Derivation of the time-dependent Ginzburg-Landau equation on the line *

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January 16, 1997

Abstract

We give a rigorous derivation of the time-dependent one-dimensional Ginzburg-Landau equation.

As in the work of Iooss, Mielke and Demay (who derived the steady Ginzburg-Landau equation) our derivation leads to a pseudodifferential complex amplitude equation with nonlocal terms of all orders that yields the cubic order Ginzburg-Landau equation when truncated. The truncation step itself is not justified by our methods.

Furthermore, we prove that the nontruncated Ginzburg-Landau equation has a normal form $\mathbf{SO}(2)$ symmetry to arbitrarily high order. The normal form symmetry forces the equation to be odd with constant coefficients. This structure is broken in the tail.

1 Introduction

Systems of partial differential equations (PDEs) such as the Navier-Stokes equations, the Boussinesq equations (modeling the planar Bénard problem), the Kuramoto-Sivashinsky equation. and reaction-diffusion equations are often posed on an unbounded domain. For an overview of such spatially extended systems of PDEs, see Cross and Hohenberg [4].

*Appeared: *J. Nonlinear. Sci.* **8** (1998) 1–15
†Supported in part by NSF Grant DMS-9403624 and by ONR Grant N00014-94-1-0317

It is possible to derive finite-dimensional ordinary differential equations (ODEs) or 'Landau equations' for bifurcating solutions with a prescribed spatial periodicity and even to justify these equations rigorously via Liapunov-Schmidt or center manifold reduction. Of course, solutions to spatially extended systems need not be spatially periodic and these techniques are somewhat limited. To include more general classes of solutions, it is customary to consider infinite-dimensional modulation equations such as the 'Ginzburg-Landau equation', [21, 26, 5, 4]. The ansatz is that there is some 'basic' spatially periodic state bifurcating at criticality. The Ginzburg-Landau equation is a slowly varying amplitude or modulation equation around this basic state. As yet there is no mathematical justification of the planar Ginzburg-Landau (or Newell-Whitehead-Segel) equation. Center manifold theory is not applicable due to difficulties with continuous spectra.

Newell and Whitehead [21] consider also the one-dimensional Ginzburg-Landau equation

$$A_T = A + D^2A + c|A|^2A. \tag{1.1}$$

Recently, there has been progress towards a mathematical understanding of this equation. Kirchgässner [13] and Mielke [16, 17, 18] restrict attention to steady-state equations and view the single unbounded spatial direction as an evolution variable. Then a center manifold reduction leads to an ODE for steady-state solutions that are small and bounded in space. In particular, equilibrium solutions that are not spatially periodic can be obtained in a very elegant manner [10, 11].

In the context of hydrodynamic instabilities, Iooss, Mielke and Demay [11] by similar means derive a four dimensional ODE that captures time-independent, yet spatially complex, solutions near criticality. A change of coordinates leads to the complex pseudodifferential equation

$$\begin{aligned} 0 = c_0\lambda A + ie_1\lambda DA + e_2D^2A + ie_3D^3A \\ + d_0|A|^2A + id_1|A|^2DA + id_2A^2D\bar{A} + \cdots \end{aligned} \tag{1.2}$$

The 'lowest order' terms in this expansion have the form of the *steady* Ginzburg-Landau equation. We note that the Ginzburg-Landau equation is a PDE only when truncated at some specified order. The full asymptotic expansion is a pseudodifferential equation, see [11, 19] and Section 3 of this paper. From now on, we use the term 'Ginzburg-Landau equation' quite generally for equations of the form (1.2) (with or without time-dependencies) and refer to equation (1.1) as the standard truncation of the Ginzburg-Landau equation.

In this paper, we give a rigorous derivation of the full time-dependent Ginzburg-Landau equation. Our philosophy is similar to that of [11] in that we are interested in the nontruncated equation. The new contribution is that we do not restrict to the steady equation. To circumvent the continuous spectrum, our function space is chosen so as to allow spectral splittings in the absence of a spectral gap, enabling a (generalized) Liapunov-Schmidt reduction. In addition, our methods cast light on the significance of terms that are neglected in equations (1.1) and (1.2). In particular, equation (1.2) is odd with constant coefficients. We show that this *normal form symmetry* is present to arbitrarily high but finite orders. On the other hand, it is not clear how to use our results to prove rigorously the existence of new solutions.

Much attention has focused on an approximate justification of the truncated time-dependent Ginzburg-Landau equation (1.1) near criticality over arbitrarily long timescales, see [2, 8, 14, 24, 25]. This is an important and surprising determinacy result. Our results on the normal form structure indicate that certain aspects may be lost in the approximation. We elaborate on this point in Section 5.

In Section 2, we recall the standard multiple scaling reduction to the truncated Ginzburg-Landau equation (1.1) and we sketch the main ideas behind our rigorous derivation of the full time-dependent equation. This derivation is carried out in Section 3 for the Swift-Hohenberg equation (including a quadratic term) on the line. In Section 4, we indicate the general applicability of our methods to systems of PDEs on larger domains with one unbounded spatial direction. Finally, in Section 5 we discuss the normal form structure of the Ginzburg-Landau equation.

Throughout this paper, we work with the one-dimensional Ginzburg-Landau equation. However, the only part of our work that does not generalize to higher dimensions is the step contained in Subsection 3.3. The consequences of our methods for higher-dimensional problems will be considered in [15].

2 Sketch of a rigorous derivation

In this section, we sketch the main ideas behind our rigorous derivation of the Ginzburg-Landau equation. We begin by recalling the formal derivation from the Swift-Hohenberg equation.

Formal derivation Consider the Swift-Hohenberg equation on the line

$$u_t = L_\lambda u + \beta u^2 + \gamma u^3, \qquad L_\lambda = \lambda - (D^2+1)^2 \tag{2.1}$$

where $u : \mathbb{R} \to \mathbb{R}$, λ is a real bifurcation parameter, β and γ are real constants, and $D = d/dx$. The trivial solution $u = 0$ undergoes a steady-state bifurcation at $\lambda = 0$. Indeed, the wave function ansatz e^{ikx}, $k \in \mathbb{R}$, has eigenvalue $\lambda - (k^2-1)^2$, and the spectrum of L_λ is the real interval $(-\infty, \lambda]$.

