GENERALIZATIONS OF A RESULT ON SYMMETRY GROUPS OF ATTRACTORS

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Abstract. The admissible symmetry groups of attractors for continuous equivariant mappings were classified in Ashwin and Melbourne [1994] and Melbourne, Dellnitz and Golubitsky [1993]. We consider extensions of these results to include attractors in fixed-point subspaces, attractors for equivariant diffeomorphisms and flows, and attractors in the presence of a continuous symmetry group. Our results lead to surprising (if somewhat speculative) implications for both theory and applications of equivariant dynamical systems.

1. Introduction

Suppose that $\Gamma \subset \mathbf{O}(n)$ is a compact Lie group acting on $\mathbb{R}^n$. We are interested in the possible symmetry groups for attractors of Γ-equivariant dynamical systems. If A is a subset of $\mathbb{R}^n$, define the symmetry group of A to be the subgroup of Γ

$$\Sigma_A = \{\gamma \in \Gamma; \gamma A = A\}.$$

Also, define the subgroup of elements that fix A pointwise

$$T_A = \{\gamma \in \Gamma; \gamma x = x \text{ for all } x \in A\}.$$

It is easily seen that T_A is a normal subgroup of Σ_A (see Proposition 3).

Following Melbourne et al [1993], we define an attractor to be a Liapunov stable (not necessarily asymptotically stable) ω-limit set. An attractor A is said to be Σ-symmetric if $\Sigma_A = \Sigma$ and $T_A = \mathbf{1}$. When Γ is finite, the combined results of Melbourne, Dellnitz and Golubitsky [1993] and Ashwin and Melbourne [1994] yield a classification of the (strongly) admissible subgroups Σ: those subgroups for which there exists a continuous Γ-equivariant mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ with a (connected) Σ-symmetric attractor. We note that when Σ is admissible, the attractor can be constructed so as to be asymptotically stable.

In order to state these results, we recall some notation. An element $\tau \in \Gamma$ is a reflection if $\dim \operatorname{Fix}(\tau) = n - 1$. Define K_Σ to be the set of reflections in $\Gamma - \Sigma$ and set $L_\Sigma = \bigcup_{\tau \in K_\Sigma} \operatorname{Fix}(\tau)$. If I is a subgroup of Γ, define I_R to be the subgroup generated by reflections in I. Finally, we say that Σ is a cyclic extension of a subgroup Δ if Δ is normal in Σ and Σ/Δ is cyclic.

Theorem 1. *Suppose that Γ is a finite subgroup of $\mathbf{O}(n)$, $n \geq 3$, and that Σ is a subgroup of Γ. Then the following statements are equivalent.*

(a) *Σ is strongly admissible,*

(b) *Σ fixes a connected component of $\mathbb{R}^n - L_\Sigma$, and*

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(c) *There is an isotropy subgroup $I \subset \Gamma$ such that $I_R \subset \Sigma \subset I$.*

A subgroup Σ is admissible if and only if it is a cyclic extension of a strongly admissible subgroup.

Remark 1. (a) Theorem 1 is proved in Melbourne et al [1993] and Ashwin and Melbourne [1994]. Condition (b) is the more geometric characterization of strong admissibility, while condition (c) is convenient for doing calculations.
(b) Conditions (b) and (c) are equivalent for all $n \geq 1$. In addition, the last statement of Theorem 1 is valid for $n \geq 1$. In fact the theorem fails only when $n = 2$ and $\Gamma \subset \mathbf{O}(2)$ is cyclic. In this case, conditions (b) and (c) hold for all subgroups Σ and indeed all subgroups are admissible. However, only Γ and $\mathbf{1}$ are strongly admissible (Ashwin and Melbourne [1994], Theorem 7.2(a)).

The setting in Theorem 1 can be generalized to include various combinations of the following three issues.

(i) Attractors in proper fixed-point subspaces ($T_A \neq \mathbf{1}$).
(ii) Attractors for differentiable/invertible mappings or flows.
(iii) Attractors in the presence of a continuous group of symmetries.

In each of these situations, the results in Melbourne et al [1993] place (nonoptimal and sometimes vacuous) restrictions on the possible symmetry groups of attractors. Eventually, we would like to have classifications of the admissible and strongly admissible subgroups in each of these situations. The purpose of this paper is to describe some partial progress in these directions.

Our results on (i) are complete and are described in Section 2. The important issue here is to take account of *hidden symmetries*. There is an additional subtlety because of recent work of Alexander et al [1992] and Buescu and Stewart [1993] indicating that ω-limit sets that lie in proper invariant subspaces often enjoy attracting properties in a strong measure-theoretic sense and yet violate our topological notion of stability. Thus it is important to rule out such ω-limit sets when proving inadmissibility of a subgroup Σ. In Section 2 we give an appropriate generalization of the definition of attractor that includes these examples and for which our theorems still hold.

In Section 3 we describe some partial results on (ii). Additional restrictions on symmetry groups are obtained for invertible mappings and we conjecture that these conditions are optimal (at least in high enough dimensions) even for diffeomorphisms and flows. Moreover, in the presence of differentiability assumptions, we expect that admissible symmetry groups may occur in a structurally stable fashion.

So far we have dealt with finite groups Γ and divided the subgroups of Γ into those that are admissible and those that are not admissible. In contrast, in situation (iii) where Γ is not assumed to be finite but an arbitrary compact Lie group, it is possible that some admissible subgroups occur as the symmetry group of an attractor only in degenerate situations. Examples of this phenomenon for specific actions of Γ are given in Ashwin and Chossat [1992] and Ashwin and Stewart [1993]. These results are extended in Section 4.

Our simplest result (Theorem 12) is that if Γ is connected and abelian (a torus), then ω-limit sets typically are fully symmetric. The situation is more subtle if Γ is not connected and abelian, since there are bounds on the symmetry groups of relative periodic orbits (Field [1980], Krupa [1990], Field [1991]). However in

Theorem 15 we show that 'sufficiently chaotic' ω-limit sets typically have symmetry at least Γ^0 (the connected component of the identity in Γ).

