A HETEROCLINIC MODEL OF GEODYNAMO REVERSALS AND EXCURSIONS

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Abstract. The Earth's magnetic field is by and large a steady dipole, but its history has been punctuated by intermittent excursions and reversals. This is at least superficially similar to the behaviour of differential equations containing structurally stable heteroclinic cycles. We present a model of the geodynamo that is based on the symmetries of velocity fields in a rotating spherical shell, and that contains such a cycle. Patterns of excursions and reversals that resemble the geomagnetic record can be obtained by introducing small symmetry-breaking terms.

1. Introduction

It has long been known that the Earth's magnetic field is due to dynamo action in its liquid core. An important part of the observational evidence for a self-excited field is the presence in palaeomagnetic data of reversals of the main axial dipolar part of the field. The timescale of these reversals (10^5y–10^6y) is very long compared with the natural diffusion time scale of the core, which is about $15\,000y$. There is an increasing amount of palaeomagnetic data on reversals, and it is clear that the distribution of reversal intervals is highly non-uniform – suggesting some sort of intermittent process [1]. More recently it has been discovered that between each reversal epoch there are many oscillations of the dipole direction and orientation which do not result in full reversals [2]; these have been termed *excursions*.

It is only recently that detailed numerical simulations of the geodynamo have been able to show evidence of both these phenomena (see [3, 4, 5] and references therein). The time dependence of the fields in these experiments is spatially complex, and the difficulty of the calculation means that only a few reversals have been observed. It seems natural then to sacrifice spatial structure in favour of a model that can be integrated over many reversals. This has been done many times in the past, starting with the work of Bullard [6] and Rikitake [7], and continuing with many others [8, 9, 10, 11, 12]. A common feature is that the dynamo is modelled by low-order sets of ODEs that show oscillations and reversals of the field, but do not exhibit separation of timescales between quasi-steady polarity configurations and rapid reversal events. The equations are derived as highly truncated low-order models, or as descriptions of mechanical circuits very different from the Earth's core. Recent efforts [13, 14] to capture the separation of timescales rely on heteroclinic cycles in the underlying velocity field, which occur for convection in slowly rotating spheres. However, the Earth's rotation period is much less than the timescale for convection in the core.

Here we consider the more realistic situation of convection in a rotating sphere without restricting to slow rotation rates. We suppose that buoyancy forces drive a fully nonlinear flow, and focus on secondary dynamo instabilities. We use the symmetry of the velocity to write down a set of normal form equations as a model of reversals. Our model has structurally stable heteroclinic cycles, and when the symmetries are weakly broken, we can identify the residence time near these cycles with the excursion time. The reversal period in the model is many multiples of the excursion time, and appears to depend sensitively on details of the system.

2. Symmetries of the geodynamo

Our supposition is that the velocity field that leads to dynamo action has symmetries that can be used to distinguish different magnetic field patterns. Analytical [15] and numerical [5] convection studies yield 'cartridge belt' roll configurations with pairs of columnar cells aligned with the rotation axis, having definite rotation and reflection symmetries. There is some observational evidence (see [16]) that the actual velocity field $\mathbf{v}$ has this structure, though the number of rolls is not well-defined by the data. We look at the case of three pairs of rolls, though any number of pairs could be considered and would yield similar results. Then $\mathbf{v}$ is invariant under equatorial reflections (κ) and 120° rotations (ρ) about the Earth's axis. The growth of the magnetic field is described by the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}\,, \tag{1}$$

which clearly has the symmetry $\mathcal{B}: \ \mathbf{B} \leftrightarrow -\mathbf{B}$. Because of the symmetries of $\mathbf{v}$, solutions to (1) can be of the following symmetry types:

		ρ	κ
Axial Dipole (x_3)	D_a	$+1$	-1
Axial Quadrupole (x_2)	Q_a	$+1$	$+1$
Equatorial Dipole (z_1)	D_e	$\mathrm{e}^{\frac{2i\pi}{3}}$	$+1$
Equatorial Quadrupole	Q_e	$\mathrm{e}^{\frac{2i\pi}{3}}$	-1

The axial modes retain the 120° rotation symmetry of the underlying velocity field, while the equatorial modes break this symmetry.