The kernel of the linearized equations at $(u, \lambda) = (0, 0)$ is thus two-dimensional, spanned by the critical eigenfunctions $e^{ix} + e^{-ix}$ and $i(e^{ix} - e^{-ix})$. However, it is desirable (and necessary) to incorporate eigenfunctions with eigenvalues close to zero. The standard approach by physicists is to consider slowly varying modulations of the critical eigenfunctions, making the 'multiple scaling'

$$T = \epsilon^2 t, \quad X = \epsilon x, \quad \lambda = \epsilon^2, \quad u(x) = \epsilon\big(A(X)e^{ix} + \bar{A}(X)e^{-ix}\big),$$

where $A : \mathbb{R} \to \mathbb{C}$ is a complex amplitude. This ansatz is substituted into the Swift-Hohenberg equation. Equating coefficients of $\epsilon^3 e^{ix}$ we obtain a modulation equation or *Ginzburg-Landau equation* $A_T = A + 4D^2 A + c|A|^2 A$ for the slowly-varying complex amplitude A, where $D = d/dX$ and $c \in \mathbb{R}$ is a constant. (In fact, $c = 3\gamma + \frac{38}{9}\beta^2$, see Subsection 3.3.)

An important aspect of this formal reduction is that certain terms are considered to be 'damped' or 'slaved'. For example the cubic term in the Swift-Hohenberg equation contributes $\epsilon^3 e^{3ix}$ terms (and complex conjugates). However, modes e^{ikx} are damped for $|k|$ far from ± 1 and can be solved for, at least formally. This observation is even more crucial for the quadratic terms which contribute terms of order ϵ^2 but with k near ± 2 and 0.

The main ideas We now sketch the main ideas that make rigorous the formal derivation described above. Two immediate consequences of our approach are

(i) The identification of 'slave modes' such as the quadratic terms and certain cubic terms which can be solved for using the implicit function theorem.

(ii) A precise definition of the amplitude function A (which was not well-defined in the formal argument above).

For simplicity, we consider first the *steady* Swift-Hohenberg equation

$$0 = \Phi(u, \lambda) = L_\lambda u + \beta u^2 + \gamma u^3, \qquad L_\lambda = \lambda - (D^2 + 1)^2.$$

As before, the spectrum of $L = L_0$ is the half-line $(-\infty, 0]$ and the critical modes are of the form $e^{\pm ix}$.

Step 1 Since the spectrum is continuous, we cannot solve for all of the noncritical modes. However, using the implicit function theorem, we can in principle solve for modes e^{ikx} with $|k|$ bounded away from 1 (and hence eigenvalues bounded away from zero). More precisely, under certain technical hypotheses, for any $\delta > 0$ we can solve for the modes e^{ikx} with $\big||k| - 1\big| \geq \delta$.

For ease of notation, we let $\mathcal{X}$ denote both the domain and range of the nonlinear operator Φ (eventually the domain will be a dense subspace of the range as usual). We require that there is a closed splitting of $\mathcal{X}$ into a subspace $\mathcal{X}^\delta(1)$ spanned by modes e^{ikx} with $\big||k|-1\big| < \delta$ and a complementary space $\mathcal{X}^c$ where $\big||k|-1\big| \geq \delta$. The existence of such a *closed* splitting may seem nonintuitive, but it is just a matter of choosing the right function space, see Subsection 3.1.

Let E denote projection onto $\mathcal{X}^c$ and $I-E$ the complementary projection onto $\mathcal{X}^\delta(1)$. The restricted linear map EL is an isomorphism on $\mathcal{X}^c$ and Liapunov-Schmidt reduction leads to a reduced equation $\phi(v,\lambda) = 0$ on $\mathcal{X}^\delta(1)$. The reduced equation is easily written down to quadratic order in v:

$$\phi(v,\lambda) = (I-E)\{L_\lambda v + \beta v^2 + \cdots\}.$$

Now v consists of Fourier modes near ± 1 and hence v^2 consists of modes near ± 2 and 0. Provided δ is chosen small enough ($\delta < 1/3$), such modes are projected away by $I-E$. Hence, there are no quadratic terms in ϕ as promised in point (i) above. (However, the quadratic terms in Φ contribute complicated cubic terms to ϕ.)

Again for simplicity, we set $\beta = 0$ and $\gamma = -1$ in the Swift-Hohenberg equation for the remainder of this section. Up to cubic order we have

$$\phi(v,\lambda) = (I-E)\{L_\lambda v - v^3 + \cdots\}.$$

Step 2 The operator ϕ is posed on $\mathcal{X}^\delta(1)$ and involves Fourier modes k lying in small intervals around ± 1. It is therefore natural to write v in the form

$$v(x) = B(x)e^{ix} + \bar{B}(x)e^{-ix},$$

where $B : \mathbb{R} \to \mathbb{C}$ is a complex amplitude function involving Fourier modes k with $|k| < \delta$. Note that B is well-defined (point (ii) above) and that this transformation is now simply a change of coordinates. (This is the only place where we use the fact that there is a single unbounded spatial variable.) We write $B \in \mathcal{X}^\delta(0)$.

The change of coordinates leads to an equation in B, $\bar{B}$ and λ. The domain and range of this equation involves modes e^{ikx} with k within distance δ of 1 and -1 and hence splits into two equations supported near 1 and -1. The two equations are related by complex conjugation and are equivalent to the equation supported near $k = 1$. Dividing by e^{ix} leads to a complex amplitude equation $\psi(B,\lambda) = 0$ defined on $\mathcal{X}^\delta(0)$, Choosing δ small enough ($\delta < 1/2$), we ensure that the 'slaved' cubic terms are projected away by $I-E$ and obtain the operator equation

$$\psi(B,\lambda) = \mathcal{P}\{4D^2B - 4iD^3B - D^4B + \lambda B - 3|B|^2B + \cdots\}, \tag{2.2}$$

where $\mathcal{P}$ is the projection onto $\mathcal{X}^\delta(0)$.