For example, if $\Gamma = \mathbf{SO}(2)$ then ω-limit sets typically are fully symmetric. If $\Gamma = \mathbf{O}(2)$, sufficiently chaotic ω-limit sets typically have symmetry $\mathbf{SO}(2)$ or $\mathbf{O}(2)$. On the other hand, there can exist robust $\mathbb{D}_1$-symmetric period two points. We conjecture that these subgroups $\mathbf{O}(2)$, $\mathbf{SO}(2)$ and $\mathbb{D}_1$ are typically the only ones that arise as the symmetry group of an attractor.

Finally, in Section 5 we consider the implications of our results both for theory (detectives, see Barany, Dellnitz and Golubitsky [1993] and Dellnitz, Golubitsky and Nicol [1993]) and applications (turbulent Taylor vortices, see Brandstater and Swinney [1987] and the Faraday experiment, see Gluckman et al [1993]). These implications are rather surprising, yet necessarily speculative due to the vagueness of the phrase 'sufficiently chaotic'.

Throughout this paper we shall make frequent use of the following result of Chossat and Golubitsky [1998].

Proposition 2. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and commutes with a matrix ρ. Let $A \subset \mathbb{R}^n$ be an attractor for f. If $A \cap \rho A \neq \emptyset$, then $\rho A = A$.*

Remark 2. Proposition 2 was originally proved under a different definition of attractor in Chossat and Golubitsky [1988], Proposition 1.1, and was reproved under the current definition in Melbourne et al [1993], Proposition 4.8.

2. Attractors in fixed-point subspaces

In this section we extend the results of Melbourne et al [1993] and Ashwin and Melbourne [1994] to include attractors that lie in proper fixed-point subspaces.

Proposition 3. *Suppose that $\Gamma \subset \mathbf{O}(n)$ is a finite group and that A is a subset of $\mathbb{R}^n$. Then T_A is an isotropy subgroup of Γ and Σ_A is contained in the normalizer $N(T_A)$ of T_A.*

Proof. Let $V = \mathrm{Fix}(T_A)$. Then it is immediate from the definition of T_A that $A \subset V$. We claim that $V = \mathrm{Fix}(I)$ for some isotropy subgroup $I \subset \Gamma$. Then $T_A \subset I$ but since $A \subset \mathrm{Fix}(I)$ we have also that $I \subset T_A$. It follows that $T_A = I$ is an isotropy subgroup of Γ.

To prove the claim, observe that if $v \in V$ and Σ_v is the isotropy subgroup of v, then $v \subset \mathrm{Fix}(\Sigma_v) \subset \mathrm{Fix}(T_A) = V$. It follows that

$$V = \bigcup_{v \in V} \mathrm{Fix}(\Sigma_v).$$

Since Γ is finite, there are finitely many isotropy subgroups and so $V = \mathrm{Fix}(\Sigma_{v_0})$ for some $v_0 \in V$. The claim follows with $I = \Sigma_{v_0}$.

Finally we show that $\Sigma_A \subset N(T_A)$. Suppose that $t \in T_A$ and $\sigma \in \Sigma_A$. If $x \in A$, $\sigma x \in A$ and hence $t\sigma x = \sigma x$. It follows that $\sigma^{-1} t \sigma x = x$ and $\sigma^{-1} t \sigma \in T_A$ as required. □

Definition 1. A pair of subgroups (Σ, T) of Γ is *(strongly) admissible* if there is a continuous Γ-equivariant map $f : \mathbb{R}^n \to \mathbb{R}^n$ with a (connected) attractor A such that $\Sigma_A = \Sigma$ and $T_A = T$.

Remark 3. Thanks to Proposition 3 we may restrict attention exclusively to pairs (Σ, T) where T is an isotropy subgroup and $\Sigma \subset N(T)$. If $T = \mathbf{1}$ then we recover the notion of (strong) admissibility in Ashwin and Melbourne [1994].

As pointed out in Melbourne et al [1993] the results there extend to give restrictions on the symmetries of attractors in fixed-point subspaces. The argument is standard. It is well-known that $f : \mathbb{R}^n \to \mathbb{R}^n$ restricts to a mapping $g : \mathrm{Fix}(T) \to \mathrm{Fix}(T)$. Moreover, g commutes with the action of the normalizer $N(T)$ on $\mathrm{Fix}(T)$. Let $\Gamma' = N(T)/T$. Then we can regard Γ' as a finite subgroup of $\mathbf{O}(m)$ where $m = \dim \mathrm{Fix}(T)$ and g as a Γ'-equivariant mapping. If A is an attractor for f with $T_A = T$ and $\Sigma_A = \Sigma$ then A is a Σ'-symmetric attractor for g where $\Sigma' = \Sigma/T$. In addition, the restrictions in Melbourne et al [1993] are obtained without using the fact that $T_A = \mathbf{1}$. Hence we have the following result.

Proposition 4. *Suppose that (Σ, T) is a (strongly) admissible pair of subgroups in Γ. Let $\Gamma' = N(T)/T$ and $\Sigma' = \Sigma/T$. Then Σ and Σ' are (strongly) admissible subgroups of Γ and Γ' respectively.*

The conditions in Proposition 4 are not optimal. This is due to the existence of *hidden symmetries*, see Golubitsky et al [1984] or Golubitsky et al [1988]. These are elements $\gamma \in \Gamma$ that do not lie in $N(T)$ (and hence do not preserve $\mathrm{Fix}(T)$), yet have the property that $\gamma\mathrm{Fix}(T) \cap \mathrm{Fix}(T) \neq \{0\}$. A simple example of this phenomenon occurs for the 12 element group $\Gamma = \mathbb{T} \subset \mathbf{O}(3)$. Since Γ contains no reflections, every subgroup Σ is strongly admissible. We consider the pair $(\Sigma, T) = (\mathbb{D}_2, \mathbb{Z}_2)$. The isotropy subgroup $\mathbb{Z}_2$ has a one-dimensional fixed-point subspace and $\mathbb{D}_2$ is the normalizer of $\mathbb{Z}_2$ in $\mathbb{T}$. Hence $\Sigma' = \Gamma'$ $(= \mathbb{Z}_2)$ so that Σ' is a strongly admissible subgroup of Γ'. Thus both conditions in Proposition 4 are satisfied. However a connected attractor A with $\mathbb{D}_2$-symmetry in $\mathrm{Fix}(\mathbb{Z}_2)$ must intersect the origin so that $\gamma A \cap A \neq \emptyset$ for all $\gamma \in \mathbb{T}$. By Proposition 2, $\Sigma_A = \mathbb{T}$. It follows that the pair $(\mathbb{D}_2, \mathbb{Z}_2)$ is not strongly admissible.