One can imagine taking a flow with the symmetries above and changing one physical parameter (e.g. the magnetic diffusivity η) to investigate (codimension 1) bifurcations to dynamo action. Because of the symmetry $\mathcal{B}$ such bifurcations are either pitchfork or Hopf bifurcations (in fact the latter is generic for the equatorial modes with an odd number of pairs of rolls). It turns out, however, that for some dynamo models (based on Kumar–Roberts flows [17]), there can be a near degeneracy between the critical parameter for three modes, namely D_a, D_e and Q_a [18]. While there can be no rigorous proof of this (though see e.g. [19]), we shall make use of it as a motivation for investigating the effects of a codimension 3 bifurcation involving modes with these three symmetries. Three modes with different symmetries are required for a heteroclinic cycle.

We represent the magnetic field by the ansatz

$$\mathbf{B}(\mathbf{r},t) = z_1(t)\mathbf{D}_e(\mathbf{r}) + x_2(t)\mathbf{Q}_a(\mathbf{r}) + x_3(t)\mathbf{D}_a(\mathbf{r})\,. \tag{2}$$

Note that the D_e mode is oscillatory. The action of the symmetries ρ and κ on z_1, x_2 and x_3 are in the table above. Supposing that the bifurcations are all supercritical, we have the following truncated normal form:

$$\dot{z}_1 = (\mu_1 + i\omega_1)z_1 + z_1(-|z_1|^2 + A_{12}x_2^2 + A_{13}x_3^2)\,, \tag{3}$$
$$\dot{x}_2 = \mu_2 x_2 + x_2(A_{21}x_1^2 - x_2^2 + A_{23}x_3^2)\,, \tag{4}$$
$$\dot{x}_3 = \mu_3 x_3 + x_3(A_{31}x_1^2 + A_{32}x_2^2 - x_3^2)\,. \tag{5}$$

These equations are nonlinear, in spite of the linearity of (1), because of the dynamical effects of the magnetic field on the flow. We note that at this stage the phase of z_1 decouples from the other variables (normal form symmetry), and so the dynamics may be described in terms of $x_1 = |z_1|$. It is easy to find conditions for a cycle of the standard type between three axial equilibria. For example, to achieve a connection between the equilibria $D_e = (\sqrt{\mu_1}, 0, 0) \to Q_a = (0, \sqrt{\mu_2}, 0)$, we need $\mu_2 + A_{21}\mu_1 > 0$, $\mu_3 + A_{31}\mu_1 < 0$. Similar connections can be found between the other pairs of equilibria under similar conditions. We can ensure that the cycle is attracting if the product of the moduli of the contracting eigenvalues at the fixed points is greater than the product of the corresponding expanding eigenvalues [20]. There is little difficulty in meeting all these conditions.

We have shown that with proper choices of the coefficients A_{ij} the dynamics exhibits a structurally stable heteroclinic cycle, which takes the form of long period fluctuations in the amplitudes; these can perhaps be identified with excursions. Nonetheless, the model is unsatisfactory, not only because there are no reversals, but because an attracting cycle is characterised by ever increasing intervals between excursion-type events. Thus our definitive model consists of a refinement of the simple system (3–5).

