-1}_{x,n+1} \circ f \circ Q_{x,n} : D_\rho \to \mathbb{R}^d$. By construction, $Df_{x,n}(0)$ is identified with $Df(f^n x)$. Also, the maps $f_{x,n}$ are uniformly C^r close to $Df_{x,n}(0)$ on D_ρ. Hence for any $\delta > 0$ there exists $\rho > 0$ and a family of (surjective) C^r diffeomorphisms $g_{x,n} : \mathbb{R}^d \to \mathbb{R}^d$, $n \geq 0$, such that $\|g_{x,n} - Df_{x,n}(0)\|_{C^1} < \delta$ and $g_{x,n} = f_{x,n}$ on D_ρ. (See for example [10, Lemma 6.2.7].)

Proposition 4.3. *For all $n \geq 0$,*

$$\|Dg_{x,n}(0) \mid \mathbb{R}^{d_s}\| \leq \hat{\lambda}, \quad \|Dg_{x,n}(0) \mid \mathbb{R}^{d_s}\| \cdot \|Dg_{x,n}(0)^{-1} \mid \mathbb{R}^{d_{cu}}\| \leq \hat{\lambda}.$$

Proof. Choose a as in Proposition 3.1. By construction, $Dg_{x,n}(0) = Df_{x,n}(0)$ is identified with $Df(f^n x)$ and moreover,

$$\|Dg_{x,n}(0) \mid \mathbb{R}^{d_s}\| = \|Df \mid E^s_{f^n x}\| = \|DX_T \mid DX_{-T}E^s_{X_T f^n x}\|,$$
$$\|Dg_{x,n}(0)^{-1} \mid \mathbb{R}^{d_{cu}}\| = \|Df^{-1} \mid Df^{n+1}E^{cu}_x\| \leq \|DX_{-T} \mid DX_T(\mathcal{C}^{cu}_{f^n x}(a))\|,$$

where we have used invariance of E^s and forward invariance of $\mathcal{C}^{cu}(a)$. The second estimate follows from (3.4). The first estimate is immediate from Proposition 3.2. □

We require a slightly modified version of the Hadamard-Perron Invariant Manifold Theorem from [10, Theorem 6.2.8, pp 242-257]:

Lemma 4.4. *Let $r \geq 1$. Fix $\lambda_{min} > 0$, $\sigma \in (0,1)$. Then there exists γ, $\delta > 0$ arbitrarily small so that the following holds:*

For each n let $g_n : \mathbb{R}^d \to \mathbb{R}^d$ be a C^r diffeomorphism such that

$$g_n(u,v) = (A_n u + \alpha_n(u,v), B_n v + \beta_n(u,v)), \quad (u,v) \in \mathbb{R}^{d_s} \oplus \mathbb{R}^{d_{cu}}, \tag{4.1}$$

for linear maps $A_n : \mathbb{R}^{d_s} \to \mathbb{R}^{d_s}, B_n : \mathbb{R}^{d_{cu}} \to \mathbb{R}^{d_{cu}}$ and C^r maps $\alpha_n : \mathbb{R}^d \to \mathbb{R}^{d_s}, \beta_n : \mathbb{R}^d \to \mathbb{R}^{d_{cu}}$ with $\alpha_n(0,0) = 0$, $\beta_m(0,0) = 0$ and $\|\alpha_n\|_{C^1} < \delta$, $\|\beta_n\|_{C^1} < \delta$.

Define $\lambda_n = \|A_n\|$, $\mu_n = \|B_n^{-1}\|^{-1}$ and suppose that $\lambda_n \geq \lambda_{min}$ and $\lambda_n/\mu_n \leq \sigma$. Set $\lambda'_n = (1+\gamma)(\lambda_n + \delta(1+\gamma))$, $\mu'_n = \frac{\mu_n}{1+\gamma} - \delta$ and suppose that $\lambda'_n < \nu_n < \mu'_n$ for all $n \in \mathbb{Z}$.

Then there exists a unique family of d_s-dimensional C^1 manifolds $Z_n = \{(x, \varphi_n(x)) : x \in \mathbb{R}^{d_s}\}$, where $\varphi_n : \mathbb{R}^{d_s} \to \mathbb{R}^{d_{cu}}$ satisfies $\varphi_n(0,0) = 0$, $D\varphi_n(0,0) = 0$, and $\|D\varphi_n\|_{C^0} < \gamma$ for all $n \in \mathbb{Z}$, and the following properties hold for all $n \in \mathbb{Z}$:

1. *$g_n(Z_n) = Z_{n+1}$,*
2. *$\|g_n(q)\| \leq \lambda'_n \|q\|$ for $q \in Z_n$,*
3. *If $\|g_{n+k-1} \circ \cdots \circ g_n(q)\| \leq C\nu_{n+k-1} \ldots \nu_n \|q\|$ for all $k \geq 0$ and some $C > 0$, then $q \in Z_n$.*

If $\sup_n \lambda_n < 1$, then the manifolds Z_n are C^r.