Step 3 The operator ψ has the desired form at low order except for the presence of the projection operator $\mathcal{P}$. Also, ψ is defined on $\mathcal{X}^\delta(0)$ whereas $\mathcal{X}$ would be more natural. To overcome these problems, we apply a 'reverse' Liapunov-Schmidt reduction. Consider an operator Ψ on $\mathcal{X}$ of the form

$$\Psi(A,\lambda) = 4D^2A - 4iD^3A - D^4A + \lambda A - 3|A|^2A + \cdots$$

Liapunov-Schmidt reduction leads to an operator of the form (2.2). In fact, the high order terms in Ψ can be chosen so that the 'reverse' reduction produces precisely the operator (2.2).

There is one obstruction to immediately applying the reverse Liapunov-Schmidt reduction step: the critical modes for Ψ at $\lambda = 0$ include modes e^{ikx} with $k = 0$ as required but also include modes with $k = -2$. For this time-independent equation, there is a simple resolution of this difficulty whereby we transform to higher order the third and fourth order derivatives in the linear terms. An approach which applies to the time-dependent case is given in Subsection 3.4.

Steps 1 to 3 are detailed in Section 3 and establish that equilibria of the Swift-Hohenberg equation are locally (near $(u,\lambda) = (0,0)$) in one-to-one correspondence with equilibria of the steady Ginzburg-Landau equation $\Psi(A,\lambda) = 0$. Our methods generalize to the time-dependent Ginzburg-Landau equation thanks to the notion of *essential solution* [1, 19, 9]. We incorporate the term u_t in the right-hand-side of the PDE so that $L_\lambda = -\partial/\partial t + \lambda - (D^2+1)^2$ and again solve for zeros using Liapunov-Schmidt reduction. The zeros now correspond not only to equilibria that are small over space but also time-dependent solutions that are small over time and space.

3 Reduction of the Swift-Hohenberg equation

In this section, we carry out the procedure outlined in Section 2 and rigorously derive the full (nontruncated) Ginzburg-Landau equation from the Swift-Hohenberg equation (2.1). Although the underlying PDE is highly idealized, out results apply quite generally as shown in Section 4.

In Subsection 3.1, we describe a function space that is suitable for our analysis. Steps 1, 2 and 3 from Section 2 are carried out in Subsections 3.2, 3.3 and 3.4.

3.1 The functional-analytic setting

In this subsection, we introduce suitable function spaces and recall the basic properties that we will require. To avoid measure-theoretic technicalities, we work initially with a space of spatially and temporally quasiperiodic functions.

Let $J \subset \mathbb{R}^2$. We consider real-valued functions $u : \mathbb{R}^2 \to \mathbb{R}$ of the form

$$u(x,t) = \sum_{(k,\ell)\in J} u_{k,\ell} e^{ikx} e^{i\ell t},$$

where the coefficients $u_{k,\ell} \in \mathbb{C}$ satisfy the reality condition $u_{-k,-\ell} = \overline{u_{k,\ell}}$, and $u_{k,\ell} = 0$ for all but finitely many (k,ℓ). We denote the vector space of all such sums by $X(J)$ and define the 'one-norm' $\|u\| = \sum_{k,\ell} |u_{k,\ell}|$. The completion of $X(J)$ with respect to this norm is then, by definition, a Banach space of spatially and temporally quasiperiodic functions containing $X(J)$ as a dense subset. We denote this completion by $\mathcal{X}(J)$. The two most important properties of $\mathcal{X} = \mathcal{X}(\mathbb{R}^2)$ are listed in the following proposition.

Proposition 3.1 (a) *The space $\mathcal{X}$ is a Banach algebra under pointwise multiplication, that is*

$$\|uv\| \le \|u\|\|v\| \text{ for all } u, v \in \mathcal{X}. \tag{3.1}$$

(b) *If J is any subset of $\mathbb{R}^2$ then there is a closed splitting*

$$\mathcal{X} = \mathcal{X}(\mathbb{R}^2) \oplus \mathcal{X}(\mathbb{R}^2 - J).$$

Proof The map $h : \mathcal{X} \to \ell^1(\mathbb{R}^2)$, $h(u) = \{u_{k,\ell}\}_{(k,\ell)\in\mathbb{R}^2}$, is an isomorphism converting pointwise multiplication of functions into convolution of sequences. It is well-known that $\ell^1(\mathbb{R}^2)$ is a convolution algebra and that $\ell^1(\mathbb{R}^2) = \ell^1(J) \oplus \ell^1(\mathbb{R}^2 - J)$. Alternatively, it is a simple calculation to verify (3.1) for $u, v \in X(\mathbb{R}^2)$ and hence the result holds in $\mathcal{X}$. The second part of the proposition is immediate except perhaps for the fact that the sum is direct. But suppose that $u \in \mathcal{X}(J) \cap \mathcal{X}(\mathbb{R}^2 - J)$. Then there are sequences $v_n \in X(J)$ and $w_n \in X(\mathbb{R}^2 - J)$ that converge to u. It follows that $\|v_n\| + \|w_n\| = \|v_n - w_n\| \to 0$. In particular, $v_n \to 0$ and so $u = 0$ as required. ∎

Uniformly continuous solutions The function space $\mathcal{X}$ considered so far consists only of quasiperiodic functions. We now show how to incorporate solutions with more complicated spatial and temporal dependence. Indeed, we obtain a large subspace of $C_{\text{unif}}(\mathbb{R}^2)$, the space of uniformly continuous functions on $\mathbb{R}^2$. By taking the norm in Fourier transform space, we preserve the crucial properties in Proposition 3.1.

