Nonasymptotically stable attractors in $\mathbf{O}(2)$ mode interactions *

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August 3, 1993

Abstract

Heteroclinic cycles are a natural source of nonasymptotically stable attractors in systems with symmetry. In this paper, stability properties are completely classified for a large class of heteroclinic cycles. In particular, we establish the existence of several nonasymptotically stable attractors in codimension two mode interactions with $\mathbf{O}(2)$ symmetry, in the process explaining the results of some numerical experiments. In the Hopf/Hopf mode interaction we show that two heteroclinic cycles can coexist as nonasymptotically stable attractors.

1 Introduction

It is now well-known following the work of Field [4] and in particular Guckenheimer and Holmes [5] that structurally stable heteroclinic cycles occur naturally in low-codimension bifurcation theory when there is a group of symmetries present. The majority of these heteroclinic cycles may be asymptotically stable. For example, codimension two mode interactions with $\mathbf{O}(2)$ symmetry provide a rich supply of asymptotically stable heteroclinic cycles between equilibria and/or periodic solutions, see Armbruster et al [2], Proctor and Jones [12] and Melbourne et al [10].

*Appeared: *Fields Inst. Comm.* **4** (1995) 219-232

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Unpublished numerical experiments by the authors of [10] indicated that certain heteroclinic cycles are numerically observable even when they are not asymptotically stable. Motivated by these observations, Melbourne [9] gave an example of a heteroclinic cycle in three dimensions that is not asymptotically stable but is an attractor. This example is somewhat idealized, though it occurs naturally in work of Lauterbach and Roberts [8]. The examples of [10] have complicating factors such as high dimension and continuous group actions which serve to disguise the underlying ideas in [9]. However the framework introduced in Krupa and Melbourne [7] for analyzing asymptotic stability of heteroclinic cycles is designed to circumvent such difficulties. In this paper, we show that the combination of the ideas in [9] and the framework in [7] leads to a proof that the numerics described above are indeed explained by the existence of a nonasymptotically stable attractor (at the same time improving upon the results in [9]).

These nonasymptotically stable heteroclinic cycles have very strong attractivity properties which we now describe. Although there is not an open basin of attraction, the cycles satisfy a measure-theoretic notion of attractor (see Milnor [11]) where nearby trajectories remain close and are asymptotic to the attractor provided that they lie in sets of positive measure. In fact, we claim much more — the measure of these sets of initial conditions becomes arbitrarily close to full measure in small enough neighborhoods of the cycle. This property is called essential asymptotic stability in [9] and is formalized in the following definition.

Definition 1.1 A flow-invariant set X is said to be *essentially asymptotically stable* if there is a set D such that for any open neighborhood U of X and any $\epsilon > 0$, there exists a (smaller) open neighborhood V such that

(a) trajectories starting in $V - D$ remain in U for all forward time and are asymptotic to X.

(b) $\lambda(V - D)/\lambda(V) > 1 - \epsilon$, where λ is Lebesgue measure.

Note that if we choose $D = \emptyset$ then we recover the notion of asymptotic stability. In the examples in [9] and in this paper, the invariant set X is a heteroclinic cycle and D is a cuspoidal region abutting one or more (but not all) of the heteroclinic connections that make up the cycle. Recently, it has been shown by Alexander et al [1] and Ashwin et al [3] that flow-invariant subspaces (forced for example by symmetry) may contain essentially asymptotically stable (but not asymptotically stable) invariant sets. Here, the set D is considerably more complicated than in our examples.

Usually, an invariant set is defined to be unstable if there are trajectories that start arbitrarily close but do not remain close. In the light of our examples this definition is very weak since unstable sets can be essentially asymptotically stable. These considerations suggest the utility of a stronger notion of instability. We propose a definition that demands instability on the same scale that we demanded asymptotic stability in our definition of essential asymptotic stability.

Definition 1.2 A flow-invariant set X is *almost completely unstable* if there is a set D and an open neighborhood U of X such that for any $\epsilon > 0$ there exists an open neighborhood V of X such that

(a) no trajectory starting in $V - D$ remains in U for all forward time,

(b) $\lambda(V - D)/\lambda(V) > 1 - \epsilon$.

The set X is *completely unstable* if D is a set of measure zero (in which case condition (b) is automatic).

Standard examples of completely unstable invariant sets are provided by saddle points and sources. (We do not distinguish between saddles and sources; for a source $D = \emptyset$, for a saddle D is nonempty but of measure zero.) More generally, any invariant set that has a hyperbolic structure is either asymptotically stable or completely unstable.

Our main result in this paper is to show for a large class of heteroclinic cycles that essential asymptotic stability and almost complete instability are typically the only possibilities. It is of interest to isolate the asymptotically stable cycles (rather less to isolate the completely unstable cycles) so the cycles fall into one of the following three mutually exclusive categories

- asymptotically stable,
- unstable but essentially asymptotically stable,
- almost completely unstable.