Proof. The only difference from [10, Theorem 6.2.8, pp 242-257] is that the rates λ_n, μ_n may depend on n. However, the ratios λ_n/μ_n are controlled uniformly, and it is easy to check that the proof in pp 242-257 of [10] is valid in this slightly more general setting with no change in the arguments. □

Remark 4.5. The constraints on γ and δ can be made explicit:

$$\gamma < \min\{1, \sigma^{-1/2} - 1\}, \quad \delta < \lambda_{min} \min\Big\{\frac{\sigma^{-1} - 1}{\gamma + \gamma^{-1} + 2} + \frac{\sigma^{-1} - (1+\gamma)^2}{(2+\gamma)(1+\gamma)}\Big\}.$$

Remark 4.6. In Lemma 4.4, there exists also a unique family of d_{cu}-dimensional C^1 manifolds $\tilde{Z}_n = \{(x, \psi_n(x)) : x \in \mathbb{R}^{d_{cu}}\}$ satisfying analogous properties to the family Z_n. This leads to a family of center-unstable manifolds $\{W^{cu}_x,\, x \in \Lambda\}$ each of which is tangent at x to E^{cu}_x. These manifolds do not play a role in this paper. (Unlike the case for stable manifolds, there is no useful notion of W^{cu}_x for $x \notin \Lambda$.)

Next, we verify the hypotheses of Lemma 4.4. Fix $x \in U_0$. The sequence of diffeomorphisms $g_{x,n} : \mathbb{R}^d \to \mathbb{R}^d$ is defined for $n \geq 0$. For $n < 0$, we set $g_{x,n} = g_{x,0}$. The diffeomorphisms $g_{x,n}$ have the structure in (4.1). Take $\sigma = \hat{\lambda} \in (0,1)$ and $\lambda_{min} = \inf_{x \in U_0} \|DX_T \mid E^s_x\| > 0$. By Proposition 4.3, the linear maps A_n, B_n satisfy the constraints $\lambda_{min} \leq \lambda_n \leq \sigma$ and $\lambda_n/\mu_n \leq \sigma$. Choose γ, $\delta > 0$ so small that $\sup_n \lambda'_n < 1$ and $\sup_n \lambda'_n/\mu'_n < 1$. Choose $\nu_n \in (\lambda'_n, \mu'_n)$ such that $\nu = \sup_n \nu_n < 1$. Finally, shrink ρ so that $\|\alpha_n\|_{C^1} < \delta$, $\|\beta_n\|_{C^1} < \delta$.

We have shown that the hypotheses of Lemma 4.4 are satisfied, with $\nu_n \leq \nu < 1$ for all n. Let $Z_{x,n}$ denote the family of d_s-dimensional C^r manifolds in Lemma 4.4 and define

$$W^s_x = Q_{x,0}(Z_{x,0} \cap D_\rho).$$

Repeating the construction for every $x \in U_0$, we obtain a family $\{W^s_x,\, x \in U_0\}$ of d_s-dimensional C^r manifolds covering U_0. We claim that this is the desired family of stable manifolds.

Lemma 4.7. *Let $x, y \in U_0$.*

- (a) *If $d(x,y) < C_2^{-1}\rho$ and $y \in W^s_x$ then $d(f^n x, f^n y) \leq C_2^2 \nu^n d(x,y)$ for all $n \geq 0$.*
- (b) *Let $C > 0$. If $d(x,y) < C_2^{-1}C^{-1}\rho$ and $d(f^n x, f^n y) \leq C\nu^n d(x,y)$ for all $n \geq 0$, then $y \in W^s_x$.*
- (c) *There exists $\varepsilon > 0$ such that if $d(x,y) < \varepsilon$ and $y \in W^s_x$ then $fy \subset W^s_{fx}$.*

Proof. Let

$$F_{x,n} = f_{x,n-1} \circ \cdots \circ f_{x,0}, \quad G_{x,n} = g_{x,n-1} \circ \cdots \circ g_{x,0}.$$

Note that if $F_{x,n}(q) \in D_\rho$ for all $0 \le n \le N_0$, or if $G_{x,n}(q) \in D_\rho$ for all $0 \le n \le N_0$, then $F_{x,n}(q) = G_{x,n}(q)$ for all $0 \le n \le N_0$.

(a) Let $y \in W^s_x$ with $d(x,y) < C_2^{-1}\rho$. Then $q = Q_{x,0}^{-1}(y) \in Z_{x,0}$, so by Lemma 4.4(1,2),

$$\|G_{x,n}(q)\| \le \nu^n \|q\| = \nu^n \|Q_{x,0}^{-1}(y)\| \le \nu^n C_2 d(x,y) < \rho,$$

for all $n \ge 0$. Now $f^n = Q_{x,n} \circ F_{x,n} \circ Q_{x,0}^{-1}$, so

$$f^n y = Q_{x,n} \circ F_{x,n}(q) = Q_{x,n} \circ G_{x,n}(q).$$

Hence

$$d(f^n x, f^n y) = d(f^n x, Q_{x,n} \circ G_{x,n}(q)) \le C_2 \|G_{x,n}(q)\| \le C_2^2 \nu^n d(x,y).$$

(b) Suppose that $d(x,y) < C_2^{-1}C^{-1}\rho$ and $d(f^n x, f^n y) \le C\nu^n d(x,y)$ for all $n \ge 0$. Let $q = Q_{x,0}^{-1}(y)$ so $d(x,y) \le C_2\|q\|$. Now $F_{x,n} = Q_{x,n}^{-1} \circ f^n \circ Q_{x,0}$, so

$$\|F_{x,n}(q)\| = \|Q_{x,n}^{-1} \circ f^n(y)\| \le C_2 d(f^n x, f^n y) \le C_2 C \nu^n d(x,y) < \rho.$$

Hence

$$\|G_{x,n}(q)\| = \|F_{x,n}(q)\| \le C_2 C \nu^n d(x,y) \le C_2^2 C \nu^n \|q\|.$$

By Lemma 4.4(3), $q \in Z_{x,0} \cap D_\rho$ and so $y = Q_{x,0}(q) \subset W^s_x$.