Let $\mathcal{B}$ denote the σ-algebra of Borel subsets in $\mathbb{R}^2$. For any $J \in \mathcal{B}$, the space $\mathcal{M}(J)$ of complex Borel measures on J is a Banach space with total variation norm $\|\mu\|$. We note that $\mathcal{M}(\mathbb{R}^2)$ has properties analogous to those of the space $\ell^1(\mathbb{R}^2)$ introduced in the proof of Proposition 3.1: $\mathcal{M}(\mathbb{R}^2)$ is a convolution algebra and $\mathcal{M}(\mathbb{R}^2) = \mathcal{M}(J) \oplus \mathcal{M}(\mathbb{R}^2 - J)$ for all $J \in \mathcal{B}$.

If $\mu \in \mathcal{M}(\mathbb{R}^2)$, the Fourier-Stieltjes transform $\mathcal{F}\mu$ [12] is defined to be

$$(\mathcal{F}\mu)(x,t) = \int_{(k,\ell)\in\mathbb{R}^2} e^{-ikx} e^{-i\ell t} d\mu(k,\ell).$$

Define $\mathcal{X} = \mathcal{X}(\mathbb{R}^2)$ to consist of the set of functions $u : \mathbb{R}^2 \to \mathbb{R}$ obtained in this way. Then $\mathcal{X}$ is a proper subspace of $C_{\text{unif}}(\mathbb{R}^2)$. Instead of the uniform norm, we use the norm inherited from $\mathcal{M}(\mathbb{R}^2)$. That is, we define $\|u\| = \|\mu\|$ where $\mu \in \mathcal{M}(\mathbb{R}^2)$ is the unique measure such that $\mathcal{F}\mu = u$. With this norm, $\mathcal{X}$ is a Banach space. Since $\mathcal{M}(\mathbb{R}^2)$ is a convolution algebra and Fourier transform converts convolution of measures into pointwise multiplication of functions, $\mathcal{X}$ satisfies the conclusions of Proposition 3.1.

Remark 3.2 Note that $\ell^1(\mathbb{R}^2)$ is the closed subspace of $\mathcal{M}(\mathbb{R}^2)$ spanned by the Dirac measures. Hence, the quasiperiodic functions form a closed subspace of the enlarged version of $\mathcal{X}$. Similarly, the absolutely continuous measures in $\mathcal{M}(\mathbb{R}^2)$ can be identified with their Radon-Nikodym derivatives in $L^1(\mathbb{R}^2)$ and it follows that $L^1(\mathbb{R}^2)$ is a closed subspace of $\mathcal{M}(\mathbb{R}^2)$. According to the Riemann-Lebesgue lemma, the corresponding space of Fourier transforms is a subalgebra of $C_0(\mathbb{R}^2)$ (the continuous functions that decay at infinity). In fact, this space is a dense proper subalgebra of $C_0(\mathbb{R}^2)$.

Next we consider the Swift-Hohenberg equation as a nonlinear operator on the function space $\mathcal{X}$. Define

$$\Phi(u,\lambda) = L_\lambda u + \beta u^2 + \gamma u^3, \qquad L_\lambda = -\partial/\partial t + \lambda - (D^2+1)^2. \tag{3.2}$$

Observe that zeroes of Φ correspond to solutions of the Swift-Hohenberg equation (2.1). We are interested in the small-norm zeroes of Φ which correspond to the *essential* solutions [1, 19] of (2.1).

Let $L = L_0 = -\partial/\partial t - (D^2+1)^2$. The linear operator $L : \mathcal{X} \to \mathcal{X}$ is unbounded. Let $\mathcal{X}^L$ denote the completion of X with respect to the graph norm $\|u\|^L = \|u\| + \|Lu\|$. Then $L : \mathcal{X}^L \to \mathcal{X}$ is a bounded operator. The remaining terms in Φ are analytic even on $\mathcal{X}$ (since by Proposition 3.1(a) $\mathcal{X}$ is a Banach algebra). Hence we obtain an analytic nonlinear operator $\Phi : \mathcal{X}^L \times \mathbb{R} \to \mathcal{X}$.

3.2 Liapunov-Schmidt reduction

In this subsection we show how to apply Liapunov-Schmidt reduction to the operator Φ in (3.2). Observe that $\ker L = \mathcal{X}^L(1,0)$. Since 0 is not isolated in the spectrum of L, it is not possible to apply the implicit function theorem on any complement of $\ker L$. However, we may apply the implicit function theorem on the complement of the sum of the eigenspaces of eigenvalues close to and including zero. Fix $\delta > 0$ and let $J_\delta = B_\delta(1,0) \cup B_\delta(-1,0)$ where $B_\delta(\pm 1,0) \subset \mathbb{R}^2$ is the open ball of radius δ, center $(\pm 1, 0)$. Define the subspaces

$$\mathcal{X}^{L,\delta}(1,0) = \mathcal{X}^L(J_\delta), \qquad \mathcal{X}^{L,c} = \mathcal{X}^L(\mathbb{R}^2 - J_\delta),$$
$$\mathcal{X}^{\delta}(1,0) = \mathcal{X}(J_\delta), \qquad \mathcal{X}^{c} = \mathcal{X}(\mathbb{R}^2 - J_\delta).$$

By Proposition 3.1(b), we have the closed splittings

$$\mathcal{X}^L = \mathcal{X}^{L,\delta}(1,0) \oplus \mathcal{X}^{L,c}, \qquad \mathcal{X} = \mathcal{X}^{\delta}(1,0) \oplus \mathcal{X}^{c}.$$

Define the complementary projections $E : \mathcal{X} \to \mathcal{X}^c$, $I - E : \mathcal{X} \to \mathcal{X}^\delta(1,0)$.