We note that our conditions for a heteroclinic cycle to be a nonasymptotically stable attractor are an improvement on these in [9]. (Conditions (i) and (ii) in [9, Theorem 1.2] are necessary and sufficient, condition (iii) is superfluous.) In addition, the fact that there is no middle ground between essential asymptotic stability and almost complete instability for the heteroclinic cycles under consideration was not at all apparent from the treatment in [9]. Kirk and Silber [6] consider examples, some of which do not fall into the class of heteroclinic cycles described in this paper (specifically, hypothesis (S2) below is invalid) and show that such examples may lead to heteroclinic cycles that are neither essentially asymptotically stable nor almost completely unstable.

In [7], necessary and sufficient conditions for asymptotic stability were derived for heteroclinic cycles satisfying certain hypotheses ((H1)–(H3) in [7]). If the conditions for asymptotic stability fail the cycle is unstable. The conditions are stated in terms of the real parts of eigenvalues of the linearized vector field at the equilibria (or relative equilibria) on the heteroclinic cycle. In this paper, we consider the unstable case in more detail and demonstrate that the cycle is either essentially asymptotically stable or almost completely unstable. Again, conditions are given in terms of the real parts of eigenvalues.

In Section 2 we state our main results. The conditions for asymptotic stability in [7] are obtained by considering the significant components of a Poincaré map defined around the heteroclinic cycle. Essentially this reduces to the analysis of the stability properties of the origin for a class of mappings of $\mathbb{R}^2$. In Section 3 we concentrate on these stability issues when the origin is unstable. Then in Section 4 we apply our results to the examples that arise in $\mathbf{O}(2)$ mode interactions. Finally, in Section 5 we show that two of the cycles discussed in Section 4 can simultaneously exist and be essentially asymptotically stable. Moreover, for this to happen each cycle must be unstable.

2 Eigenvalue data for heteroclinic cycles

For a general vector field without symmetry, a heteroclinic cycle is necessarily structurally unstable. However, symmetry may force the flow-invariance of certain subspaces and this may permit structural stability to occur. In this section, we describe the setup surrounding such heteroclinic cycles, and define the eigenvalue data associated with these cycles. Once we have done this we are able to state our main results.

This section is similar to [7, Sections 2 and 3]. (All numbered references to [7] are to the revised version.) However there are several differences due to the fact that the interest in [7] centered exclusively around asymptotic stability. We shall emphasize these differences and refer to [7] and references therein for details on such objects as isotropy subgroups and their fixed-point spaces, isotypic decompositions, relative equilibria, normal vector fields and so on.

Let Γ be a compact Lie group acting on $\mathbb{R}^n$ and suppose that $f:\mathbb{R}^n\to\mathbb{R}^n$ is a Γ-equivariant vector field. Suppose that ξ_j, $j=1,\ldots,m$ are hyperbolic relative equilibria with stable and unstable manifolds $W^s(\xi_j)$ and $W^u(\xi_j)$. The set of group orbits of heteroclinic connections

$$X=\{W^u(\gamma\xi_j)\cap W^s(\delta\xi_{j+1}),\ j=1,\ldots,m,\ \delta,\gamma\in\Gamma\},$$

forms a *heteroclinic cycle* provided

$$(W^u(\xi_j)-\{\xi_j\})\cap W^s(\gamma\xi_{j+1})\neq\emptyset \text{ for some } \gamma\in\Gamma.$$

This is more general than the corresponding definition in [7] where we demanded that $W^u(\xi_j)-\{\xi_j\}$ be completely contained in $\bigcup_{\gamma\in\Gamma}W^s(\gamma\xi_{j+1})$. Clearly, the two definitions coincide when X is asymptotically stable.

We shall make several hypotheses which we label (S1)–(S4) to distinguish them from (H1)–(H3) and (H3)$'$ in [7]. The first hypothesis guarantees that the heteroclinic cycle is robust (cf. [7, Proposition 2.5]).

(S1) There is an isotropy subgroup Σ_j with fixed-point subspace $P_j=\mathrm{Fix}(\Sigma_j)$ such that $W^u(\xi_j)\cap P_j\subset W^s(\xi_{j+1})$ and ξ_{j+1} is a sink in P_j.

Remark 2.1 In our definition of heteroclinic cycle and in hypothesis (S1) we have relaxed the assumptions on $W^u(\xi_j)$ in [7]. It should be noted that any heteroclinic cycle that is asymptotically stable satisfies the definition of heteroclinic cycle in [7]. Also, (S1) together with asymptotic stability implies (H1) in [7]. (Kirk and Silber [6] have studied an example where (S1) fails. In their example, $W^u(\xi_j)\cap P_j$ is a one-dimensional manifold and they demand only that one branch of this manifold is contained in $W^s(\xi_{j+1})$.)

It is possible also to relax the assumption in hypothesis (S1) that ξ_{j+1} is a sink in P_j. Then the cycle could still be robust provided the intersection of $W^u(\xi_j)\cap P_j$ with $W^s(\xi_{j+1})\cap P_j$ is a transverse intersection of manifolds in P_j. However, if ξ_{j+1} is not a sink, it is clear that the cycle is almost completely unstable.