Nevertheless, it is possible to state a theorem that is completely analogous to Theorem 1 and which takes account of these hidden symmetries. We begin by refining the terminology used in the introduction.

Define $K_{\Sigma,T}$ to be the set of elements $\tau \in \Gamma - \Sigma$ such that $\mathrm{Fix}(\tau)$ intersects $\mathrm{Fix}(T)$ in a codimension one subspace. (Think of $K_{\Sigma,T}$ as being made up elements in $N(T) - \Sigma$ that act as reflections on $\mathrm{Fix}(T)$ (cf. the definition of K_Σ) together with 'hidden reflections' in $\Gamma - N(T)$.) Then we set $L_{\Sigma,T} = \bigcup_{\tau \in K_{\Sigma,T}} \mathrm{Fix}(\tau)$. If I is a subgroup of Γ, define I_T to be the subgroup generated by elements in $K_{T,T} \cap I$. Finally, we say that a pair (Σ, T) is a cyclic extension of a pair (Δ, T) if Σ is a cyclic extension of Δ.

Theorem 5. *Suppose that Γ is a finite subgroup of $\mathbf{O}(n)$, that T is an isotropy subgroup of Γ with $\dim \mathrm{Fix}(T) \geq 3$, and that Σ is a subgroup of Γ satisfying $T \subset \Sigma \subset N(T)$. Then the following statements are equivalent.*

(a) *The pair (Σ, T) is strongly admissible.*
(b) *Σ fixes a connected component of $\mathrm{Fix}(T) - L_{\Sigma,T}$.*
(c) *There is an isotropy subgroup $I \subset \Gamma$ such that $I_T \subset \Sigma \subset I$.*

A pair (Σ, T) is admissible if and only it is a cyclic extension of a strongly admissible pair.

Proof. Many of the details of the proof are exactly the same as for the corresponding results in Melbourne et al [1993] and Ashwin and Melbourne [1994]. In these cases, we give the references and omit the details.

The proof of equivalence of (b) and (c) is identical to that required to prove equivalence of (b) and (c) in Theorem 1, (changing $\mathbb{R}^n$ to $\mathrm{Fix}(T)$, L_Σ to $L_{\Sigma,T}$ and

I_R to I_T) see Ashwin and Melbourne [1994], Theorem 3.2. Similarly, the fact that condition (a) implies condition (b) and necessity of the condition for admissibility are proved by adapting the arguments used to prove Melbourne et al [1993], Proposition 4.9, and Melbourne et al, Theorem 4.10, respectively. (Alternatively, see the end of this section where these implications are rederived for a more general notion of attractor).

It remains to prove that condition (b) implies condition (a) and that the condition for admissibility is sufficient. Suppose that C is a connected component of $\mathrm{Fix}(T)-L_{\Sigma,T}$ fixed by Σ. Let C' consists of those points in C with isotropy subgroup T. Following the argument (and using the terminology) in Section 6 of Ashwin and Melbourne [1994], an Eulerian, extendable Σ-graph may be embedded in C'. Let A be the embedded graph. Then $\Sigma_A = \Sigma$, $T_A = T$ and $\gamma A \cap A = \emptyset$ for $\gamma \in N(T)-\Sigma$. Moreover, A is topologically transitive under a suitable Σ-equivariant mapping (Ashwin and Melbourne [1994] Theorem 4.3). Provided $\gamma A \cap A = \emptyset$ for all $\gamma \in \Gamma - N(T)$ we may prove as in Ashwin and Melbourne [1994], Theorem 5.4, that (Σ, T) is strongly admissible and that any cyclic extension of (Σ, T) is admissible.

Let $H = \{x \in \mathrm{Fix}(T);\ \gamma x \in \mathrm{Fix}(T) \text{ for some } \gamma \in \Gamma - N(T)\}$. Then H is a finite union of proper subspaces of $\mathrm{Fix}(T)$ and A can be embedded so that $A \cap H$ is finite. A problem arises only if A intersects two points (not necessarily distinct) of H related by an element $\gamma \in \Gamma - \Sigma$. Such a situation can be avoided by changing the embedding slightly unless these points are also related by an element $\sigma \in \Sigma$. But this means that A contains a point x with $\sigma x = \gamma x$, that is $x \in \mathrm{Fix}(\delta)$ where $\delta = \gamma^{-1}\sigma$. But $\delta \notin \Sigma$ contradicting the fact that x has isotropy T. □

There are two further statements of the characterization of strong admissibility in Theorem 5. Define $M_\Sigma = \bigcup_{\gamma \in \Gamma} \mathrm{Fix}(\gamma)$. We have the conditions

(d) Σ fixes a connected component of $\mathrm{Fix}(T) - M_\Sigma$,
(e) There is an isotropy subgroup $I \subset \Gamma$ such that $\Sigma \subset I$ and such that if $J \subset I$ is an isotropy subgroup and $\mathrm{Fix}(J)$ intersects $\mathrm{Fix}(T)$ in a codimension one subspace then $J \subset \Sigma$,

which are clearly equivalent to conditions (b) and (c) respectively. Note that condition (d) is particularly easy to state.

Remark 4. (a) When $T = \mathbf{1}$ we have $\mathrm{Fix}(T) = \mathbb{R}^n$, $L_{\Sigma,T} = L_\Sigma$ and $I_T = I_R$ and recover Theorem 1.
(b) If $K_{T,T} \subset N(T)$ (that is, there are no hidden reflections) then Proposition 4 is optimal. The easiest way to verify this condition is to check that there are no isotropy subgroups J with $\dim \mathrm{Fix}(J) = \dim \mathrm{Fix}(T) - 1$.
(c) As in Ashwin and Melbourne [1994] there is the possibility of additional topological restrictions when $\mathrm{Fix}(T)$ is one- or two-dimensional. Again as in Ashwin and Melbourne [1994], these restrictions are very mild. Theorem 5 fails only when $\dim \mathrm{Fix}(T) = 2$, $\Gamma' = N(T)/T$ is a cyclic subgroup of $\mathbf{O}(2)$, and (Σ, T) is a pair with $\Sigma \neq T$ and $\Sigma \neq N(T)$. Such pairs are admissible (as cyclic extensions of the strongly admissible pair (T, T)) but are not strongly admissible by Proposition 4 and Ashwin and Melbourne [1994], Theorem 7.2(a). This is the case even though conditions (b) and (c) may be valid. Finally, the pair $(N(T), T)$ is sometimes but not always strongly admissible depending on the validity of conditions (b) and (c).
(d) If Σ is an isotropy subgroup of Γ, then (Σ, T) is strongly admissible (with $I = \Sigma$ in condition (c)).