3. A model for reversals

In order to obtain a model with the required properties we make the following changes: (a) we make the z_1 periodic orbit non-circular. This corresponds to breaking the normal form symmetry of the oscillatory D_e mode – this is entirely natural since there are higher order harmonics generated naturally in the nonlinear regime. The effect of this is that the secular increase in transit times produced by the simpler system is replaced by chaotic/intermittent cycling behaviour [21]; (b) we weakly break the ρ and κ symmetries; this is also natural in the context of the Earth as there are lateral inhomogeneities in mantle heat fluxes, topography etc. This has the effect of allowing reversals of the main D_a mode, on a timescale that differs from that for the excursions. The symmetry breaking terms in the model below are proportional to ϵ_1 (normal form symmetry), ϵ_2 (normal form and ρ), and ϵ_3 (κ). The $\mathbf{B} \to -\mathbf{B}$ symmetry is of course retained. After some redefinition to bring out the role of the expanding and contracting eigenvalues near the fixed points, we obtain our model system

$$\begin{aligned}
\dot{z}_1 &= (\mu_1 + \mathrm{i}\omega) z_1 - |z_1|^2 z_1 - \frac{c_2+\mu_1}{\mu_2} x_2^2 z_1 + \frac{e_3-\mu_1}{\mu_3} x_3^2 z_1 \\
&\quad + \epsilon_1 \bar{z}_1^5 + \epsilon_2 |z_1|^4 x_2 + \epsilon_3 z_1 x_2 x_3^3 , \qquad (6)\\
\dot{x}_2 &= \mu_2 x_2 - x_2^3 - \frac{c_3+\mu_2}{\mu_3} x_3^2 x_2 + \frac{e_1-\mu_2}{\mu_1} |z_1|^2 x_2 \\
&\quad + \epsilon_1 \,\mathrm{Re}(z_1^3)\, x_2^2 + \epsilon_2 x_1^5 + \epsilon_3 x_3^5 , \qquad (7)\\
\dot{x}_3 &= \mu_3 x_3 - x_3^3 - \frac{c_1+\mu_3}{\mu_1} |z_1|^2 x_3 + \frac{e_2-\mu_3}{\mu_2} x_2^2 x_3 \\
&\quad + \epsilon_1 \,\mathrm{Re}(z_1^3)\, x_2 x_3 + \epsilon_2 x_1^3 x_2 x_3 + \epsilon_3 x_2^5 , \qquad (8)
\end{aligned}$$

where $z_1 = x_1 + \mathrm{i}y_1$, μ_1, μ_2, μ_3 and ω are growth rate parameters and a frequency, c_1, c_2 and c_3 are the contracting eigenvalues at the three fixed points of the cycle and e_1, e_2 and e_3 are the expanding eigenvalues.