Let f_N denote the normal vector field of f defined in a neighborhood of the relative equilibria ξ_j. Choose $x_j \in \xi_j$ and consider the linearization $(df_N)_{x_j}$. The results in this paper and in [7] are stated in terms of the real parts of the eigenvalues of $(df_N)_{x_j}$, $j = 1, \ldots, m$. Recall that the real parts are independent of the choice of $x_j \in \xi_j$ (Field [4]).

As in [7] the geometry of a heteroclinic cycle satisfying hypothesis (S1) allows us to divide the eigenvalues into four classes, *radial*, *contracting*, *expanding* and *transverse*. Let $L_j = P_{j-1} \cap P_j$. We divide up the eigenvalues as follows.

Eigenvalues	Description
radial	eigenvalues in L_j
contracting	nonradial eigenvalues in P_{j-1}
expanding	nonradial eigenvalues in P_j
transverse	remaining eigenvalues

Note that the collections of radial, contracting and expanding eigenvalues are always nonempty, that the radial and contracting eigenvalues have negative real part and that there is at least one expanding eigenvalue with positive real part. Set $-c_j$, e_j and t_j to be the maximum real parts of the contracting, expanding and transverse eigenvalues respectively. Hence we have selected the weakest contracting, strongest expanding and most unstable transverse eigenvalues. If there are no transverse eigenvalues, set $t_j = -\infty$. Observe that $c_j > 0$, $e_j > 0$, $t_j \neq 0$.

Remark 2.2 In [7], the transverse eigenvalues are assumed to have negative real part, this being a necessary condition for asymptotic stability of the cycle. This assumption is relaxed here and the transverse eigenvalues may have positive real part. It is this greater generality that leads to the occurrence of nonasymptotically stable attractors.

Corresponding to each isotropy subgroup Σ_j in (S1) is the isotypic decomposition $\mathbb{R}^n = W_0 \oplus \cdots \oplus W_q$ of $\mathbb{R}^n$ into isotypic components. We may choose $W_0 = P_j$. Let $N(\Sigma_j)$ denote the normalizer of Σ_j in Γ.

(S2) the eigenspaces corresponding to c_j, t_j, e_{j+1} and t_{j+1} lie in the same Σ_j-isotypic component.

(S3) $\dim W^u(\xi_j) \cap P_j = \dim (N(\Sigma_j)/\Sigma_j) + 1$.

(S4) All transverse eigenvalues of ξ_j with positive real part lie in the same Σ_j-isotypic component.

Hypotheses (S2) is identical to hypotheses (H2) in [7]. In addition, (S3) reduces to (H3)′ in [7] when there are no transverse eigenvalues with positive real part. Of course (S4) is vacuous in the context of [7].

Set $\rho_j = \min(c_j/e_j, 1 - t_j/e_j)$ and define $\rho = \rho_1 \cdots \rho_m$. We make the nondegeneracy assumptions $c_j \neq e_j - t_j$, $t_j \neq e_j$ and $\rho \neq 1$. Then [7, Theorem 3.3] states that, provided $t_j < 0$ for each j, generically a heteroclinic cycle satisfying hypotheses (S1)–(S4) is asymptotically stable if and only if $\rho > 1$. In fact it follows easily from the proof in [7] that the cycle is completely unstable when $\rho < 1$. (Complete instability is established on cones that can be made arbitrarily wide).

In this paper, we prove the following result.

Lemma 2.3 *Suppose that X is a heteroclinic cycle satisfying hypotheses (S1)–(S4). Generically X is essentially asymptotically stable if and only if $\rho > 1$ and $t_j < e_j$, $j = 1, \ldots, m$. Otherwise X is almost completely unstable.*

Combining Lemma 2.3 with [7, Theorem 3.3] yields the following.

Theorem 2.4 *Suppose that X is a heteroclinic cycle satisfying hypotheses (S1)–(S4). Then generically the stability of X is described by precisely one of the following possibilities.*

(a) asymptotically stable ($\rho > 1$ and $t_j < 0$ for each j),

(b) unstable but essentially asymptotically stable ($\rho > 1$, $t_j < e_j$ for each j and $t_j > 0$ for some j),

(c) almost completely unstable ($\rho < 1$ or $t_j > e_j$ for some j).

Moreover, if $\rho < 1$ the cycle is completely unstable.

Remark 2.5 It follows from Theorem 2.4 that if $t_j > 0$ for each j then X is completely unstable. For then $\rho_j < 1$ for each j, and hence $\rho < 1$.

3 Stability of a class of planar maps

In [7], stability of structurally stable heteroclinic cycles under an equivariant vector field is related to the stability of certain dynamically invariant regions under a Poincaré map g. Moreover, it is shown that certain components of this Poincaré map govern the stability. Ultimately, this leads to the analysis of a rather unusual class of planar maps and an investigation of the stability of the origin under these maps. The mappings are not necessarily defined in an open neighborhood of the origin, but are defined at least on an open dense full measure subset of a neighborhood of the origin.

Suppose that X is a heteroclinic cycle satisfying hypotheses (S1)–(S4) with eigenvalue data $c_j > 0$, $e_j > 0$, and $t_j \neq 0$ for $j = 1, \ldots, m$. To prove Lemma 2.3 we may reduce as in [7] to proving a similar result about the stability of the origin under the mapping $g = g_m \circ \cdots \circ g_1$ where g_j has to lowest order the form

$$g_j(w, z) = (A_j w^{c_j/e_j} + B_j w^{-t_j/e_j} z, C_j w^{c_j/e_j} + D_j w^{-t_j/e_j} z).$$

For completeness, we describe quickly how this is arrived at, the details can be found in [7].