Two situations in which Proposition 4 is optimal are given in Remark 4(a) and (b). We give two further instances in which Proposition 4 is optimal.

Corollary 6. (a) *If Γ is generated by reflections, (Σ, T) is strongly admissible if and only if Σ is an isotropy subgroup (equivalently Σ is strongly admissible).*

(b) *Suppose that* $\dim \mathrm{Fix}(T) = n-1$. *Then (Σ, T) is strongly admissible if and only if Σ is strongly admissible.*

Proof. (a) By Ashwin and Melbourne [1994], Corollary 3.3, the isotropy subgroups of Γ are the same as the strongly admissible subgroups. Now apply Proposition 4 and Theorem 5(c).
(b) Since $\dim \mathrm{Fix}(T) = n-1$, $L_{\Sigma,T} = L_\Sigma$. □

If $T = \mathbf{1}$ we can proceed as in Ashwin and Melbourne [1994]. Also if T is generated by a single reflection, we can apply Corollary 6(b). At the other extreme, when $T = \Gamma$, there is the single pair (Γ, Γ) which is trivially strongly admissible. The next case to consider is when T is a maximal isotropy subgroup of Γ.

Proposition 7. *Suppose that Γ is a finite subgroup of $\mathbf{O}(n)$ and that T is a maximal isotropy subgroup of Γ. Let $d = \dim \mathrm{Fix}(\Gamma) - \dim \mathrm{Fix}(T)$. If $d \geq 2$ any pair (Σ, T) is strongly admissible. If $d = 1$ the only strongly admissible pairs are (Γ, T) (if $N(T) = \Gamma$) and (T, T). Any other pair (Σ, T) is admissible if Σ/T is cyclic and inadmissible if the quotient is noncyclic.*

Proof. The pairs (Σ, T) with $\Sigma = \Gamma$ (if applicable) and $\Sigma = T$ are strongly admissible by Remark 4(d) so we may suppose that $\Sigma \neq \Gamma$ and $\Sigma \neq T$. Since T is a maximal isotropy subgroup, our only choice of isotropy subgroup in condition (c) is $I = \Gamma$. If $d \geq 2$, Γ_T is trivial and condition (c) is satisfied. However, if $d = 1$, then condition (e) fails with $J = \Gamma$. □

Remark 5. (a) If $\dim \mathrm{Fix}(T)$ is odd, then $N(T)/T$ is isomorphic to $\mathbb{Z}_2$ or $\mathbf{1}$. In addition, if $\dim \mathrm{Fix}(T) = 1$ we have $d = 1$ and there are three possibilities:

(i) $N(T) = T$. There is a single pair (T, T) which is strongly admissible.
(ii) $N(T) = \Gamma$. There are two pairs (T, T) and (Γ, T) both of which are strongly admissible.
(iii) $N(T) \neq T, \Gamma$. There is a pair (T, T) which is strongly admissible and a pair $(N(T), T)$ that is nonstrongly admissible.

(b) Part (a) applies to the groups $\mathbb{D}_m \subset \mathbf{O}(2)$ as follows. The only nontrivial isotropy subgroup is $T = \mathbb{D}_1$ with $\dim \mathrm{Fix}(T) = 1$. Possibilities (i), (ii) and (iii) correspond to the case m odd, the case $m = 2$ and the case m even, $m \geq 4$ respectively.

Finally, we address the subtlety concerning the definition of attractor that was mentioned in the introduction. Results of Alexander et al [1992] and Buescu and Stewart [1993] indicate that the definition of attractor as a stable ω-limit set is often inappropriate for attractors in proper invariant subspaces. However, these results suggest that the following definition is appropriate.

Definition 2. Suppose that $T_A = T$. Then A is an *attractor in* $\mathrm{Fix}(T)$ if

(i) A is an ω-limit set,
(ii) A is Liapunov stable for the map f restricted to $\mathrm{Fix}(T)$,

(iii) For any point $y \in A$ there is an open neighborhood V of y (in $\mathbb{R}^n$) such that any smaller neighborhood V' of y has a subset V'' of (Lebesgue) measure greater than half that of V' such that $\omega(z) = A$ for $z \in V''$.

In the remainder of this section we show that Theorem 5 remains valid when we allow this different notion of an attractor in $\mathrm{Fix}(T)$. From now on we take the obvious generalization of the definition of (strong) admissibility of a pair (Σ, T). Of course, it is sufficient to show that condition (b) implies condition (a) and that the necessary condition for admissibility is valid. We begin by generalizing Proposition 2.

Proposition 8. *Let $\Gamma \subset \mathbf{O}(n)$ with isotropy subgroup T and $\rho \in \Gamma$. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and Γ-equivariant. Let $A \subset \mathrm{Fix}(T)$ be an attractor in $\mathrm{Fix}(T)$. If $A \cap \rho A \neq \emptyset$ then $\rho A = A$.*

Proof. First note that by equivariance, ρA is an attractor in $\mathrm{Fix}(\rho T \rho^{-1})$. Suppose that $A \cap \rho A \neq \emptyset$ and let $y \in A \cap \rho A$. Choose neighborhoods V for y respect to A and ρA as in part (iii) of Definition 2 and let V' be the intersection of these neighborhoods. Then the two subsets V'' corresponding to the two attractors must intersect and there is a point z such that $\omega(z) = A$ and $\omega(z) = \rho A$. □

It is now an easy matter to show that condition (b) implies condition (a) in Theorem 5. Suppose that A is a connected attractor in $\mathrm{Fix}(T)$ and that $\Sigma_A = \Sigma$. By Proposition 8, we deduce that $A \cap L_{\Sigma,T} = \emptyset$. Hence the connected set A lies in a connected component of $\mathrm{Fix}(T) - L_{\Sigma,T}$ which is therefore fixed by Σ as required.