(c) Let $x' = fx$, $y' = fy$ and choose $E \ge 1$ such that $d(x,y) \le E d(x',y')$ for all $x, y \in U_0$.

Suppose that $y \in W^s_x$ and $d(x,y) < C_2^{-5}E^{-1}\rho$. Then certainly, $d(x,y) < C_2^{-1}\rho$, so by part (a),

$$d(f^n x', f^n y') = d(f^{n+1}x, f^{n+1}y) \le C_2^2 \nu^{n+1} d(x,y) \le C_2^2 E \nu^n d(x',y') = C\nu^n d(x',y'),$$

where $C = C_2^2 E$. Also, $d(x',y') \le C_2^2 d(x,y) < C_2^{-3}E^{-1}\rho = C_2^{-1}C^{-1}\rho$, so the result follows from part (b). □

Lemma 4.8. *The C^r embedded disks W^s_x depend continuously on x in the C^0 topology: there is a continuous map $\gamma : U_0 \to \mathrm{Emb}^0(\mathbb{D}^{d_s}, M)$ such that $\gamma(x)(0) = x$ and $\gamma(x)(\mathbb{D}^{d_s}) = W^s_x$. Moreover, there exists $L \ge 1$ such that $\mathrm{Lip}\,\gamma(x) \le L$ for all $x \in U_0$, where $\mathrm{Lip}\,\gamma(x) = \sup_{u \ne u'} d(\gamma(x)(u), \gamma(x)(u'))/\|u - u'\|$.*

Proof. Fix $x \in U_0$ and recall that $W^s_x = Q_{x,0}(Z_{x,0} \cap D_\rho)$. For y close to x, let $A_y = Q_{x,0}^{-1}(W^s_y)$. Let $p_y = Q_{x,0}^{-1}(y) = Q_{x,0}^{-1} \circ Q_{y,0}(0) \in A_y$. In particular $A_x = Z_{x,0} \cap D_\rho$ and $p_x = 0$. Moreover, $y \mapsto p_y$ is continuous.

Now $T_{p_y}A_y = DQ_{x,0}^{-1}(y) T_y W^s_y = DQ_{x,0}^{-1}(y) E^s_y$, so it follows from smoothness of $Q_{x,0}$ and continuity of E^s that A_y can be viewed as a graph over $\mathbb{D}^{d_s} \subset \mathbb{R}^{d_s}$ for y close to x. In particular, $A_y = \{(u, \phi_y(u)) : u \in \mathbb{D}^{d_s}\}$ where $\phi_y : \mathbb{D}^{d_s} \to \mathbb{R}^{d_{cu}}$, see Figure 1. Hence $W^s_y = \{Q_{x,0}(u, \phi_y(u)) : u \in \mathbb{D}^{d_s}\}$. The family of functions ϕ_y are C^r with uniform Lipschitz constant. Since $p_y \in A_y$, there exists $u_y \in \mathbb{D}^{d_s}$ such that $p_y = (u_y, \phi_y(u_y))$.

Define the family of embeddings $\gamma : U_0 \to \mathrm{Emb}^r(\mathbb{D}^{d^s}, M)$ given by

$$\gamma(y)(u) = Q_{x,0}(u, \phi_y(u)).$$

We claim that $y \mapsto \phi_y$ is continuous at x in the C^0 topology, and hence the embedding γ is continuous at x in the C^0 topology.

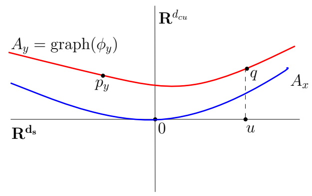

FIGURE 1. A_y as the graph of ϕ_y near A_x.

It remains to verify the claim. Suppose that $y_n \to x$. By Arzelà-Ascoli, we can pass to a further subsequence such that $\lim_{n\to\infty} \sup_{u\in\mathbb{D}^{d_s}} \|\phi_{y_n}(u) - \psi(u)\| = 0$ for some continuous function $\psi : \mathbb{R}^{d_s} \to \mathbb{R}^{d_{cu}}$.