The map g_j represents the significant components of the part of the Poincaré map that begins and ends in neighborhoods of ξ_j and ξ_{j+1}. In the notation of [7], $g_j = \psi_j \circ \phi_j$ where ϕ_j is a first hit map defined in a small neighborhood of ξ_j and ψ_j is a diffeomorphism connecting the neighborhoods of ξ_j and ξ_{j+1}. The coordinates w and z correspond to the most expanding and largest transverse directions. Finally, the constants A_j, B_j, C_j, D_j are drawn from the linearization of ψ_j (taking only the significant components into account) and are generically nonzero by hypothesis (S2), see [7, Remark 4.11]). Hypothesis (S4) implies that we can neglect transverse directions corresponding to the other unstable transverse eigenvalues (if any).

As before we set $\rho_j = \min(c_j/e_j, 1 - t_j/e_j)$ and $\rho = \rho_1 \cdots \rho_m$. Also we assume the nondegeneracy conditions $c_j \neq e_j - t_j$, $t_j \neq e_j$ and $\rho \neq 1$. We must prove the following result.

Lemma 3.1 *Generically,* 0 *is essentially asymptotically stable under* g *if and only if* $\rho > 1$ *and* $t_j < e_j$ *for each* j. *Otherwise* 0 *is almost completely unstable under* g.

To simplify the notation it is convenient to generalize the class of mappings and allow g_j to have to lowest order the form

$$g_j(w,z) = (A_j w^{a_j} z^{b_j} + B_j w^{c_j} z^{d_j}, C_j w^{a_j} z^{b_j} + D_j w^{c_j} z^{d_j}),$$

where A_j, B_j, C_j, D_j are nonzero constants and $a_j + b_j \neq c_j + d_j$. Lemma 3.1 is then a special case of Corollary 3.3 below.

Without loss of generality we may assume that $a_j + b_j < c_j + d_j$. Accordingly, we set $\rho_j = a_j + b_j$ and $\rho = \rho_1 \cdots \rho_m$. Choose constants $\beta > \alpha > 0$ such that

$$4\alpha|A_j| \leq |C_j| \leq \frac{1}{4}\beta|A_j|, \quad 4\alpha|B_j| \leq |D_j| \leq \frac{1}{4}\beta|B_j|.$$

For $\mu > 1$ and $0 < \nu < 1$ define the *cuspoidal cone*

$$\mathcal{C}_{\mu,\nu} = \{\alpha|w|^{\mu} < |z| < \beta|w|^{\nu}\}.$$

The portion of the cuspoidal cone that lies in the first quadrant is illustrated in Figure 1.

Figure 1: The cuspoidal cone in the first quadrant

Observe that if B_ϵ is the open ball of radius ϵ and center 0 in $\mathbb{R}^2$ and λ is Lebesgue measure, then $\lambda(B_\epsilon \cap \mathcal{C}_{\mu,\nu})/\lambda(B_\epsilon) \to 1$ as $\epsilon \to 0$. (The easiest way to check this is to use square neighborhoods and explicitly integrate.) We shall take D to be the complement of such a cuspoidal cone in Definitions 1.1 and 1.2.

Lemma 3.2 *There are numbers $\mu_0 > 1$, $\nu_0 \in (0,1)$, an open neighborhood V of 0 and positive constants k, K such that for $1 < \mu < \mu_0$, $\nu_0 < \nu < 1$,*

(a) $g_j(\mathcal{C}_{\mu,\nu} \cap V) \subset \mathcal{C}_{\mu,\nu}$.

(b) If $b_j \geq 0$

$$k|w|^{a_j + b_j\mu} \leq |g_j(w,z)| \leq K|w|^{a_j+b_j\nu} \text{ for } (w,z) \in \mathcal{C}_{\mu,\nu} \cap V.$$

A similar inequality holds for $b_j < 0$ with μ and ν interchanged.

Proof For ease of notation we suppress the index j throughout the proof.
(a) For $(w,z) \in \mathcal{C}_{\mu,\nu}$, we have

$$|w^a z^b| \geq \begin{cases} \alpha^b |w|^{a+b\mu}; & b \geq 0 \\ \beta^b |w|^{a+b\nu}; & b < 0 \end{cases}$$

Similarly, we have

$$|w^c z^d| \leq \begin{cases} \beta^d |w|^{c+d\nu}; & d \geq 0 \\ \alpha^d |w|^{c+d\mu}; & d < 0 \end{cases}$$

Hence for μ and ν close enough to 1, we have $|w^c z^d| = o(|w^a z^b|)$. To lowest order, $g(w,z) = (Aw^a z^b, Cw^a z^b)$.