Finally, suppose that (Σ, T) is admissible. We follow the proof of Melbourne et al [1993], Theorem 4.10, to show that (Σ, T) is a cyclic extension of a strongly admissible pair. Let A be a Σ-symmetric attractor in $\mathrm{Fix}(T)$. Let H be the union of all fixed-point subspaces not intersected by A:

$$H = \bigcup_{\mathrm{Fix}(\gamma)\cap A=\emptyset} \mathrm{Fix}(\gamma),$$

where $\gamma \in \Gamma$, and set $L = H \cap \mathrm{Fix}(T)$. The Γ-equivariant map f restricts to a map $g : \mathrm{Fix}(T) \to \mathrm{Fix}(T)$. Let $\mathcal{P}_L \subset \mathrm{Fix}(T)$ denote the preimage set

$$\mathcal{P}_L = \bigcup_{n\geq 0} g^{-n}(L).$$

By Definition 2(iii) A is an attractor (stable ω-limit set) for g and we may apply Melbourne et al [1993], Corollary 2.5. Since $A \cap L = \emptyset$, A is covered by finitely many connected components $C_0, \ldots , C_{r-1}$ of $\mathrm{Fix}(T) - \mathcal{P}_L$ and these connected components are permuted cyclically by g. Moreover Σ acts on the connected components and this action commutes with the cyclic action of g.

Define $\Delta_i = \{\sigma \in \Sigma; \sigma C_i = C_i\}$. As in Melbourne et al [1993] we deduce that the Δ_i are all equal (to Δ say), and that Σ is a cyclic extension of Δ. It remains to show that the pair (Δ, T) is strongly admissible, that is, Δ fixes a connected component of $\mathrm{Fix}(T) - L_{\Delta,T}$. This reduces to showing that $L_{\Delta,T} \subset L$ and the argument is identical to that in Melbourne et al [1993] (except that we use Proposition 8 instead of Proposition 2 at the appropriate moment).

3. Attractors for homeomorphisms and diffeomorphisms

Let $\Gamma \subset \mathbf{O}(n)$ be a finite group acting on $\mathbb{R}^n$. Recall that $\tau \in \Gamma$ is a reflection if $\dim \mathrm{Fix}(\tau) = n-1$. Let K denote the set of reflections in Γ and set

$$L = \bigcup_{\tau \in K} \mathrm{Fix}(\tau).$$

Observe that elements of Γ permute the connected components of $\mathbb{R}^n - L$.

Proposition 9. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous one-to-one Γ-equivariant map. Then f^2 fixes each connected component of $\mathbb{R}^n - L$.*

Proof. Let $\tau \in K$. Since f is one-to-one and Γ-equivariant, the subspace $\mathrm{Fix}(\tau)$ is backward as well as forwards invariant under f, and hence f permutes the two connected component of $\mathbb{R}^n - \mathrm{Fix}(\tau)$. Suppose that $H \subset L$ is a reflection hyperplane. Then $\mathbb{R}^n - H$ consists of two connected components permuted by f. Since f is one-to-one, f either fixes each connected component or interchanges them. In each case, the components are fixed by f^2.

Now suppose that C is a connected component of $\mathbb{R}^n - L$. We may write $L = \mathrm{Fix}(\tau_1) \cup \cdots \cup \mathrm{Fix}(\tau_k)$ and hence $C = C_1 \cap \cdots \cap C_k$ where C_j is a connected component of $\mathbb{R}^n - \mathrm{Fix}(\tau_j)$. Then $f^2(C) = f^2(C_1) \cap \cdots \cap f^2(C_k) = C_1 \cap \cdots \cap C_k = C$ as required. □

Theorem 10. *Suppose that A is a Σ-symmetric ω-limit set for a continuous one-to-one Γ-equivariant map $f : \mathbb{R}^n \to \mathbb{R}^n$. Then either*

(i) *Σ contains no reflections and fixes a connected component of $\mathbb{R}^n - L$, or*
(ii) *Σ contains no reflections and has an index two subgroup that fixes a connected component of $\mathbb{R}^n - L$, or*
(iii) *Σ contains one reflection τ and $\Sigma = \Delta \oplus \mathbb{Z}_2$ where Δ fixes a connected component of $\mathbb{R}^n - L$ and $\mathbb{Z}_2$ is the subgroup generated by τ.*

In particular, if Σ is an admissible subgroup, then one of conditions (i)–(iii) is valid. Moreover, if Σ is strongly admissible, then either condition (i) or condition (iii) is valid.

Proof. By Proposition 9, the ω-limit set A intersects at most two connected components of $\mathbb{R}^n - L$. Let C be a component intersected by A. Since each reflection $\tau \in \Sigma$ maps C into distinct components τC, it follows that Σ contains at most one reflection and that if A contains a reflection then A intersects precisely two connected components.

If A is contained in a single connected component C of $\mathbb{R}^n - L$, then Σ fixes A and hence C so that condition (i) is satisfied. Otherwise, A intersects two connected components C_1, C_2 of $\mathbb{R}^n - L$. Define $\Delta_i = \{\sigma \in \Sigma; \sigma C_i = C_i\}$. Since f is equivariant and interchanges C_1 and C_2, $\Delta_1 = \Delta_2$.

Set $\Delta = \Delta_1 = \Delta_2$. Then Δ is the kernel of the action of Σ on $\{C_1, C_2\}$ and hence is a normal subgroup. Moreover Σ/Δ acts fixed-point freely on $\{C_1, C_2\}$ so that Δ is of index one or index two in Σ.

If Σ contains no reflections we are in one of the situations described in (i) and (ii). We show that if Σ contains a reflection τ then condition (iii) is valid. Indeed Δ and the subgroup of Σ generated by reflections are normal subgroups of Σ and we obtain the required direct sum decomposition for Σ.

Finally, we rule out condition (ii) when Σ is strongly admissible. If a connected attractor A intersects two connected components of $\mathbb{R}^n - L$ then $A \cap L \neq \emptyset$. Hence $A \cap \text{Fix}(\tau) \neq \emptyset$ for some reflection $\tau \in \Sigma$. By Proposition 2 Σ contains the reflection τ. □

To apply Theorem 10 it is important to have a characterization of those subgroups that fix a connected component of $\mathbb{R}^n - L$. The required characterization is obtained by an argument identical to that in Ashwin and Melbourne [1994], Theorem 3.2.