Since $p_{y_n} \to 0$, for n large enough we have that $p_{y_n} \in D_{\frac{1}{2}C_2^{-5}\rho}$. Now fix $u \in \mathbb{D}^{d_s}$ (see Figure 1). Shrinking the disk $\mathbb{D}^{d_s}$, we can ensure that $q_n = (u, \phi_{y_n}(u)) \in D_{\frac{1}{2}C_2^{-5}\rho}$ for n sufficiently large. Hence

$$d(Q_{x,0}(q_n), y_n) \le d(Q_{x,0}(q_n), x) + d(x, y_n) \le C_2^{-3}\rho \le C_2^{-1}\rho.$$

By construction, $Q_{x,0}(q_n) \in W^s_{y_n}$, so by Lemma 4.7(a),

$$d(f^k \circ Q_{x,0}(q_n), f^k y_n) \le C_2^2 \nu^k d(Q_{x,0}(q_n), y_n) \quad \text{for all } k \ge 0.$$

Letting $n \to \infty$, we obtain that

$$d(f^k \circ Q_{x,0}(u, \psi(u)), f^k x) \le C_2^2 \nu^k d(Q_{x,0}(u, \psi(u)), x) \quad \text{for all } k \ge 0.$$

By Lemma 4.7(b), $Q_{x,0}(u, \psi(u)) \in W^s_x$ so $(u, \psi(u)) \in A_x$. It follows that $\psi(u) = \phi_x(u)$. Hence all subsequential limits of ϕ_y (as $y \to x$) coincide with ϕ_x so $\lim_{y\to x} \phi_y = \phi_x$ in the C^0 topology as required. □

Lemma 4.9. *The family of disks $\{W^s_x : x \in U_0\}$ defines a topological foliation.*

Proof. Let $x \in U_0$ and choose an embedded d^{cu}-dimensional disk $Y \subset M$ containing x and transverse to W^s_x. By continuity of E^s, we can shrink Y so that Y is transverse to W^s_y at y for all $y \in Y$. Let $\psi : \mathbb{D}^{cu} \to Y$ be a choice of embedding.

Now define $\chi : \mathbb{D}^s \times \mathbb{D}^{cu} \to U_0$ by setting $\chi(u, v) = \gamma(\psi(v))(u)$. Note that χ maps horizontal lines $\{v = \text{const.}\}$ homeomorphically onto stable disks; see Figure 2.

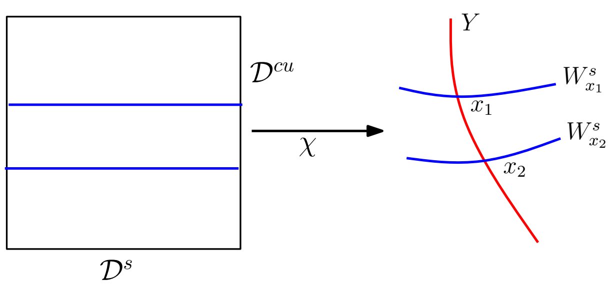

FIGURE 2. Topological foliation chart

By Lemma 4.8, each of these embeddings is Lipschitz with uniform Lipschitz constant L. Using this together with the continuity statement in Lemma 4.8,

$$\begin{aligned} d(\chi(u,v),\chi(u_0,v_0)) &\le d(\gamma(\psi(v))(u),\gamma(\psi(v))(u_0)) + d(\gamma(\psi(v))(u_0),\gamma(\psi(v_0))(u_0)) \\ &\le L\|u-u_0\| + \|\gamma(\psi(v))-\gamma(\psi(v_0))\|_{C^0} \to 0, \end{aligned}$$

as $(u,v)\to(u_0,v_0)$, establishing continuity of χ.

Suppose that $\chi(u_1,v_1)=\chi(u_2,v_2)$ with common value $y\in U_0$. Then $y\in W^s_{x_1}\cap W^s_{x_2}$ where $x_j=\psi(v_j)$. We claim that $x_1=x_2$ with common value $\hat{x}$. In particular $v_1=v_2$. But now $\gamma(\hat{x})(u_1)=\gamma(\hat{x})(u_2)$ and so $u_1=u_2$. It follows that χ is injective and hence is a homeomorphism onto a neighborhood of x as required.

It remains to prove the claim. Note that $W^s_{x_2}$ can be viewed as a graph over $W^s_{x_1}$. Let $A=W^s_{x_1}\cap W^s_{x_2}$. We show that A is open and closed in $W^s_{x_1}$. Since $y\in A$ and $W^s_{x_1}$ is connected, $A=W^s_{x_1}$ and in particular, $x_2=x_1$ as required.

It is clear that A is closed in $W^s_{x_1}$. To prove that A is open, suppose that $z\in A$. Since $W^s_{x_j}$ are tangent to $E^s_{x_j}$ with uniform Lipschitz constant, there exists $C>0$ such that $d(x_1,x_2)\le Cd(z,x_j)$. Let $z'\in W^s_{x_1}$ be such that $d(z,z')\le(1/2C)d(x_1,x_2)$. (This implies that $d(x_1,x_2)\le 2Cd(z',x_2)$.) We must show that $z'\in A$. Now

$$\begin{aligned} &d(f^nz',f^nx_2)\le d(f^nz',f^nx_1)+d(f^nx_1,f^nz)+d(f^nz,f^nx_2) \\ &\le C_2^2\nu^n\{d(z',x_1)+d(x_1,z)+d(z,x_2)\} \\ &\le C_2^2\nu^n\{d(z',x_2)+d(x_2,x_1)\ +\ d(x_1,x_2)+d(x_2,z')+d(z',z)\ +\ d(z,z')+d(z',x_2)\} \\ &= C_2^2\nu^n\{3d(z',x_2)+2d(x_1,x_2)+2d(z,z')\}\le C_2^2\nu^n\{3d(z',x_2)+4d(x_1,x_2)\} \\ &\le (3+8C)C_2^2\nu^n d(z',x_2). \end{aligned}$$