Write g in components, $g = (g^w, g^z)$. For (w,z) close enough to 0 we compute that

$$\alpha |g^w(w,z)|^\mu \leq 2\alpha |Aw^a z^b| \leq \frac{1}{2}|Cw^a z^b| \leq |g^z(w,z)|.$$

A similar computation shows that $|g^z(w,z)| \leq \beta |g^w(w,z)|^\nu$ so that $g(w,z) \in \mathcal{C}_{\mu,\nu}$ as required.
(b) Proceeding as in (a) we have $g(w,z) = (Aw^a z^b, Cw^a z^b)$ at lowest order. Using the estimates in (a) for $|w^a z^b|$ we take $k = \frac{1}{2}\min(|A|,|C|)\min(\alpha^b, \beta^b)$. This yields the required lower bound. A similar argument gives the upper bound. ■

Corollary 3.3 *Suppose that $\rho \neq 1$ and that $\rho_j \neq 0$ for each j. Then 0 is essentially asymptotically stable if and only $\rho > 1$ and $\rho_j > 0$ for each j. Otherwise 0 is almost completely unstable.*

Proof Choose μ and ν close enough to 1 so that parts (a) and (b) of Lemma 3.2 are valid. Part (a) of the lemma implies that points in $\mathcal{C}_{\mu,\nu}$ remain in $\mathcal{C}_{\mu,\nu}$ under iteration by g for as long as they are close to 0. Hence we may consider the stability of 0 under the mapping g restricted to $\mathcal{C}_{\mu,\nu}$. Note that asymptotic stability in $\mathcal{C}_{\mu,\nu}$ implies essential asymptotic stability in $\mathbb{R}^2$. Similarly, complete instability in $\mathcal{C}_{\mu,\nu}$ corresponds to almost complete instability in $\mathbb{R}^2$.

Suppose first that $\rho_j < 0$ for some j. Then $a_j + b_j < 0$ and we can choose μ and ν close enough to 1 so that $a_j + b_j\mu < 0$ and $a_j + b_j\nu < 0$. Then it follows from the lower estimates in Lemma 3.2(b) that for V a small enough neighborhood of 0, $g_j(V \cap \mathcal{C}_{\mu,\nu}) \cap V = \emptyset$. Hence 0 is completely unstable in $\mathcal{C}_{\mu,\nu}$.

Now suppose that $\rho_j > 0$ for each j. This time the upper estimates imply (for ν close enough to 1) that each g_j extends to a continuous map on $\overline{\mathcal{C}_{\mu,\nu}}$ with 0 a fixed point. Combining the estimates in Lemma 3.2(b) for each of the g_j we obtain an estimate for g

$$k^m |w|^{\rho+\epsilon} \leq |g(w,z)| \leq K^m |w|^{\rho-\epsilon},$$

for $(w,z) \in \mathcal{C}_{\mu,\nu}$, where $\epsilon > 0$ can be made as small as we wish by choosing μ and ν close enough to 1. It follows that g is uniformly contracting in $\mathcal{C}_{\mu,\nu}$ if $\rho > 1$ and uniformly expanding in $\mathcal{C}_{\mu,\nu}$ if $\rho < 1$. Hence 0 is asymptotically stable in $\mathcal{C}_{\mu,\nu}$ or completely unstable in $\mathcal{C}_{\mu,\nu}$ respectively as required. ∎

4 Cycles in $\mathbf{O}(2)$ mode interactions

Codimension two mode interactions with $\mathbf{O}(2)$ symmetry have provided a rich source of examples of structurally stable heteroclinic cycles, see [2], [12] and [10]. Necessary and sufficient conditions for asymptotic stability of many of these heteroclinic cycles were obtained in [7, Section 6]

For convenience we give a quick (but incomplete) description of the heteroclinic cycles that occur in $\mathbf{O}(2)$ mode interactions. There are three codimension two mode interactions.

- steady state/steady state ([2], [12])
- steady state/Hopf ([10])
- Hopf/Hopf ([10])

The cycles occur on center manifolds whose dimension is determined by the multiplicity of the eigenvalues passing through the imaginary axis. Generically these eigenvalues are simple or have multiplicity two corresponding to the absolutely irreducible representations of $\mathbf{O}(2)$. The existence of structurally stable heteroclinic cycles requires all eigenvalues to have multiplicity two and the center manifolds for the three mode interactions have dimension 4, 6 and 8 respectively.

4.1 Steady state/steady state interaction

In the steady state/steady state interaction the center manifold is four-dimensional and coordinates can be chosen on $\mathbb{C}^2$ so that the action of $\mathbf{O}(2)$ is given by

$$\theta \cdot z = (e^{ki\theta} z_1, e^{\ell i\theta} z_2),$$

$$\kappa \cdot z = (\bar{z}_1, \bar{z}_2),$$

where k and ℓ are positive coprime integers and $k \leq \ell$.

Structurally stable heteroclinic cycles occur only when $k = 1$, $\ell = 2$, and consist of two equilibria and two heteroclinic connections between these equilibria. The pair of equilibria are related by elements of the group, as are the heteroclinic connections. The equilibria have isotropy subgroup $\mathbb{D}_2$ generated by rotation through π and the reflection κ. The heteroclinic connections lie in the two-dimensional flow-invariant subspaces $\mathrm{Fix}(\kappa)$ and $\mathrm{Fix}(\pi\kappa)$. In fact, there is a continuous group orbit of such heteroclinic cycles.