Proposition 11. *A subgroup $\Sigma \subset \Gamma$ fixes a connected component of $\mathbb{R}^n - L$ if and only if there is an isotropy subgroup $I \subset \Gamma$ such that $\Sigma \subset I$ and I contains no reflections.*

Theorem 10 gives necessary conditions for (strong) admissibility. It is natural to ask whether they are also sufficient, and also whether they depend further on the kind of dynamical system being studied (invertible and/or differentiable mapping, ODE or PDE). We conjecture that (at least in high enough dimensions) there is the following very positive answer to these questions.

Conjecture 1. Suppose that Γ is a finite subgroup of $\mathbf{O}(n)$ with subgroup Σ and that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous one-to-one Γ-equivariant map. Provided n is large enough, the conditions in Theorem 10 for (strong) admissibility of Σ are sufficient as well as necessary. Moreover, these conditions remain necessary and sufficient even if f is a diffeomorphism (possibly for n even larger).

Remark 6. (a) For flows there are further restrictions since attractors are connected and lie in a single connected component of $\mathbb{R}^n - L$. We conjecture therefore that Σ is admissible (and hence strongly admissible) for a flow if and only if Σ contains no reflections and fixes a connected component of $\mathbb{R}^n - L$. On the other hand, at the level of PDEs we do not expect any reflections (there should not be any codimension one invariant subspaces in the infinite-dimensional space of solutions) and so all subgroups of Γ are (strongly) admissible.
(b) Pete Ashwin and Mike Field have pointed out that it should be possible to prove the conjecture (for $n \geq 4$) by adapting the methods in Williams [1967]. This is currently being pursued in Field, Melbourne and Nicol [1993]. Note that this approach should lead to structurally stable (indeed Axiom A) attractors with the required symmetry group.

4. Continuous groups

In this section, we consider ω-limit sets for mappings of $\mathbb{R}^n$ that are equivariant with respect to a continuous group of symmetries $\Gamma \subset \mathbf{O}(n)$. Our results in this section are valid for all classes of mappings (from continuous mappings to diffeomorphisms) and even for flows. Throughout, we restrict attention to ω-limit sets that contain points with trivial isotropy. There are obvious generalizations to ω-limit sets that lie in fixed-point subspaces.

Denote by Γ^0 the connected component of the identity in Γ. We begin by considering the case when Γ^0 is abelian.

Theorem 12. *Suppose that $\Gamma \subset \mathbf{O}(n)$ is a connected abelian group (a torus). Then ω-limit sets for Γ-equivariant dynamical systems in $\mathbb{R}^n$ typically are fully symmetric.*

Corollary 13. *Suppose that $\Gamma \subset \mathbf{O}(n)$ and that Γ^0 is abelian. Let $\omega(x)$ be an ω-limit set in $\mathbb{R}^n$ and suppose that*
(i) $\Sigma_{\omega(x)} \subset \Gamma^0$,
(ii) $\gamma\omega(x) \cap \omega(x) = \emptyset$ *for* $\gamma \in \Gamma - \Gamma^0$.
Then typically $\Sigma_{\omega(x)} = \Gamma^0$.

Remark 7. (a) Theorem 12 and its corollary will be made precise below. In particular, typicality can be precisely interpreted in both a topological and a measure-theoretic sense.
(b) Assumption (ii) in Corollary 13 is redundant if $\omega(x)$ is (Liapunov) stable or even orbitally stable ($\omega(x)$ is orbitally stable if $\Gamma\omega(x)$ is stable).

To prove Theorem 12 we need a technical lemma.

Lemma 14. *Suppose that n_k is a strictly increasing sequence of positive integers and that $g_k : T^p \to T^p$ is a mapping of the p-torus defined by $g_k(\theta_1, \dots, \theta_p) = (n_k\theta_1, \dots, n_k\theta_p)$. For $x \in T^p$, define*

$$O(x) = \{g_k(x);\ k = 1, 2, \dots\}.$$

Then $\overline{O(x)} = T^p$ for a residual full (Haar) measure subset of points $x \in T^p$.

Proof. Passing to a subsequence, we may suppose that $n_k \geq 2^k$. Consider the case when $n_k = 2^k$. Then $g_k(x) = g^k(x)$ where g is the expanding map that doubles angles. It is well-known that T^p is topologically transitive (even mixing) under such a map g: a residual set of points in T^p have dense orbits in T^p under iteration by g. Moreover, this subset is of full measure. It is clear that the arguments used to prove these statements will still go through if $n_k \geq 2^k$. □

Proof of Theorem 12. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Γ-equivariant mapping and that $x_0 \in \mathbb{R}^n$. We prove that typically $\omega(x_0)$ is a Γ-symmetric set. The result for flows is proved similarly.

Let $X = \bigcup_{\gamma\in\Gamma} \gamma\omega(x_0)$ and let U be a Γ-invariant neighborhood of X. Choose a smaller neighborhood V with $\overline{V} \subset U$, and for $\rho \in \Gamma$ define a smooth Γ-invariant map

$$g_\rho : \mathbb{R}^n \to \Gamma,$$

satisfying $g_\rho(x) = \rho$ for $x \in V$ and $g_\rho(x) = 1$ for $x \in \mathbb{R}^n - U$.

We consider perturbations of f of the form $f_\rho = g_\rho \cdot f$. It is easy to check that f_ρ is invertible if f is invertible. Let $\omega_\rho(x_0)$ denote the ω-limit set of x_0 under f_ρ. Clearly $\omega_\rho(x_0) \subset X$. We construct a residual full measure subset $R \subset \Gamma$ such that $\omega_\rho(x_0) = X$ for $\rho \in R$, thus proving the theorem.

Choose a dense sequence $z_1, z_2, \dots$ in $\omega(x_0)$. Fix $r \geq 1$, and note that since $z_r \in \omega(x_0)$, there is an increasing sequence n_k such that $f^{n_k}(x_0) \to z_r$. We can assume that $f^{n_k}(x_0) \in V$ for all k. Hence $f_\rho^{n_k}(x_0) = \rho^{n_k} f^{n_k}(x_0)$. By the lemma, there is a full measure residual set $R_r \subset \Gamma$ such that the sequence $\{\rho^{n_k}\}$ is dense in Γ for $\rho \in R_r$.