We can arrange that χ takes values in $B_\varepsilon(x)$ where ε is as small as required. By Lemma 4.7(b), $z'\in W^s(x_2)$ and hence $z'\in A$ completing the proof. □

Corollary 4.10. *There exists $\varepsilon>0$ such that $X_t(W^s_x\cap B_\varepsilon(x))\subset W^s_{X_tx}$ for all $t\ge 0$, $x\in U_0$.*

Proof. Choose $n_0\ge 1$ such that $C_2^2\nu^{n_0}<1$. Shrinking ε, it follows from Lemma 4.7(a,c) that $f^{n_0}(W^s_x\cap B_\varepsilon(x))\subset W^s_{f^{n_0}x}\cap B_\varepsilon(f^{n_0}x)$ and inductively that $f^{kn_0}(W^s_x\cap B_\varepsilon(x))\subset W^s_{f^{kn_0}x}\cap B_\varepsilon(f^{kn_0}x)$ for all $k\ge 0$.

Next choose $C\ge 1$ such that $d(X_rx,X_ry)\le Cd(x,y)$ for all $x,y\in U_0$, $r\in[-n_0T,n_0T]$. Suppose that $y\in W^s_x$ and let $x'=X_rx$, $y'=X_ry$. By Lemma 4.7(a), for y sufficiently close to x,

$$d(f^nx',f^ny')=d(X_rf^nx,X_rf^ny)\le Cd(f^nx,f^ny)\le CC_2^2\nu^nd(x,y)\le C^2C_2^2\nu^nd(x',y'),$$

for all $n\ge 0$. By Lemma 4.7(b), $X_ry\in W^s_{X_rx}$ for y sufficiently close to x. Hence there exists $\varepsilon>0$ such that $X_r(W^s_x\cap B_\varepsilon(x))\subset W^s_{X_rx}$ for all $r\in[0,n_0T]$, $x\in U_0$.

The result for general t follows by writing $t=kn_0T+r$ where $k\ge 0$, $r\in[0,n_0T)$. □

Recall that $f=X_T$. Choose C such that $\sup_{r\in[0,T]}d(X_rx,X_ry)\le Cd(x,y)$ for all $x,y\in U$. Write $t=nT+r$, $n\ge 0$, $r\in[0,T)$. By Lemma 4.7(a), if $d(x,y)<C_2^{-1}\rho$ and

$y \in W^s_x$, then

$$d(X_tx, X_ty) = d(X_{nT+r}x, X_{nT+r}y) \le C_2^2 C\nu^n d(x, y) \le C'\tilde{\nu}^t d(x, y),$$

where $C' = C_2^2 C\nu^{-1}$ and $\tilde{\nu} = \nu^{1/T}$.

Passing to an adapted metric, we can arrange that there are constants $\varepsilon > 0$, $\nu \in (0, 1)$ such that if $d(x, y) < \varepsilon$ and $y \in W^s_x$, then $d(X_tx, X_ty) \le \nu^t d(x, y)$ for all $t \ge 0$. From now on, we write W^s_x instead of $W^s_x \cap B_\varepsilon(x)$. With this notation, Corollary 4.10 states that $X_t(W^s_x) \subset W^s_{X_tx}$ for all $x \in U_0$, $t \ge 0$.

This completes the proof of Theorem 4.2.

4.2. **Regularity of the stable foliation.** In this subsection, we prove results on the regularity of the stable foliation $\{W^s_x\}$ in a neighborhood of Λ under an appropriate bunching condition. We follow [8, Theorem 6.5], adapting and applying the results of [7] in our setting.

We continue to suppose that X_t is the flow generated by a C^r vector field G where $r \ge 1$. Let $q \in [0, r]$. We suppose that there exists $t > 0$ such that the following bunching condition holds:

$$\|DX_t \mid E^s_x\| \cdot \|DX_{-t} \mid E^{cu}_{X_tx}\| \cdot \|DX_t \mid E^{cu}_x\|^q < 1 \quad \text{for all } x \in \Lambda. \tag{4.2}$$

Choose t as in (4.2) and let $f = X_t$. Increasing t and shrinking U_0 if necessary, we can ensure that

$$\|Df \mid E^s_x\| \cdot \|Df^{-1} \mid E^{cu}_{fx}\| \le \|Df \mid E^s_x\| \cdot \|Df^{-1} \mid E^{cu}_{fx}\| \cdot \|Df \mid T_xM\|^q < 1, \tag{4.3}$$

for all $x \in U_0$.

Let $T_{U_0}M = E^s \oplus E^{cu}$ be the continuous splitting with E^s invariant as in Proposition 3.2. Let $T_{U_0}M = F^s \oplus F^{cu}$ be a C^r approximation of this splitting. For each $x \in U_0$, let $L(F^s_x, F^{cu}_x)$ denote the space of linear maps from F^s_x to F^{cu}_x, and let $\mathbb{D}_x$ denote the unit disk in $L(F^s_x, F^{cu}_x)$. Define the corresponding disk bundle $\mathcal{D}_0 = \{\mathbb{D}_x,\, x \in U_0\}$.