Since the equilibria on the cycle are related by the group of symmetries we may view this as a cycle with $m = 1$. There are no transverse eigenvalues and the eigenvalue data consists of $c > 0$ and $e > 0$. It was shown in [7] and in [12] that generically the heteroclinic cycle is asymptotically stable if $c > e$ and unstable if $c < e$. It follows from Theorem 2.4 that when $c < e$ the cycle is completely unstable. In particular the cycle is never a nonasymptotically stable attractor.

4.2 Steady state/Hopf interaction

We can choose coordinates on the six-dimensional center manifold $\mathbb{C}^3$ so that the $\mathbf{O}(2)$ action is given by

$$\phi \cdot (z_0, z_1, z_2) = (e^{ki\phi} z_0, e^{\ell i\phi} z_1, e^{-\ell i\phi} z_2),$$

$$\kappa. \cdot (z_0, z_1, z_2) = (\bar{z}_0, z_2, z_1),$$

where k and ℓ are positive coprime integers and $k \leq \ell$. There is also an $\mathbf{SO}(2)$ phase shift symmetry

$$\theta \cdot (z_0, z_1, z_2) = (z_0, e^{i\theta} z_1, e^{i\theta} z_2).$$

Structurally stable heteroclinic cycles occur when $k = \ell = 1$.

A schematic diagram of the heteroclinic cycle is given in Figure 2. There is an equilibrium (1) lying in an invariant line and a periodic solution (2) in an invariant plane. These are joined by heteroclinic connections in the three-dimensional invariant subspaces (3) and (4). The equilibrium has eigenvalue data $c_1 > 0$, $e_1 > 0$ and $t_1 = -\infty$ (no transverse eigenvalues). The periodic solution has a single transverse eigenvalue and hence data $c_2 > 0$, $e_2 > 0$ and $t_2 \neq 0$.

Theorem 4.1 *Generically, the heteroclinic cycle in the steady state/Hopf interaction is a nonasymptotically stable attractor if and only if the following conditions are valid.*

(a) $0 < t_2 < e_2$,

(b) $c_1 \min(c_2, e_2 - t_2) > e_1 e_2$.

More precisely, the cycle is unstable yet essentially asymptotically stable if these conditions are satisfied.

If the second inequality in (a) or the inequality in (b) is reversed, the cycle is almost completely unstable.

If the first inequality in (a) is reversed, but the others maintained, the cycle is asymptotically stable.

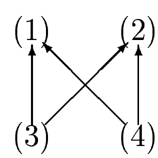

Figure 2: The cycle in the steady state/Hopf interaction

4.3 Hopf/Hopf interaction

Effectively, the symmetry group is $\mathbf{O}(2) \times T^2$, the T^2-symmetry arising from the simultaneous Hopf bifurcations. We can choose coordinates so that the action of $\mathbf{O}(2) \times T^2$ is as follows:

$$\begin{aligned} \phi \cdot z &= (e^{ik\phi} z_1, e^{-ik\phi} z_2, e^{i\ell\phi} z_3, e^{-i\ell\phi} z_4), \quad \phi \in \mathbf{SO}(2), \\ (\psi_1, \psi_2) \cdot z &= (e^{i\psi_1} z_1, e^{i\psi_1} z_2, e^{i\psi_2} z_3, e^{i\psi_2} z_4), \quad (\psi_1, \psi_2) \in T^2, \\ \kappa \cdot z &= (z_2, z_1, z_4, z_3), \end{aligned}$$

where k and ℓ are positive coprime integers and $k \leq \ell$.

Heteroclinic cycles occur for all values of k and ℓ, but the cycles for $\ell > 1$ violate hypothesis (S2), see [7]. Hence we restrict attention to the case $k = \ell = 1$.

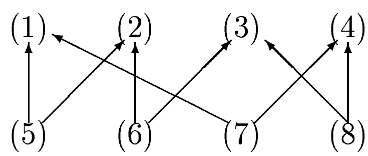

Figure 3: The cycle of rotating and standing waves in the Hopf/Hopf interaction.

There are three heteroclinic cycles to consider, one that connects a pair of rotating waves, another connecting a pair of standing waves and a third connecting all four of these solutions. Schematically, the first two cycles look like the one in the steady state/Hopf interaction, see Figure 2. The third cycle is shown schematically in Figure 3. In the notation of [10] modes (1) and (4) correspond to the rotating waves, and modes (2) and (3) to the standing waves. Each of the modes (j), $j = 1, \cdots, 4$ has a pair of transverse eigenvalues and hence data $c_j > 0$, $e_j > 0$ and $t_j \neq 0$.

Theorem 4.2 *Generically, the cycles in the Hopf/Hopf interaction are unstable but essentially asymptotically stable if and only if*

(a) $t_j < e_j$, *for each* j,

(b) $t_j > 0$ *for at least one* j,

(c) $\prod_{j=1}^{m} \min(c_j, e_j - t_j) > \prod_{j=1}^{m} e_j$ *($m = 2$ for the first two cycles and $m = 4$ for the third).*

If an inequality in (a) or (c) is reversed, then the cycle is almost completely unstable. If (a) and (c) are valid but $t_j < 0$ for each j then the cycle is asymptotically stable.