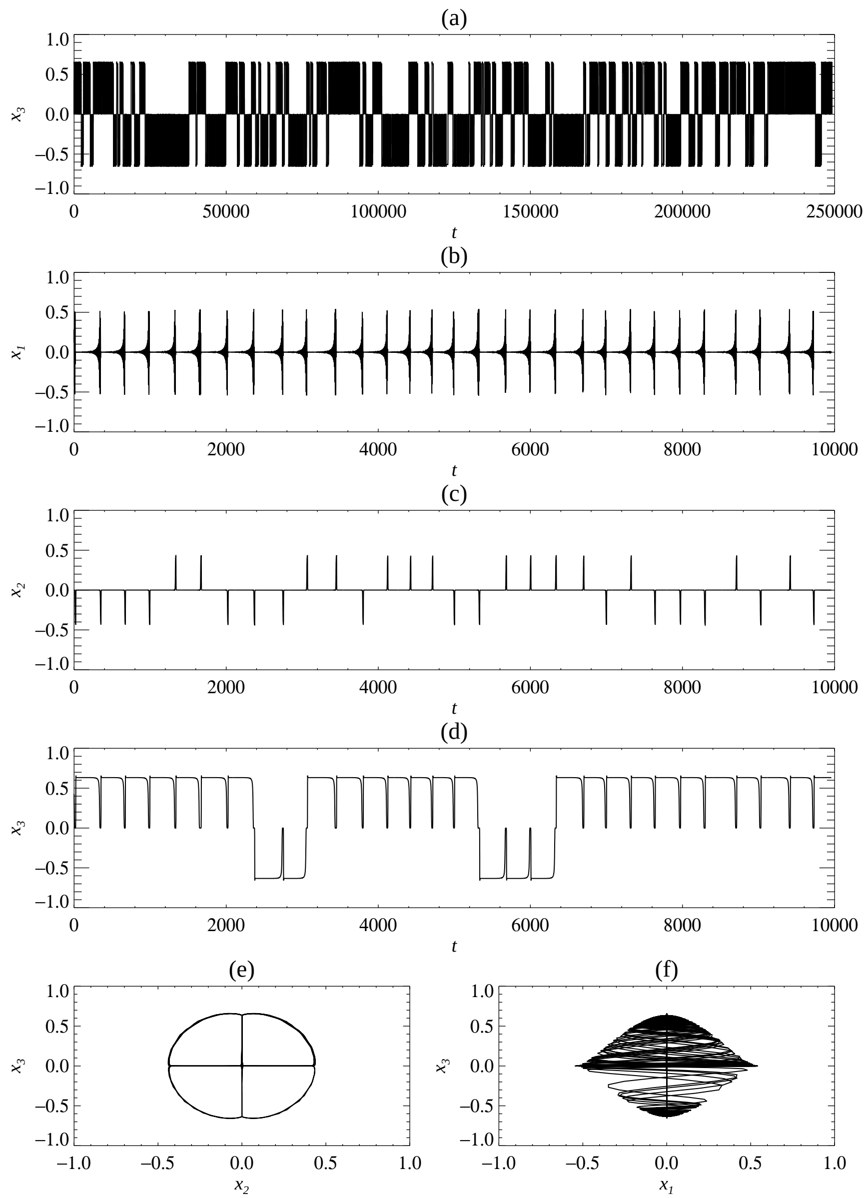

Figure 1. Behaviour of (6–8) for the parameters given in the text. (a) shows a long time series for the dipole mode x_3, and (b–d) show the x_1, x_2 and x_3 evolution in the first part of the time series. (e–f) show phase portraits: (e) x_3 vs. x_2 (f) x_3 vs. x_1.