Let $U_1 = f(U_0) \subset U_0$ and set $\mathcal{D}_1 = \{\mathbb{D}_x,\, x \in U_1\}$. Let $h = f^{-1}|_{U_1} : U_1 \to U_0$. Since $h(U_1) = U_0 \supset U_1$, the C^r diffeomorphism h is *overflowing* in the sense of [7, p. 30].

Represent $Dh(x) : T_xM \to T_{hx}M$ using the splitting $F^s \oplus F^{cu}$ by writing

$$Dh(x) = \begin{pmatrix} A_x & B_x \\ C_x & D_x \end{pmatrix} : F^s_x \times F^{cu}_x \to F^s_{hx} \times F^{cu}_{hx}, \quad x \in U_1.$$

We define the graph transform $\Gamma : \mathcal{D}_1 \to \mathcal{D}_0$,

$$\Gamma_x(\ell) = (C_x + D_x\ell)(A_x + B_x\ell)^{-1}, \quad \ell \in \mathbb{D}_x,\, x \in U_1.$$

Lemma 4.11. *The neighborhood U_0 of Λ and the C^r splitting $F^s \oplus F^{cu}$ can be chosen so that $\Gamma : \mathcal{D}_1 \to \mathcal{D}_0$ is well-defined and* $\mathrm{Lip}(\Gamma_x) \cdot \|Dh^{-1}|T_{hx}M\|^q < 1$ *for all $x \in U_1$.*

Proof. By (4.3), we can choose $\lambda_x \in (0, 1)$ such that

$$\|Df \mid E^s_x\| \cdot \|Df^{-1} \mid E^{cu}_{fx}\| < \lambda_x \quad \text{and} \quad \lambda_x \|Df \mid T_xM\|^q < 1 \quad \text{for all } x \in U_0.$$

Since f is C^1 and $\overline{U_0}$ is compact, there exists $\delta \in (0, 1)$ such that $(\lambda_{hx} + 2\delta)(1 - \delta)^{-2} < 1$ and

$$(\lambda_{hx} + 2\delta)(1 - \delta)^{-2}\|Dh^{-1} \mid T_{hx}M\|^q < 1 \quad \text{for all } x \in U_1. \tag{4.4}$$

Since F^s is close to the Df-invariant contracting bundle E^s, we can arrange that $\|C_x\| \le 1$ and $\|A_x^{-1}\| \le 1$ for all $x \in U_1$. Also, F^{cu} is close to E^{cu} which is invariant when restricted to Λ so we can arrange that $\|B_x\| < \delta$. Moreover, A_x^{-1} is close to $Df \mid E^s_{hx}$ and D_x is close to $Df^{-1} \mid E^{cu}_x$ so we can ensure that $\|A_x^{-1}\|\|D_x\| \le \lambda_{hx}$ for all $x \in U_1$.

Let $\ell, \ell' \in \mathbb{D}_x$. Note that $\|A_x^{-1}B_x\ell\| \le \delta$, so $\|(I + A_x^{-1}B_x\ell)^{-1}\| \le (1-\delta)^{-1}$. Similarly, $\|(I + A_x^{-1}B_x\ell')^{-1}\| \le (1-\delta)^{-1}$. It follows that

$$\begin{aligned}\|(A_x + B_x\ell)^{-1} - (A_x + B_x\ell')^{-1}\| &= \|(A_x + B_x\ell)^{-1}(B_x(\ell' - \ell))(A_x + B_x\ell')^{-1}\| \\ &\le \|A_x^{-1}\|^2\delta(1-\delta)^{-2}\|\ell' - \ell\| \le \|A_x^{-1}\|\delta(1-\delta)^{-2}\|\ell' - \ell\|.\end{aligned}$$

Hence

$$\begin{aligned}\|\Gamma_x(\ell) - \Gamma_x(\ell')\| &\le \|D_x(\ell - \ell')\|\|(A_x + B_x\ell)^{-1}\| \\ &\qquad + \|(C_x + D_x\ell')\|\|(A_x + B_x\ell)^{-1} - (A_x + B_x\ell')^{-1}\| \\ &\le \|A_x\|^{-1}\|D_x\|(1-\delta)^{-1}\|\ell - \ell'\| + (1 + \|D_x\|)\|A_x^{-1}\|\delta(1-\delta)^{-2}\|\ell - \ell'\| \\ &\le \lambda_{hx}(1-\delta)^{-1}\|\ell - \ell'\| + 2\delta(1-\delta)^{-2}\|\ell - \ell'\|,\end{aligned}$$

and so $\mathrm{Lip}(\Gamma_x) \le (\lambda_{hx} + 2\delta)(1-\delta)^{-2}$ for all $x \in U_1$. In particular, $\mathrm{Lip}(\Gamma_x) < 1$ so $\Gamma_x(\mathbb{D}_x) \subset \mathbb{D}_{hx}$, and hence Γ is well-defined. The result follows from this estimate combined with (4.4). □

Theorem 4.12. *Let $q \in [0, [r]\,]$. If condition (4.2) holds for some $t > 0$, then the bundle E^s is C^q over U_1. That is, the map $x \mapsto E^s_x$ is a C^q map from U_1 to $\mathcal{D}_1$.*

Proof. Recall that we can regard E^s_x as the graph of an element $\omega \in L(F^s_x, F^{cu}_x)$ with $\|\omega\|$ as close to zero as desired. In particular, $\|\omega\| \le 1$, and hence E^s is identified with a continuous Df-invariant section of $\mathcal{D}_1$.