5 Bistability in the Hopf/Hopf interaction

In Section 4 we saw that there are three distinct heteroclinic cycles that may occur in the Hopf/Hopf mode interaction with $\mathbf{O}(2)$ symmetry when $k = \ell = 1$. These cycles connect the rotating waves (1) and (4), the standing waves (2) and (3), and all four periodic solutions (1)–(4) respectively. It is natural to ask whether these cycles can coexist and be asymptotically stable or at least essentially asymptotically stable together.

In fact it is clear that if the cycle connecting all four modes is even essentially asymptotically stable then the other two cycles are automatically almost completely unstable. To see this observe that if two cycles have a mode in common, then the expanding eigenvalues for one cycle are transverse eigenvalues for the other cycle and vice versa. Hence one of the cycles must have a transverse eigenvalue dominating the expanding eigenvalues so that the cycle is almost completely unstable.

The only possibility for bistability lies with the cycle joining rotating waves and the cycle joining standing waves. We show that this possibility is realized but only with both cycles being unstable (but essentially asymptotically stable).

The vector field on the eight-dimensional manifold $\mathbb{C}^4$ can be written in the form $f = (f_1, f_2, f_3, f_4)$ where

$$\begin{aligned} f_1(z) &= P^1 z_1 + R^1 z_2 z_3 \bar{z}_4, \\ f_2(z) &= f_1(z_2, z_1, z_4, z_3), \\ f_3(z) &= P^3 z_3 + R^3 z_1 \bar{z}_2 z_4, \\ f_4(z) &= f_3(z_2, z_1, z_4, z_3). \end{aligned}$$

Here, P^1, P^3, R^1 and R^3 are complex valued functions of parameters $\lambda, \mu \in \mathbb{R}$ and the invariants

$$\rho_i = z_i \bar{z}_i,\ i = 1, \dots, 4, \quad \Re\alpha, \quad \Im\alpha,$$

where $\alpha = z_1 \bar{z}_2 \bar{z}_3 z_4$. Let p^i denote the real part of P^i, and r^i the real part of R^i for $i = 1, 3$. Existence results for the heteroclinic cycles are stated in [10].

Following the treatment there we set $p^1_\mu(0) = p^3_\mu(0) = 0$. From now on we shall take it as given that the various Taylor coefficients are to be evaluated at 0. We write p^1_μ instead of $p^1_\mu(0)$ and so on. By scaling we may assume that $p^1_\lambda = p^3_\lambda = 1$.

Each of the modes (1)–(4) has four eigenvalues up to multiplicities forced by the group action. We are interested in the lowest order coefficient in the real part of each eigenvalue. These are determined by the Taylor coefficients of the real parts p^1, p^3 up to cubic order. Following [10] we list these lowest order coefficients corresponding to the mode (j) as ϵ_{ij}, $i = 1, \dots, 4$.

$$\begin{array}{llll}
\epsilon_{11} = 2p^1_{\rho_1} & \epsilon_{12} = 2(p^1_{\rho_1} + p^1_{\rho_2}) & \epsilon_{13} = 2(p^3_{\rho_3} + p^3_{\rho_4}) & \epsilon_{14} = 2p^3_{\rho_3} \\
\epsilon_{21} = p^1_{\rho_2} - p^1_{\rho_1} & \epsilon_{22} = 2(p^1_{\rho_1} - p^1_{\rho_2}) & \epsilon_{23} = 2(p^3_{\rho_3} - p^3_{\rho_4}) & \epsilon_{24} = p^3_{\rho_4} - p^3_{\rho_3} \\
\epsilon_{31} = p^3_{\rho_1} - p^1_{\rho_1} & \epsilon_{32} = p^3_{\rho_1} + p^3_{\rho_2} + r^3 - p^1_{\rho_1} - p^1_{\rho_2} & \epsilon_{33} = p^1_{\rho_3} + p^1_{\rho_4} + r^1 - p^3_{\rho_3} - p^3_{\rho_4} & \epsilon_{34} = p^1_{\rho_4} - p^3_{\rho_3} \\
\epsilon_{41} = p^3_{\rho_2} - p^1_{\rho_1} & \epsilon_{42} = p^3_{\rho_1} + p^3_{\rho_2} - r^3 - p^1_{\rho_1} - p^1_{\rho_2} & \epsilon_{43} = p^1_{\rho_3} + p^1_{\rho_4} - r^1 - p^3_{\rho_3} - p^3_{\rho_4} & \epsilon_{44} = p^1_{\rho_3} - p^3_{\rho_3}
\end{array}$$

We also define numbers ϵ_5, ϵ_6, ϵ_7 and ϵ_8 as follows.