We claim that for $\rho \in R_r$, $\Gamma z_r \subset \omega_\rho(x_0)$. Now set $R = \cap_{r\geq 1} R_r$ and observe that for any $\rho \in R$,

$$\bigcup_{r\geq 1} \Gamma z_r \subset \omega_\rho(x_0) \subset X.$$

The left-hand-side of this inequality is dense in X and since $\omega_\rho(x_0)$ is closed we have $\omega_\rho(x_0) = X$ as required.

It remains to prove the claim. Let $\gamma \in \Gamma$ and $z_r \in \omega(x_0)$. We have $f^{n_k}(x_0) \to z_r$. Since $\rho \in R_r$, we can pass to a subsequence so that

$$f^{n_k}(x_0) \to z_r, \quad \rho^{n_k} \to \gamma.$$

Using the fact that $\Gamma \subset \mathbf{O}(n)$, we compute that $f_\rho^{n_k}(x_0) \to \gamma z_r$:

$$\begin{aligned} |f_\rho^{n_k}(x_0) - \gamma z_r| &= |\rho^{n_k} f^{n_k}(x_0) - \gamma z_r| \\ &\leq |\rho^{n_k}(f^{n_k}(x_0) - z_r)| + |(\rho^{n_k} - \gamma) z_r| \\ &\leq |f^{n_k}(x_0) - z_r| + \|\rho^{n_k} - \gamma\| |z_r| \\ &\to 0. \end{aligned}$$

□

It is a nontrivial problem to weaken the hypotheses in Theorem 12 and Corollary 13. Some progress has been made by Ashwin and Chossat [1992] and Ashwin and Stewart [1993]. This work shows that results for 'sufficiently chaotic' ω-limit sets are significantly different from known results for relative periodic orbits. We generalize this work in Theorem 15 below.

A set X is a *relative periodic orbit* if it is dynamically-invariant and consists of finitely many group orbits permuted cyclically by the dynamics. Equivalently, on passing to the orbit space X collapses to a periodic orbit. We denote by P a (dynamic) trajectory in X and by abuse of notation call P the relative periodic orbit. We shall say that P is *asymmetric* if $\Sigma_P \subset \Gamma^0$.

It follows from results of Field [1980] and Krupa [1990] that if P is an asymmetric relative periodic orbit then typically Σ_P is a maximal torus in Γ^0. Of course, if Γ^0 is abelian this is a special case of Corollary 13. The situation is more subtle for relative periodic orbits that are not asymmetric, see Field [1991]. The simplest example is for $\Gamma = \mathbf{O}(2)$ (acting on $\mathbb{R}^n$). If P is not asymmetric then $\Sigma_P = \mathbb{D}_1$ (Field [1991]). In particular, Σ_P does not contain any continuous symmetries. Contrast that with the following generalization of a result in Ashwin and Stewart [1993].

Theorem 15. *Suppose that $\Gamma \subset \mathbf{O}(n)$. If $\omega(x)$ is not a relative periodic orbit and asymmetric relative periodic orbits are dense in $\omega(x)$ then typically $\Gamma^0 \subset \Sigma_{\omega(x)}$.*

Proof. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Γ-equivariant map with ω-limit set $\omega(x)$. Set $W^0 = \bigcup_{\gamma \in \Gamma^0} \gamma\omega(x)$ and $W = \bigcup_{\gamma \in \Gamma} \gamma W^0$. We consider perturbations of f of the form $\tilde{f} = g \cdot f$ where $g : \mathbb{R}^n \to \Gamma$ is a smooth Γ-invariant map supported in a neighborhood of W. Let $\tilde{\omega}(x)$ denote the ω-limit set of x under $\tilde{f}$. Clearly $\tilde{\omega}(x) \subset W$, indeed for small perturbations $\tilde{\omega}(x) \subset W^0$. We show that typically $\tilde{\omega}(x) = W^0$.

Suppose $z \in \omega(x)$. We show that typically $\Gamma^0 z \subset \tilde{\omega}(x)$. The theorem follows since we can take a countable dense subset $\{z_r\}$ of $\omega(x)$ as in the proof of Theorem 12.

It is sufficient to prove that $\gamma_1 z, \ldots, \gamma_k z \in \tilde{\omega}(x)$ for generators $\gamma_1, \ldots, \gamma_k$ of Γ^0 (in the sense that the elements generate a subgroup that is dense in Γ^0). As pointed out in Ashwin and Stewart [1993] there is always a finite number k of maximal tori that together generate Γ^0 and we can take the elements γ_i to be generators of such tori. Note that given this number k, almost any choice of elements $\gamma_1, \ldots, \gamma_k$ will produce a set of generators for Γ^0.

Let p_j, $j \geq 1$ be a sequence of asymmetric relative periodic points in $\omega(x)$ converging to z. Make an initial perturbation so that for each j, $M_j p_j \subset \tilde{\omega}(x)$

where M_j is a maximal torus in Γ^0. Choose a sequence $m_j \in M_j$, $j \geq 1$. Then $m_j p_j$ is a sequence of points in $\tilde{\omega}(x)$ and a convergent subsequence yields a point $\gamma_1 z \subset \tilde{\omega}(x)$. The sequence m_j can be chosen so that γ_1 generates a maximal torus in Γ^0.

The next step is to consider the sequence p_{2j}. Make a perturbation supported in a Γ-invariant neighborhood of these points such that the neighborhood does not include the points p_{2j+1}. In this way we obtain a new point $\gamma_2 z \in \tilde{\omega}(x)$ while preserving the point $\gamma_1 z \in \tilde{\omega}(x)$. Typically, we can arrange that γ_1 and γ_2 generate distinct maximal tori in Γ^0. After k steps we obtain the required set of generators $\gamma_1, \cdots, \gamma_k$. □

We note that many of the results in this section become much simpler if we restrict to stable ω-limit sets. However, it is more reasonable a priori to assume only orbital stability, an assumption that does not appear to help in the proofs.

5. Implications for theory and applications

In this section we consider some (speculative) implications of our results in this paper, especially those in Section 4.