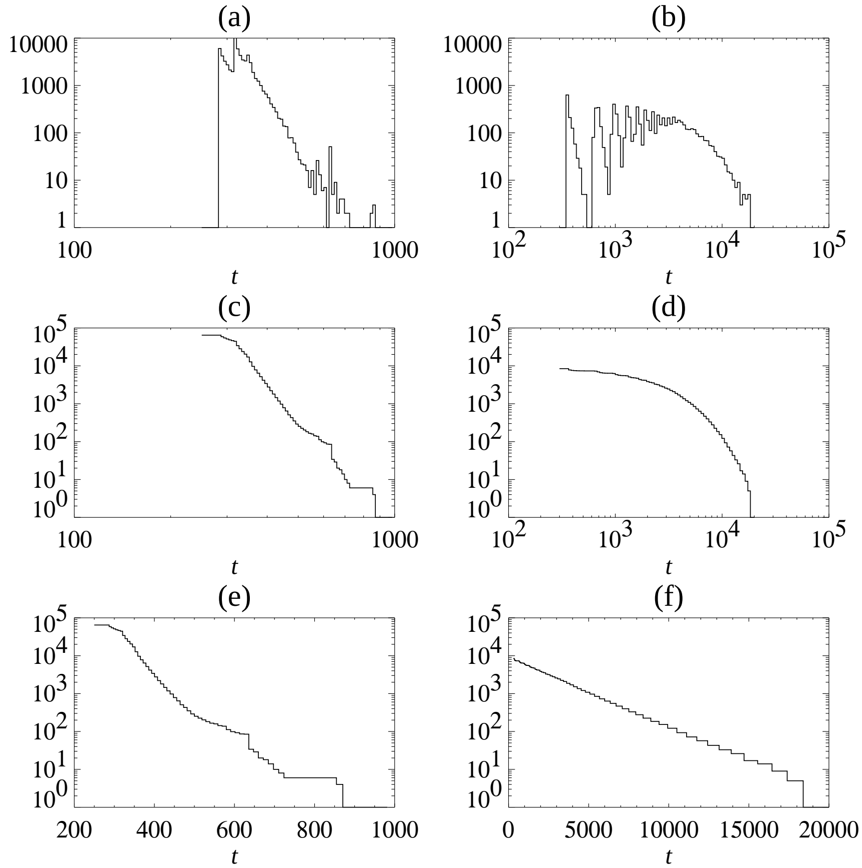

Figure 2. Distributions of durations of (a,c,e) excursions and (b,d,f) reversals. (a,b) histograms. (c,d) cumulative plot (log–log). (e,f) cumulative plot (log–linear).

In this preliminary report we focus on results for just one set of parameter values, selected to produce reversal-like behaviour. These are:

$$\begin{array}{lll}
\lambda_1 = 0.3 \ (\omega_1 = 1) & e_1 = 0.7 & c_1 = 1.1 \\
\lambda_2 = 0.2 & e_2 = 1.7 & c_2 = 1.2 \\
\lambda_3 = 0.4 & e_3 = 0.02 & c_3 = 1.3 \\
\epsilon_1 = 0.12 & \epsilon_2 = 0.1 & \epsilon_3 = 0.001
\end{array}$$

Figure 1(a) shows a typical long time series (details shown in figure 1b–d). The x_3 variable, measuring the dipole strength, vacillates many times without changing sign, punctuated by much rarer reversal events. The phase portrait in figure 1(e) shows that the dynamics is close to a heteroclinic cycle, while figure 1(f) shows the oscillatory nature of the x_1 variable. The probability distribution of reversals and excursions is in figure 2. While the excursion data show an unrepresentative peak at about 300 (arbitrary) time units, the reversal data is smoother and the cumulative picture is similar (though without 'superchron' outliers) to the real dataset (figure 3).

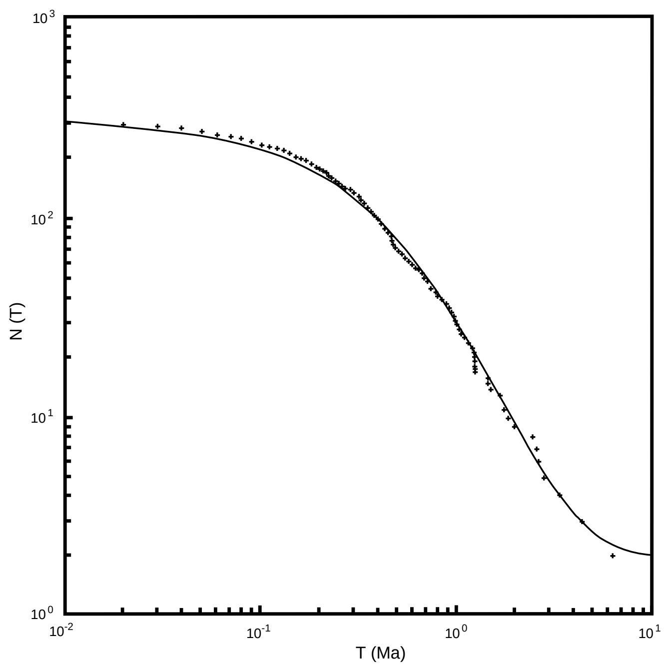

Figure 3. Distribution of reversal durations: cumulative plot (log–log) for the geomagnetic field (reproduced from [1]). Compare with figure 2(d).

4. Discussion

In this paper we have shown some results from a normal form model based on the (approximate) symmetries of the full geodynamo problem. This model captures certain intermittent phenomena associated with the geodynamo in the form of excursions and field-reversals. Our model requires magnetic fields of three different symmetry types in order to have a heteroclinic cycle, and it differs from earlier low order models of reversals in that the symmetries are paramount in obtaining the form of the model equations. Of course, as for other low-order calculations, the parameters cannot be related to Earth-like quantities at this stage. That said, transitions between magnetic fields with different types of symmetry are a prominent feature of large-scale geodynamo calculations [3, 4]. In addition, our model suggests that weakly broken symmetries may play an important role in determining the ratio between excursion and reversal timescales. More detailed investigations of the model will be reported elsewhere [22].