Note that $Dh(x)\,\mathrm{graph}(\ell) = \mathrm{graph}(\Gamma_x(\ell))$ for $\ell \in \mathbb{D}_x$. Since $h = df^{-1}$, it follows that $E^s : U_1 \to \mathcal{D}_1$ is a continuous Γ-invariant section.

By Lemma 4.11, the graph transform $\Gamma : \mathcal{D}_1 \to \mathcal{D}_0$ defines a fiber contraction over the overflowing diffeomorphism $h : U_1 \to U_0$, and this fiber contraction is *q-sharp* in the terminology of [7]. When q is an integer, we have verified the hypotheses of the "C^r section theorem" [7, Theorem 3.5] (with q playing the role of r, and vector bundles replaced by disk bundles as in [7, Remark, p. 36]). It follows from [7, Theorem 3.5] that $E^s : U_1 \to \mathcal{D}_1$ is the unique continuous Γ-invariant section and moreover that this section is C^q.

This completes the proof in the case that q is an integer. The general case follows from [7, Remark 2, p. 38]. □

Remark 4.13. (a) It is immediate from domination (2.2) that condition (4.2) holds with $q = 0$. By smoothness of the flow and compactness of Λ, condition (4.2) holds for some $q > 0$ and hence the stable bundle E^s is at least Hölder over U_1.

(b) When $q \ge 1$ in Theorem 4.12, it follows by Frobenius that the family of stable manifolds $\{W^s_x\}$ obtained in Theorem 4.2 forms a C^q foliation of U_1 in the sense that the foliation charts are C^q. Moreover the holonomy maps along the stable leaves are C^q smooth. (See [13, Section 6] for more details.)

For $q \in (0, 1)$, it remains true that the holonomy maps are C^q [13], but the stable foliation $\{W^s_x\}$ need not be Hölder continuous.

5. Strong dissipativity

In this section, we define strong dissipativity for sectional hyperbolic attractors. This is a verifiable condition for smoothness of stable foliations, extending [2] who proved strong dissipativity and hence smoothness of the stable foliation for the classical Lorenz attractor. We recover the result of [2] and moreover obtain an estimate for the smoothness.

Recall that $d_s = \dim E^s_x$. Given $A = \{a_{ij}\} \in \mathbb{R}^{d\times d}$, let $\|A\|_2 = (\sum_{ij} a_{ij}^2)^{1/2}$.

Definition 5.1. Let $q > 1/d_s$. A partially hyperbolic attractor Λ is *q-strongly dissipative* if

(a) For every equilibrium $\sigma \in \Lambda$ (if any), the eigenvalues λ_j of $DG(\sigma)$, ordered so that $\Re\lambda_1 \le \Re\lambda_2 \le \cdots \le \Re\lambda_d$, satisfy $\Re(\lambda_1 - \lambda_{d_s+1} + q\lambda_d) < 0$.

(b) $\sup_{x\in\Lambda}\big\{\operatorname{div} G(x) + (d_s q - 1)\|(DG)(x)\|_2\big\} < 0$.

Theorem 5.2. *Let Λ be a sectional hyperbolic attractor. Suppose that Λ is q-strongly dissipative for some $q \in (1/d_s, [r]\,]$. Then there exists a neighborhood U_0 of Λ such that the stable manifolds $\{W^s_x,\, x \in U_0\}$ define a C^q foliation of U_0.*

Proof. For each $t \in \mathbb{R}$, we define $\eta_t : \Lambda \to \mathbb{R}$,

$$\eta_t(x) = \log\big\{\|DX_t|E^s_x\| \cdot \|DX_{-t}|E^{cu}_{X_t x}\| \cdot \|DX_t|E^{cu}_x\|^q\big\}.$$

Note that $\{\eta_t,\ t \in \mathbb{R}\}$ is a continuous family of continuous functions each of which is subadditive, that is, $\eta_{s+t}(x) \le \eta_s(x) + \eta_t(X_s x)$.

Let $\mathcal{M}$ denote the set of flow-invariant ergodic probability measures on Λ. We claim that for each $m \in \mathcal{M}$, the limit $\lim_{t\to\infty} t^{-1}\eta(x)$ exists and is negative for m-almost every $x \in \Lambda$. It then follows from [4, Proposition 3.4] that there exists constants $C, \beta > 0$ such that $\exp\eta_t(x) \le Ce^{-\beta t}$ for all $t > 0$, $x \in \Lambda$. In particular, for t sufficiently large, $\exp\eta_t(x) < 1$ for all $x \in \Lambda$. Hence condition (4.2) is satisfied for such t and the result follows from Theorem 4.12 and Remark 4.13.