We begin by making a conjecture based on Theorem 15. Results of Field [1991] impose restrictions on the continuous symmetries of relative periodic orbits that have discrete symmetries. Let $\Pi : \Gamma \to \Gamma^0$ be the projection onto the component of the identity and suppose that S is a cyclic subgroup of Γ/Γ^0. An abelian subgroup $K \subset \Gamma$ is *of type S* if $K \cong T^r \times S$ (where T^r denotes the r-torus) and $\Pi(K) = S$. In Field [1991] it is shown that all maximal abelian subgroups of type S are conjugate. Define $\mathrm{rk}(\Gamma, S) = r$ where $T^r \times S$ is maximal. Then we have the following result.

Theorem 16 (Field [1991]). *Suppose that P is a relative periodic orbit containing points with trivial isotropy and that $\Pi(\Sigma_P) = S \subset \Gamma/\Gamma^0$. Then $\Sigma_P \cong T^r \times S$ where $r \leq \mathrm{rk}(\Gamma, S)$. Moreover, typically $r = \mathrm{rk}(\Gamma, S)$.*

We now state our conjecture which says roughly that ω-limit sets full into two classes: 'regular' and 'sufficiently chaotic', and satisfy the conclusions of Theorem 16 and Theorem 15 respectively.

Conjecture 2. Suppose that A is an ω-limit set containing points with trivial isotropy and possessing discrete symmetries $S = \Pi(\Sigma_A) \subset \Gamma/\Gamma^0$. Typically, either

(a) $\Sigma_A \cong T^r \times S$ where $r = \mathrm{rk}(\Gamma, S)$, or
(b) $\Gamma^0 \subset \Sigma_A$.

To make the connection between theory and applications, it is convenient to call T_A the 'instantaneous symmetry' and Σ_A the 'symmetry on average'. As pointed out in Dellnitz, Golubitsky and Melbourne [1992] and Melbourne, Dellnitz and Golubitsky [1993], the instantaneous symmetry corresponds to symmetry that is present at any instant in time whereas the symmetry on average includes also those symmetries that are observed only in the time average. In this terminology, Conjecture 2 suggests that in the absence of nontrivial instantaneous symmetry, the symmetry on average contains at least the continuous symmetries unless the dynamics is sufficiently regular.

Even for a group as simple as $\Gamma = \mathbf{O}(2)$ (acting on $\mathbb{R}^n$) we see that the situation is surprisingly complicated. The subgroups of $\mathbf{O}(2)$ are $\mathbb{Z}_k$, $\mathbb{D}_k$, $k \geq 1$, $\mathbf{SO}(2)$

and $\mathbf{O}(2)$. It follows from Corollary 13 that we can (typically) rule out attractors with $\mathbb{Z}_k$-symmetry. Conjecture 2 goes further and rules out attractors with $\mathbb{D}_k$-symmetry for $k \geq 2$. In fact, suppose that A is an ω-limit set for an $\mathbf{O}(2)$-equivariant dynamical system and assume that A contains points with trivial isotropy. In the terminology of Conjecture 2 we have $S = \mathbf{1}$ or $S = \mathbb{D}_1$. Accordingly we have corresponding to parts (a) and (b) of the conjecture:

(a) $\Sigma_A = \mathbf{SO}(2)$ or $\Sigma_A = \mathbb{D}_1$,
(b) $\Sigma_A = \mathbf{SO}(2)$ or $\Sigma_A = \mathbf{O}(2)$.

Our first implication is theoretical in nature. Barany et al [1993] (see also Dellnitz et al [1993]) investigate the problem of numerically detecting the symmetry on average of an attractor. In particular, a satisfying solution leading to an effective numerical algorithm is given at least for finite groups Γ. At first sight, generalizing this to arbitrary compact Lie groups is a somewhat tedious task. However, Conjecture 2 suggests that for sufficiently chaotic attractors the continuous symmetries are automatically present on average so that there is only the finite set Γ/Γ^0 to worry about. A straightforward adaptation of the methods in Barany et al [1993] is then possible. However, there is still the problem of detecting the 'instantaneous symmetries' (an a priori simpler problem).

Next we consider implications for applied problems. Gluckman et al [1993] take time averages of chaotic surface waves in the Faraday experiment in a circular domain. The time averages that they obtain have the full circular symmetry of the domain. Now these time averages cannot distinguish between $\mathbf{SO}(2)$ and $\mathbf{O}(2)$ symmetry. (A circular pattern in the plane automatically has reflection symmetry. In the terminology of Barany et al [1993], the observation that is being averaged is not a detective.) So the implication to be drawn from Gluckman et al [1993] is that the solutions have $\mathbf{SO}(2)$ or $\mathbf{O}(2)$ symmetry on average. This is precisely the prediction of Conjecture 2 for such chaotic solutions.

The point that we wish to make here is that the time average in Gluckman et al [1993] must (typically) be as is observed. This is in contrast to the experiments in square domains that are also considered in Gluckman et al [1993]. Here the time average is observed to have $\mathbb{D}_4$ symmetry but the results in Ashwin and Melbourne [1994] (more precisely, Conjecture 1) imply that the time average could in principle be any subgroup of $\mathbb{D}_4$.

Our final and most surprising application is to turbulent Taylor vortices. This is a chaotic but patterned flow that occurs in observations in the Taylor-Couette experiment. The full symmetry group of the problem is $\Gamma = \mathbf{SO}(2) \times \mathbf{O}(2)$. The present belief among physicists and mathematicians appears to be that this is a chaotic (perhaps turbulent) solution that has no symmetry at any instant in time and yet on average has the symmetry of regular Taylor vortices (circular symmetry in the azimuthal direction and a reflection symmetry in the axial direction). We have $T_A = \mathbf{1}$ and $\Sigma_A = \mathbf{SO}(2) \times \mathbb{D}_1$. This is in disagreement with our conjecture which predicts that if $T_A = \mathbf{1}$ then either $\Sigma_A = \mathbf{SO}(2) \times \mathbf{SO}(2)$ or $\Sigma_A = \mathbf{SO}(2) \times \mathbf{O}(2)$. We believe that the solution possesses nontrivial instantaneous symmetry (that is $T_A \neq \mathbf{1}$). Indeed, if one looks at turbulent Taylor vortices there is clearly some structure present. The structure is not exact, but exact enough that one 'sees' it. It seems worthwhile performing tests to determine whether the structure of Taylor vortices is present in the chaotic flow instantaneously or only on average.

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