Theorem 3.3 *For any $\delta > 0$, there is a reduced analytic nonlinear operator*

$$\phi : \mathcal{X}^{L,\delta}(1,0) \times \mathbb{R} \to \mathcal{X}^{\delta}(1,0),$$

such that locally (near $(0,0)$) zeros of ϕ are in one-to-one correspondence with zeros of Φ. The reduced operator ϕ is given by

$$\phi(v,\lambda) = (I-E)\Phi(v + W(v,\lambda),\lambda),$$

where $W : \mathcal{X}^{L,\delta}(1,0) \times \mathbb{R} \to \mathcal{X}^c$ is defined locally and implicitly by

$$E\Phi(v + W(v,\lambda),\lambda) \equiv 0, \qquad W(0,0) = 0.$$

Proof Observe that L is bounded below when restricted to $\mathcal{X}^{L,c}$. It follows that L is an isomorphism from $\mathcal{X}^{L,c}$ onto its range and that the range is closed. Moreover, the dense subspace $X \cap \mathcal{X}^c \subset \mathcal{X}^c$ is clearly contained in the range and so $L : \mathcal{X}^c \to \mathcal{X}^c$ is an isomorphism. Now the standard proof of the Liapunov-Schmidt reduction proceeds as usual, see for example [6, steps 2-4, p.293]. ∎

Remark 3.4 The Banach spaces $\mathcal{X}^{L,\delta}(1,0)$ and $\mathcal{X}^{\delta}(1,0)$ are equal as vector spaces and the norms are equivalent. Hence, we can consider the reduced operator ϕ to be an analytic operator $\phi : \mathcal{X}^{\delta}(1,0) \times \mathbb{R} \to \mathcal{X}^{\delta}(1,0)$.

The reduced nonlinear operator ϕ is only defined implicitly, but derivatives may be computed in the same way as in [6] to obtain the following.

Proposition 3.5 *Locally,*

$$\phi(0,\lambda)=0,\quad (d\phi)_{0,\lambda}v=(I-E)L_\lambda v,\quad (d^2\phi)_{0,\lambda}(v,v)=(I-E)(d^2\Phi)_{0,\lambda}(v,v),$$
$$(d^3\phi)_{0,\lambda}(v,v,v)=(I-E)\{(d^3\Phi)_{0,\lambda}(v,v,v)-3(d^2\Phi)_{0,\lambda}(v,L_\lambda^{-1}E(d^2\Phi)_{0,\lambda}(v,v))\}.$$

We now apply the reduction to the Swift-Hohenberg equation. By Proposition 3.5, the reduced operator is

$$\phi(v,\lambda)=(I-E)\{L_\lambda v+\beta v^2+\gamma v^3-2\beta^2 vL_\lambda^{-1}Ev^2+\cdots\},$$

where $\cdots$ consists of terms of order at least five in v. Since the inverse of L appears in the cubic term (and at fifth order even when $\beta=0$), ϕ contains nonlocal terms.

Observe that $(I-E)v^2=0$ for δ small enough ($\delta<1/3$). (The heuristic argument in Section 2, Step 1, is now made completely rigorous.) In fact, by choosing δ small enough, it is possible to guarantee that ϕ involves only terms of odd degree up to any prescribed order. Note also that the projection E disappears in the β part of the cubic term. Assign v weight one and λ weight two. Then the reduced operator has the form

$$\phi(v,\lambda)=(I-E)\{L_\lambda v+\gamma v^3-2\beta^2 vL^{-1}v^2+\cdots\},\tag{3.3}$$

where $\cdots$ consists of terms of (weighted) order at least five.

3.3 Equation for complex amplitudes

Recall that $\mathcal{X}^\delta(1,0)$ consists of real sums of Fourier modes with wave numbers (k,ℓ) close to $(\pm1,0)$. Hence we can split these sums into two components centered around e^{ix} and e^{-ix}. If $v\in\mathcal{X}^\delta(1,0)$, we write

$$v(x,t)=B(x,t)e^{ix}+\bar{B}(x,t)e^{-ix},\tag{3.4}$$

where $B:\mathbb{R}\to\mathbb{C}$ has the form $B(x,t)=\sum_{|(k,\ell)|<\delta}B_{k,\ell}e^{ikx}e^{i\ell t}$ (or more generally, $B(x,t)=\int_{|(k,\ell)|<\delta}e^{-ikx}e^{-i\ell t}d\mu(k,\ell)$ where μ is a complex Borel measure). We suppress the fact that B takes values in $\mathbb{C}$ and write $B\in\mathcal{X}^\delta(0,0)$. The components B and $\bar{B}$ are clearly unique provided we choose $\delta<1$.

The linear change of coordinates (3.4) leads, as in Section 2, to the analytic operator $\psi:\mathcal{X}^\delta(0,0)\times\mathbb{R}\to\mathcal{X}^\delta(0,0)$,

$$\psi(B,\lambda)=\mathcal{P}\{-B_t+4D^2B-4iD^3B-D^4B+\lambda B+C(B)+\cdots\},\tag{3.5}$$

where

$$C(B) = 3\gamma|B|^2B - 2\beta^2\left(2BL^{-1}|B|^2 + \bar{B}e^{-2ix}L^{-1}(B^2e^{2ix})\right),$$

and $\mathcal{P} : \mathcal{X} \to \mathcal{X}^\delta(0,0)$ is projection.

At this point, we can easily read off the constant c in Section 2. Ignoring the terms with derivatives (replace L^{-1} by $-I$), we have $c = 3\gamma + 4\beta^2 + \frac{2}{9}\beta^2 = 3\gamma + \frac{38}{9}\beta^2$.

3.4 Reverse Liapunov-Schmidt reduction

Suppose that L is a constant coefficient linear partial differential operator. Then we can write $Le^{ikx}e^{i\ell t} = Q(k,\ell)e^{ikx}e^{i\ell t}$ where the *symbol* Q is a polynomial function of k and ℓ. We may also consider nonpolynomial functions Q in which case L is called a pseudodifferential operator. We note that the regularity of L is independent of the regularity of Q; for example the projections E and $\mathcal{P}$ considered in this paper have discontinuous symbols.

Similar considerations apply to nonlinear operators. It is desirable to obtain a Ginzburg-Landau equation defined on the whole of $\mathcal{X}$ rather than $\mathcal{X}(0,0)$ and such that the linear and nonlinear operators have analytic symbols. (It is unrealistic to expect polynomial symbols since any reduction must involve the inversion of linear operators.)

In this subsection, we realize the first aim obtaining an equation defined on $\mathcal{X}$. The second aim is partially realized: we remove the projection $\mathcal{P}$ through at least cubic order and obtain smooth (C^∞) symbols for the linear and cubic terms.