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References

1. Merrill, R.L., McElhinny, M.W. & McFadden, P.L. (1996) *The Magnetic Field of the Earth: Paleomagnetism, the Core, and the Deep Mantle.* Academic Press: San Diego.
2. Gubbins, D. (1999) The distinction between excursions and reversals, *Geophys. J. Int.* **137** F1–F3
3. Busse, F.H. (2001) In this volume
4. Roberts, P.H. (2001) In this volume
5. Jones, C.A. (2000) Convection-driven geodynamo models, *Phil. Trans. R. Soc. Lond. A* **358** 873–897
6. Bullard, E.C. (1955) The stability of a homopolar dynamo, *Proc. Camb. Phil. Soc.* **51** 744–760
7. Rikitake, T. (1958) Oscillations in a system of disk dynamos, *Proc. Camb. Phil. Soc.* **54** 89–105
8. Allan, D.W. (1962) On the behaviour of systems of coupled dynamos, *Proc. Camb. Phil. Soc.* **58** 671–693
9. Robbins, K.A. (1975) *Disk Dynamos and Magnetic Reversal* (PhD thesis, M.I.T.)
10. Chui, A.Y.K. & Moffatt, H.K. (1993) A thermally driven disc dynamo. In *Solar and Planetary Dynamos* (ed. M.R.E. Proctor, P.C. Matthews & A.M. Rucklidge), pp. 51–58. Cambridge University Press: Cambridge
11. Hide, R., Skeldon, A.C. & Acheson, D.J. (1996) A study of two novel self-exciting single-disk homopolar dynamos: theory, *Proc. R. Soc. Lond.* A **452** 1369–1395
12. Moroz, I.M., Hide, R. & Soward, A.M. (1998) On self-exciting coupled Faraday disk homopolar dynamos driving series motors, *Physica* **117D** 128–144
13. Oprea, I., Chossat, P. & Armbruster, D. (1997) Simulating the kinematic dynamo forced by heteroclinic convective velocity fields, *Theor. Comput. Fluid Dyn.* **9** 293–309
14. Chossat, P., Guyard, F. & Lauterbach, R (1999) Generalized heteroclinic cycles in spherically invariant systems and their perturbations, *J. Nonlinear Sci.* **9** 479–524
15. Busse, F.H. (1970) Thermal instabilities in rapidly rotating systems, *J. Fluid Mech.* **44** 441–460
16. Jackson, A., Jonkers, A.R.T. & Walker, M.R. (2000) Four centuries of geomagnetic secular variation from historical records, *Phil. Trans. R. Soc. Lond. A* **358** 957–990
17. Kumar, S. & Roberts, P.H. (1975) A three-dimensional kinematic dynamo, *Proc. R. Soc. Lond.* A **344** 235–258
18. Gubbins, D., Barber, C.N., Gibbons, S. & Love, J.J. (2000) Kinematic dynamo action in a sphere. II. Symmetry selection, *Proc. R. Soc. Lond.* A **456** 1669–1683
19. Proctor, M.R.E. (1977) The role of mean circulation in parity selection by planetary magnetic fields, *Geophys. Astrophys. Fluid Dynamics* **8** 311–324
20. Krupa, M. & Melbourne, I. (1995) Asymptotic stability of heteroclinic cycles in systems with symmetry, *Ergod. Th. & Dynam. Sys.* **15** 121–147
21. Melbourne, I. (1989) Intermittency as a codimension three phenomenon, *J. Dyn. Diff. Eqns.* **1** 347–367
22. Melbourne, I., Proctor, M.R.E. & Rucklidge, A.M. (2001) In preparation