It remains to verify the claim. For each $m \in \mathcal{M}$, we label the Lyapunov exponents

$$\chi_1(m) \le \chi_2(m) \le \cdots \le \chi_d(m).$$

Since Λ is partially hyperbolic, the Lyapunov exponents $\chi_j(m)$, $j = 1, \dots, d_s$ are associated with E^s and are negative, while the remaining exponents are associated with E^{cu}.

For m-a.e. $x \in \Lambda$ we have

$$\lim_{t\to\infty} t^{-1}\log\|DX_t|E^s_x\| = \chi_1(m), \quad \lim_{t\to\infty} t^{-1}\log\|DX_{-t}|E^{cu}_{X_t x}\| = -\chi_{d_s+1}(m),$$
$$\lim_{t\to\infty} t^{-1}\log\|DX_t|E^{cu}_x\| = \lim_{t\to\infty} t^{-1}\log\|DX_t \mid T_x M\| = \chi_d(m).$$

Hence, m-almost everywhere,

$$\lim_{t\to\infty} t^{-1}\eta_t(x) = \chi_1(m) - \chi_{d_s+1}(m) + q\chi_d(m).$$

If m is a Dirac delta at an equilibrium $\sigma \in \Lambda$, then $\chi_j(m) = \Re\lambda_j$ for $j = 1, \dots, d$, where λ_j are the eigenvalues of $DG(\sigma)$. Hence, it is immediate from Definition 5.1(a) that $\lim_{t\to\infty} t^{-1}\eta_t(\sigma) < 0$.

If m is not supported on an equilibrium, then there is a zero Lyapunov exponent in the flow direction. Sectional expansion ensures that $\chi_{d_s+1}(m) = 0$ and $\chi_j(m) > 0$ for

$j = d_s + 2, \dots, d$. Hence, m-almost everywhere,

$$\begin{aligned}\lim_{t\to\infty} t^{-1}\eta_t(x) &= \chi_1(m) + q\chi_d(m) \le d_s^{-1}\textstyle\sum_{j=1}^{d_s}\chi_j(m) + q\chi_d(m)\\ &= d_s^{-1}\big(\textstyle\sum_{j=1}^{d_s}\chi_j(m) + d_s q\chi_d(m)\big) \le d_s^{-1}\big(\textstyle\sum_{j=1}^{d}\chi_j(m) + (d_s q - 1)\chi_d(m)\big)\\ &= d_s^{-1}\lim_{t\to\infty} t^{-1}\big(\log|\det DX_t(x)| + (d_s q - 1)\log\|DX_t(x)\|\big)\\ &\le d_s^{-1}\lim_{t\to\infty} t^{-1}\textstyle\int_0^t\big(\operatorname{div} DG(X_s x) + (d_s q - 1)\|DG(X_s x)\|_2\big)\,ds\\ &\le d_s^{-1}\sup_{x\in\Lambda}\big\{\operatorname{div} DG(x) + (d_s q - 1)\|DG(x)\|_2\big\}.\end{aligned}$$

By Definition 5.1(b), we again have that $\lim_{t\to\infty} t^{-1}\eta_t(x) < 0$ for m-almost every $x \in \Lambda$. This completes the proof of the claim. □

Remark 5.3. If $\sup_\Lambda \operatorname{div} G < 0$, then condition (b) holds for $q = d_s^{-1} + \varepsilon$ for ε sufficiently small. When $\dim M = 3$, we have $d_s = 1$ and hence we recover the result in [2, Lemma 2.2]. For the classical Lorenz equations [11], we have

$$\operatorname{div} G \equiv -\tfrac{41}{3}, \quad \lambda_1 \approx -22.83, \quad \lambda_2 = -\tfrac{8}{3}, \quad \lambda_3 \approx 11.83,$$

so the Lorenz attractor is $(1+\varepsilon)$-strongly dissipative for $\varepsilon > 0$ sufficiently small. Hence, the stable foliation is $C^{1+\varepsilon}$ for the classical Lorenz attractor.

In fact, we have:

Corollary 5.4. *The stable foliation for the classical Lorenz attractor is at least $C^{1.264}$.*

Proof. By definition, q-strong dissipativity holds for any $q < \min\{q_1, q_2\}$ where

$$q_1 = \frac{\lambda_2 - \lambda_1}{\lambda_3} \approx 1.704, \quad q_2 = 1 - \frac{\operatorname{div} G}{\sup\|DG\|_2} = 1 + \frac{41}{3}\frac{1}{\sup\|DG\|_2}.$$

Now

$$\|DG(x)\|_2^2 = 201 + \tfrac{64}{9} + 2x_1^2 + x_2^2 + (x_3 - 28)^2 \approx 208.11 + V,$$

where

$$V = 2x_1^2 + x_2^2 + (x_3 - 28)^2.$$

By [5], a trapping region is given by ellipsoids of the form

$$\frac{c-28}{10}x_1^2 + x_2^2 + (x_3 - 28)^2 = R,$$

provided $R \ge \frac{c^2b^2}{4(b-1)}$ where $b = 8/3$. (See in particular [5, equation (20) and Fig 10]. See also [16, 17] for related results.) Taking $c = 48$ we obtain $\frac{c^2b^2}{4(b-1)} = 2457.6$. Hence $V \le 2457.6$ and so $q_2 > 1.264$ as required. □

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