Theorem 3.6 *There is a nonlinear operator $\Psi : \mathcal{X} \times \mathbb{R} \to \mathcal{X}$ of the form*

$$\Psi(A,\lambda) = -A_t + D^2\tilde{L}A + \lambda A + \tilde{C}(A) + \widetilde{\Psi}(A,\lambda), \tag{3.6}$$

where $\tilde{L}$ is a linear isomorphism with smooth symbol, $\tilde{C}$ is an analytic cubic nonlinearity with smooth symbol, and $\widetilde{\Psi}$ denotes analytic terms of higher weight, such that zeroes for Ψ are locally in one-to-one correspondence with essential solutions of the Swift-Hohenberg equation (2.1).

Proof The operator $4D^2B - 4iD^3B - D^4B$ has symbol $Q(k) = -4k^2 - 4k^3 - k^4$. Let $\tilde{Q}$ be a smooth function of k agreeing with Q for $|k| \le \delta$, equal to one for $|k| \ge 2\delta$, and with no zeroes. The linear operator $\tilde{L}$ with symbol $\tilde{Q}$ is a linear isomorphism with smooth symbol. Moreover $\mathcal{P}\{4D^2B - 4iD^3B - D^4B\} = \mathcal{P}D^2\tilde{L}B$. Similarly, we can construct $\tilde{C}$ analytic with smooth symbol (dependent on k and ℓ) such that $\mathcal{P}\tilde{C} = \mathcal{P}C$. Operator (3.5) becomes

$$\psi(B,\lambda) = \mathcal{P}\{-B_t + D^2\tilde{L}B + \lambda B + \tilde{C}(B) + \cdots\}. \tag{3.7}$$

Liapunov-Schmidt reduction applied to an operator Ψ of the form (3.6) leads to an operator ψ of the form (3.7). The only problem is to ensure that the higher order terms are correct. First, let us consider what happens if we take $\widetilde{\Psi}(A,\lambda)\equiv 0$. The reduced equation has the form

$$\mathcal{P}\Psi(B+W(B,\lambda),\lambda)=\psi(B,\lambda)+\widetilde{\psi}(B,\lambda) \tag{3.8}$$

where $\widetilde{\psi}$ takes values in $\mathcal{X}^\delta(0,0)$ and consists of terms of weight greater than three. In particular, $\mathcal{P}\widetilde{\psi}=\widetilde{\psi}$. As usual, $W:\mathcal{X}^\delta(0,0)\times\mathbb{R}\to\mathcal{X}^c$ is defined implicitly by

$$(I-\mathcal{P})\Psi(B+W(B,\lambda))\equiv 0, \qquad W(0,0)=0. \tag{3.9}$$

Set $\widetilde{\Psi}(A,\lambda)=-\widetilde{\psi}(\mathcal{P}A,\lambda)$. Then, equation (3.9) and hence W are unaltered (since $(I-\mathcal{P})\widetilde{\psi}=0$) so that equation (3.8) becomes

$$\mathcal{P}\Psi(B+W(B,\lambda),\lambda)=\psi(B,\lambda)+\widetilde{\psi}(B,\lambda)-\mathcal{P}\widetilde{\psi}(\mathcal{P}(B+W(B,\lambda)),\lambda)=\psi(B,\lambda),$$

as required. ∎

4 Reduction of systems of PDEs

The Ginzburg-Landau equation is supposed to be *universal* and hence should be applicable in some generality. In Sections 3, we derived the Ginzburg-Landau equation from a particularly simple example: the Swift-Hohenberg equation on the line. (However, we include quadratic terms — these terms provide no additional difficulties, cf [8, 14].) Having illustrated the main ideas, we now consider a system of PDEs on $\mathbb{R}\times[0,1]$ with Dirichlet boundary conditions. In particular, we show that our method deals successfully both with systems (cf [25]) and with bounded spatial variables — there is no need in general to remove the bounded variables in a preliminary step as suggested in [19]. Indeed, we show that it is possible to reduce in a single step to a (scalar) equation $\phi=0$ on $\mathcal{X}^\delta(0,0)$. From then on, we can proceed as before to obtain a complex amplitude equation $\Psi=0$ on $\mathcal{X}$.

A precise discussion of universality of the Ginzburg-Landau equation requires serious consideration of the presence of Euclidean symmetry (which is suppressed in this paper) as well as a notion of genericity for unbounded partial differential operators. This is the subject of work in progress [15].

Brusselator on $\mathbb{R}\times[0,\pi]$ Consider the system of reaction-diffusion equations

$$\begin{aligned} u_t &= \Delta u+(8+\lambda)u+16v+N(u,v,\lambda), \\ v_t &= 4\Delta v-(9+\lambda)u-16v-N(u,v,\lambda), \end{aligned} \tag{4.1}$$

where the nonlinearity is given by $N(u,v,\lambda) = \frac{1}{4}(9+\lambda)u^2 + 8uv + u^2v$, and $x \in \mathbb{R}$, $z \in [0, 2\pi]$, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial z^2$. We impose Dirichlet boundary conditions $u = v = 0$ when $z = 0, \pi$. These equations may be obtained from those for the Brusselator [23] by choosing $D_1 = 1$, $D_2 = 4$, $A = 4$ and $B = 9 + \lambda$ and subtracting the trivial solution.

Let $\mathcal{X}(L^2)$ denote the Banach space of functions $u : \mathbb{R}^2 \to \mathbb{R}$ of the form $u(x,z) = \sum_{(k,\ell)\in\mathbb{R}^2} u_{k,\ell}(z)e^{ikx}e^{i\ell t}$ where $u_{k,\ell} \in L^2[0,\pi]$ say, and $\|u\| = \sum \|u_{k,\ell}\|$. Now let $\mathcal{X}^2(L^2)$ consist of functions $(u,v) : \mathbb{R}^2 \to \mathbb{R}^2$ where $u, v \in \mathcal{X}(L^2)$ and $\|(u,v)\| = \|u\| + \|v\|$.