$$\epsilon_5 = \frac{p^1_{\rho_3}}{p^3_{\rho_3}} + \frac{p^3_{\rho_1}}{p^1_{\rho_1}} \quad \epsilon_7 = \frac{p^3_{\rho_1} + p^3_{\rho_2} + r^3}{p^1_{\rho_1} + p^1_{\rho_2}} + \frac{p^1_{\rho_3} + p^1_{\rho_4} + r^1}{p^3_{\rho_3} + p^3_{\rho_4}}$$

$$\epsilon_6 = \frac{p^1_{\rho_4}}{p^3_{\rho_3}} + \frac{p^3_{\rho_2}}{p^1_{\rho_1}} \quad \epsilon_8 = \frac{p^3_{\rho_1} + p^3_{\rho_2} - r^3}{p^1_{\rho_1} + p^1_{\rho_2}} + \frac{p^1_{\rho_3} + p^1_{\rho_4} - r^1}{p^3_{\rho_3} + p^3_{\rho_4}}$$

According to [10, Theorem 4.1] the cycle between rotating waves exists provided

(i) $\epsilon_{11} < 0$, $\epsilon_{14} < 0$,

(ii) $\epsilon_{31} > 0$, $\epsilon_{34} > 0$, $\epsilon_{41} < 0$, $\epsilon_{44} < 0$,

(iii) $\epsilon_5 > -2$, $\epsilon_6 > -2$.

In our terminology, the data in (i) corresponds to the radial eigenvalues, the data in (ii) to the contracting and expanding eigenvalues. The cycle exists also if the signs in (ii) are reversed, interchanging the contracting and expanding eigenvalues. The conditions (iii) ensure boundedness of trajectories in the invariant subspaces so that the required connections are made.

We shall let c_j^R, e_j^R and t_j^R, $j = 1, 2$ denote the eigenvalue data for the cycle connecting rotating waves. Then

$$c_1^R = -\epsilon_{41},\ e_1^R = \epsilon_{31},\ t_1^R = \epsilon_{21},\ c_2^R = -\epsilon_{44},\ e_2^R = \epsilon_{34},\ t_2^R = \epsilon_{24}.$$

Similarly, by [10, Theorem 4.4] the cycle between standing waves exists provided

(i) $\epsilon_{12} < 0$, $\epsilon_{13} < 0$,

(ii) $\epsilon_{32} > 0$, $\epsilon_{43} > 0$, $\epsilon_{33} < 0$, $\epsilon_{42} < 0$,

(iii) $\epsilon_7 > -2$, $\epsilon_8 > -2$.

If c_j^S, e_j^S and t_j^S, $j = 1, 2$ denotes the eigenvalue data for the cycle connecting standing waves, we have

$$c_1^S = -\epsilon_{42},\ e_1^S = \epsilon_{32},\ t_1^S = \epsilon_{22},\ c_2^S = -\epsilon_{33},\ e_2^S = \epsilon_{43},\ t_2^S = \epsilon_{23}.$$

Observe that the transverse eigenvalues of the two cycles are related: $t_j^S = -2t_j^R$. It follows that if one of the cycles is asymptotically stable with negative transverse eigenvalues, then the other cycle has both transverse eigenvalues positive. That cycle is then almost completely unstable by Remark 2.5.

We show that the two cycles exist and are simultaneously unstable but essentially asymptotically stable for a nonempty open set of codimension two bifurcation problems with $\mathbf{O}(2)$ symmetry. Since the conditions for existence and stability are given by finitely many inequalities, it is sufficient to find one set of Taylor coefficients for which the inequalities are satisfied simultaneously. Consider the following values for the Taylor coefficients:

$$p^1_{\rho_1} = -1.0 \quad p^1_{\rho_2} = 0.0 \quad p^1_{\rho_3} = -4.0 \quad p^1_{\rho_4} = 0.0 \quad r^1 = -5.0$$

$$p^3_{\rho_1} = 5.1 \quad p^3_{\rho_2} = -7.0 \quad p^3_{\rho_3} = -1.0 \quad p^3_{\rho_4} = -2.0 \quad r^3 = 2.0$$

We compute that

$$\begin{array}{llll} \epsilon_{11} = -2.0 & \epsilon_{12} = -2.0 & \epsilon_{13} = -6.0 & \epsilon_{14} = -2.0 \\ \epsilon_5 = -1.1 & \epsilon_6 = 7.0 & \epsilon_7 = 2.9 & \epsilon_8 = 107/30 \\ c_1^R = 6.0 & c_2^R = 3.0 & c_1^S = 2.9 & c_2^S = 6.0 \\ e_1^R = 6.1 & e_2^R = 1.0 & e_1^S = 1.1 & e_2^S = 4.0 \\ t_1^R = 1.0 & t_2^R = -1.0 & t_1^S = -2.0 & t_2^S = 2.0 \end{array}$$

Note that conditions (i)–(iii) for existence of each cycle are satisfied. Moreover, each cycle has an unstable transverse eigenvalue (t_1^R and t_2^S) and hence is unstable. Finally it is easily checked that the condition $\rho > 1$ for essential asymptotic stability in Theorem 2.4 is satisfied for the two cycles.

Acknowledgments

The research of I.M. was supported in part by NSF Grant DMS-9101836 and by the Texas Advanced Research Program (003652037). Both authors would like to thank the Fields Institute for their support and hospitality during visits in November 1992 and February 1993.

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