We now turn to the linear stability analysis of the trivial solution $u = v = 0$. Let $u(k,\ell,m)$ denote the coefficient of $e^{ikx}e^{i\ell t}\sin mz$ in u and define $v(k,\ell,m)$ similarly. The (k,ℓ,m) component of the linearized equations leads to the matrix

$$\begin{pmatrix} -i\ell - K + 8 + \lambda & 16 \\ -(9+\lambda) & -i\ell - 4K - 16 \end{pmatrix}$$

where $K = k^2 + m^2$. The real parts of the eigenvalues are independent of the value of ℓ and are given by the trace tr $= \lambda - (8 + 5K)$ and determinant det $= 4((K-2)^2 - \lambda K)$. It follows that the trivial solution is stable provided that $\lambda < 8 + 5K$ and $\lambda < (K-2)^2/K$ for all $K \geq 0$. Hence there is a steady-state bifurcation at $\lambda = 0$ corresponding to $K = k^2 + m^2 = 2$. Since m is a positive integer, the critical modes have $m = 1$ and $(|k|, \ell)$ close to $(1, 0)$. The eigenvalues corresponding to $e^{ikx}e^{i\ell t}\sin z$ are

$$\mu_\pm(k,\ell) = -i\ell + \tfrac{1}{2}\left\{-5k^2 - 13 \pm 3\sqrt{a(k)}\right\}$$

where $a(k) = (k^2+1)(k^2+17)$. The corresponding eigenfunctions are

$$w_\pm(k,\ell) = \begin{pmatrix} 16 \\ k^2 - 7 + \mu_\pm(k,\ell) \end{pmatrix} e^{ikx}e^{i\ell t}\sin z.$$

The critical eigenfunctions are given by $w_+(k,\ell)$ for (k,ℓ) close to $(\pm 1, 0)$, and may be identified with elements of $\mathcal{X}^\delta(1,0)$ (consisting of functions $w(x,t) : \mathbb{R}^2 \to \mathbb{R}$ with restricted Fourier modes just as in Section 3). Hence we can perform a Liapunov-Schmidt reduction to a (time-dependent) operator $\phi : \mathcal{X}^\delta(1,0) \times \mathbb{R} \to \mathcal{X}^\delta(1,0)$. We then proceed as in Section 3 to obtain a complex amplitude equation $\Psi(A,\lambda) = 0$ on $\mathcal{X}$.

General systems It is worth considering which systems of PDEs are amenable to the reduction described in this paper. Observe that the projection operator $I - E$ simultaneously wipes out all but one component of the system of PDEs, all but one eigenfunction in the bounded direction z, and all but a small band of eigenfunctions in the unbounded direction x. In [15], we argue that these features should hold generically for systems of PDEs with Euclidean symmetry.

Roughly speaking, the argument goes as follows. Consider a system of s PDEs with Euclidean symmetry posed on $\mathbb{R}$ (so there are no bounded directions). In general, the stability analysis leads to a family of $s \times s$ matrices $L(k, \ell)$ parameterized by k and ℓ. At a steady-state bifurcation, for some value of the bifurcation parameter λ, there is a critical wave number $k_0 \geq 0$ such that the corresponding matrix $L(k_0, 0)$ has a zero eigenvalue. Generically the value k_0 is unique and $L(k_0, 0)$ has rank $s - 1$. Moreover, for each (k, ℓ) close to $(k_0, 0)$, $L(k, \ell)$ has a unique eigenvalue close to 0 with eigenvector $w_{k,\ell}$. We now use the implicit function theorem to reduce to the space spanned by the eigenfunctions $w_{k,\ell}e^{\pm ikx}e^{i\ell t}$ for (k, ℓ) close to $(k_0, 0)$. This space can be identified with $\mathcal{X}^\delta(k_0, 0)$. Thus, Liapunov-Schmidt reduction leads to an equation $\phi(v, \lambda) = 0$ where $\phi : \mathcal{X}^\delta(k_0, 0) \times \mathbb{R} \to \mathcal{X}^\delta(k_0, 0)$. Provided that $k_0 > 0$, we can now proceed as in Section 3. (The case $k_0 = 0$ is different and does not lead to equation (1.1).)

This methodology is illustrated in our analysis of the Brusselator example above. The bounded direction adds no complications since we solve for all but one of the corresponding eigenfunctions.

5 Normal form symmetry

The truncated one-dimensional Ginzburg-Landau equation (1.1) is odd with constant coefficients. These are symmetry properties — constant coefficients corresponding to the translation $A(x) \mapsto A(x + v)$. At first sight it is no surprise that the equation is constant coefficient. However it is the underlying function u (in the Swift-Hohenberg equation, say) that transforms under translations in the usual way: $u(x) \mapsto u(x + v)$, whereas A (via the ansatz $u(x) = A(x)e^{ix} + \bar{A}(x)e^{-ix}$) transforms as $A(x) \mapsto A(x+v)e^{iv}$. Hence, there is extra structure in the truncated equation that needs explanation. This extra structure can be described in terms of a 'normal form' $\mathbf{SO}(2)$ symmetry generated by the transformations

$$A \mapsto e^{i\theta}A, \qquad \theta \in \mathbf{SO}(2).$$

We show that the nontruncated Ginzburg-Landau equation (3.6) possesses this normal form symmetry up to arbitrary high order. Recall that our reduction depended on the number $\delta > 0$ chosen sufficiently small, see Subsection 3.2.

Proposition 5.1 *Let m be an odd integer and choose $\delta \in (0, 1/m)$. Then the Ginzburg-Landau equation* (3.6) *has the normal form* $\mathbf{SO}(2)$ *symmetry through* *order m. In particular, the equation truncated at order m is odd with constant coefficients. Ignoring the λ-dependence and terms with derivatives, the general term through order m has the form $|A|^{2k}A$.*

Proof In Subsection 3.2, we chose $\delta < 1/3$ to ensure that there are no quadratic terms in the Ginzburg-Landau equation. Moreover, the cubic terms take the form $C(A,\bar{A}) = C_0(A,A,\bar{A})$ where C_0 is a trilinear constant coefficient (except for the dependence on λ) pseudodifferential operator. Similarly, a computation shows that choosing $\delta < 1/m$ ensures that there are no even terms through order m and that the general odd term has the form $H(A,\bar{A}) = H_0(A,\dots,A;\bar{A},\dots,\bar{A})$ where H_0 is a $(2k+1)$-linear constant coefficient pseudodifferential operator. (The argument A occurs $k+1$ times, and $\bar{A}$ occurs k times.) ∎

Normal form symmetry in the Ginzburg-Landau equation appears to have been overlooked in the mathematics and physics literature with the exception of Pomeau [22] who worked formally and investigated some of the implications.

The normal form symmetry is an artifact of the analysis and is not present for the full nontruncated equation (in contrast to the physical translations and reflections which are present to all orders but in a disguised form). Eventually, for any fixed value of $\delta > 0$, terms of the form $|A|^{2m}e^{-ix}$ are unavoidable. More generally, we expect terms such as

$$A^p\bar{A}^q e^{i(p-q-1)x} \tag{5.1}$$

for $p+q$ large (as well as more complicated terms involving derivatives of A).

Remark 5.2 In our opinion, it is obvious *in retrospect* that the normal form symmetry is not present to all orders. Examine the underlying ansatz $u(x) = A(x)e^{ix} + \bar{A}(x)e^{-ix}$ (or some infinite expansion of this form). The amplitude A simply does not transform under translations in the same way that u does.

Suppose that the Ginzburg-Landau equation has the normal form $\mathbf{SO}(2)$ symmetry to all orders. In particular, the transformation $A \mapsto -A$ is a symmetry; if A is a solution then so is $-A$. Via the ansatz, we see that if u us a solution to the underlying problem, then so is $-u$. But we have not assumed that the underlying equation is odd, hence there is a contradiction. This *proves* that generally there are even order terms of the form (5.1). Note that such terms automatically break the constant coefficient structure as well.

(If the underlying equation is odd, then the corresponding Ginzburg-Landau equation is odd. Hence, the subgroup $\mathbb{Z}_2 \subset \mathbf{SO}(2)$ is maintained to all orders. The remaining $\mathbf{SO}(2)$ symmetry (constant coefficient structure) is still broken in the tail by terms (5.1) with $p-q$ odd, $p-q \neq 1$.)

The content of Proposition 5.1 is that the $\mathbf{SO}(2)$-symmetry, although not present to all orders, is present to arbitrarily high order. This result, suggested by Pomeau [22] on formal grounds, has not previously been rigorously proved.

Implications for solutions From the point of view of dynamical systems and bifurcation theory, it is evident that the terms in the tail should have significant consequences for solutions of the Ginzburg-Landau equation. For example, steady-state/Hopf mode-interaction in systems without symmetry leads to a three-dimensional ODE with $\mathbf{SO}(2)$ normal form symmetry up to arbitrarily high order. The equations in normal form can be solved fairly completely [7], but the full equations have delicate chaotic dynamics.

Coullet *et al* [3] used these ideas from ODEs to obtain time-independent spatially chaotic solutions in the Ginzburg-Landau equation. They did this by adding an external 'periodic forcing' term to the standard truncation of the Ginzburg-Landau equation, so as to break the translation invariance of the underlying problem. It follows from our results that such terms already occur *internally* and it is not necessary to break the underlying translation invariance.

There is one class of solutions that is particularly sensitive to terms that break the normal form $\mathbf{SO}(2)$ symmetry. Suppose that the operator Ψ in (3.6) has the $\mathbf{SO}(2)$ symmetry to all orders and hence is constant coefficient. It follows from standard implicit function theorem arguments that for each $\omega > 0$, there is a branch of spatially periodic equilibria with period $2\pi/\omega$ bifurcating from the trivial solution $A = 0$ at $\lambda = \lambda_\omega > 0$. Moreover, $\lambda_\omega \to 0$ as $\omega \to 0$. Provided ω is small, these spatially periodic solutions exist also for the operator ψ. However, if ω is irrational, the corresponding solutions for ϕ and hence Φ are spatially quasiperiodic with independent frequencies 1 and ω. The existence of quasiperiodic solutions for the underlying PDE is independent of any Diophantine conditions on ω! Of course, this argument breaks down because of the nonsymmetric terms in the tail. Iooss and Los [10] show that those quasiperiodic solutions with ω Diophantine exist for the underlying PDE and therefore survive the terms in the tail. Presumably, the remaining quasiperiodic solutions do not survive.

We have made the comparison with low-codimension bifurcation theory. In fact, the tail is likely to be of even more importance for the Ginzburg-Landau equations than in bifurcation theory. (i) In the bifurcation theory, the exotic behavior often occurs in thin cuspoidal wedges in parameter space. In the Ginzburg-Landau equation, there is only one parameter so the thin wedges are everything. (ii) Normal form symmetry leads to group orbits of solutions. Often these group orbits are normally hyperbolic so that breaking the symmetry in the tail picks out some of these solutions. Now, solutions A on an $\mathbf{SO}(2)$-group are essentially the same, but as pointed out in Pomeau [22] the corresponding solutions $u = Ae^{ix} + \bar{A}e^{-ix}$ need not be physically identical. (iii) In the formal derivation reviewed in Section 2, the argument of A is the 'slow' variable X whereas the exponentiated x is the 'fast' variable. Under the standard scaling, a term in the tail of the form $|A|^{2m}e^{-ix}$ becomes proportional to $|A|^{2m}e^{-iX/\epsilon}$.

These considerations put us in a position to appreciate both the achievements and the limitations of the approach of [2, 8, 14, 24, 25] whose aim is the approximate justification of solutions to the truncated Ginzburg-Landau equation over long but finite intervals of time. Perhaps these results can be used to show that the normal form equations are determined by the standard truncation, a truly remarkable result. However, the information from the tail is lost in the approximation. To return to the analogy with low-codimension bifurcation theory, it is desirable (though usually impossible) to show that the normal form equations are determined at low order, but the tail effects should not be neglected completely.

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