Asymptotic Stability of Heteroclinic Cycles in Systems with Symmetry, II

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Abstract

Systems possessing symmetries often admit *robust* heteroclinic cycles that persist under perturbations that respect the symmetry. In previous work, we began a systematic investigation into the asymptotic stability of such cycles. In particular, we found a sufficient condition for asymptotic stability, and we gave algebraic criteria for deciding when this condition is also necessary. These criteria are satisfied for cycles in $\mathbb{R}^3$.

Field & Swift and Hofbauer considered examples in $\mathbb{R}^4$ for which our sufficient condition for stability is not optimal. They obtained necessary and sufficient conditions for asymptotic stability using a *transition matrix* technique.

In this paper, we combine our previous methods with the transition matrix technique and obtain necessary and sufficient conditions for asymptotic stability for a larger class of heteroclinic cycles. In particular, we obtain a complete theory for "simple" heteroclinic cycles in $\mathbb{R}^4$ (thereby proving and extending results for homoclinic cycles that were stated without proof by Chossat, Krupa, Melbourne and Scheel). A partial classification of simple heteroclinic cycles in $\mathbb{R}^4$ is also given. Finally, our stability results generalise naturally to higher dimensions and many of the higher-dimensional examples in the literature are covered by this theory.

1 Introduction

Heteroclinic cycles connecting equilibria are atypical for general vector fields. However, dos Reis [6] and Field [7] have shown that heteroclinic cycles can occur robustly in symmetric systems (that is, the heteroclinic cycle persists under small perturbation of the vector field, provided the perturbations are also symmetric). The example of Guckenheimer & Holmes [10], based on rotating convection models analyzed by Busse & Heikes [3], gave a major impetus to the study of heteroclinic cycles in bifurcation theory. Since this paper of [10], several authors have exploited symmetry to compute examples of robust heteroclinic cycles, see the review article [15] and [5, 8] for further references. See also [12] for examples in population dynamics.

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Many of the heteroclinic cycles in the above references can be asymptotically stable. Then the cycles lead to interesting phenomena such as intermittency and bursting in the dynamics. In previous work [16], we gave a sufficient condition for asymptotic stability of robust heteroclinic cycles based on the relative magnitudes of the real parts of certain eigenvalues at each equilibrium along the cycle. The condition takes the form

$$\prod_{j=1}^{m} \min(c_j, e_j - t_j) > \prod_{j=1}^{m} e_j, \tag{1.1}$$

where the quantities $c_j, e_j > 0$, $t_j < 0$ correspond to 'contracting', 'expanding' and 'transverse' eigenvalues respectively. For a certain class of heteroclinic cycles, including cycles in $\mathbb{R}^3$, the condition (1.1) is necessary as well as sufficient [16].

Nevertheless, examples of Field & Swift [9] and Hofbauer [11] show that even in $\mathbb{R}^4$ there are cycles for which condition (1.1) is not optimal. They studied asymptotic stability using a technique based on *transition matrices.* The necessary and sufficient conditions for asymptotic stability that they obtained are quite different from condition (1.1).

In this paper, we combine our previous methods with the transition matrix technique and obtain necessary and sufficient conditions for asymptotic stability for a larger class of heteroclinic cycles. We begin by considering *simple* robust cycles in $\mathbb{R}^4$ (with heteroclinic connections lying in two-dimensional planes). These cycles can be divided into three classes. Roughly speaking, cycles of Type A are those studied in [16], whereas cycles of Type B lie within a flow-invariant subspace and reduce to a cycle of Type A within this subspace. The methods in [16] suffice for cycles of Type A and B. The conditions for stability of Type C cycles are complicated and nonintuitive, but are readily computable via the transition matrix method [9, 11, 12].

The definition of Type A cycle generalises naturally to higher dimensions. Indeed, this is precisely the class of cycles for which condition (1.1) was shown to be optimal in [16]. In this paper, we show that Type B and Type C cycles generalise to higher-dimensions in such a way that the transition matrix method gives optimal conditions for asymptotic stability.

The remainder of the paper is organised as follows. In Section 2, we recall the set up in [16] and the main results therein. In Section 3, we consider heteroclinic cycles in $\mathbb{R}^4$. *Simple* cycles in $\mathbb{R}^4$ are defined in Subsection 3(a) and divided into Types A, B and C. In Subsection 3(b), we classify the simple cycles of Types B and C in $\mathbb{R}^4$. As a byproduct of the classification, we obtain in Subsection 3(c) an alternative characterization of Types A, B and C which is the 'correct' definition for theoretical purposes. In Section 4, we obtain optimal conditions for asymptotic stability of simple cycles in $\mathbb{R}^4$. In Section 5, we generalise our results to higher-dimensional robust heteroclinic cycles, and to continuous symmetry groups. In Section 6, we consider examples that occur in codimension two mode-interactions with $\mathbf{O}(2)$ symmetry.

As the title of this paper suggests, this work follows up on the previous paper [16]. For the most part, this paper can be read independently. However, some of the proofs in Section 5 rely on technical results from [16]. In these situations, we sketch the ideas but refer to the appropriate parts of [16] for details.

Some of the results derived in this paper are quoted without proof in [16, p. 143] and [4].

2 Robust heteroclinic cycles

In this section we recall the notion of a robust heteroclinic cycle [15]. Suppose that $\Gamma \subset \mathbf{O}(n)$ is a finite Lie group acting linearly on $\mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Γ-equivariant vector field. That is, $f(\gamma x) = \gamma f(x)$, for all $\gamma \in \Gamma$.

Suppose that f has hyperbolic saddle points $\xi_1, \xi_2, \ldots, \xi_m$, where to avoid redundancies we assume that the group orbits $\Gamma\xi_j$, $j = 1, \ldots, m$ are distinct. Let $W^s(\xi_j)$ and $W^u(\xi_j)$ denote the stable and unstable manifolds of ξ_j. We assume that

$$W^u(\xi_j) - \{\xi_j\} \subset \textstyle\bigcup_{\gamma\in\Gamma} W^s(\gamma\xi_{j+1}), \tag{2.1}$$

for each j (where $j+1$ is computed mod m). Define the *heteroclinic cycle* $X = \bigcup_{\gamma\in\Gamma}\bigcup_{j=1}^m \gamma W^u(\xi_j)$. Condition (2.1) ensures that X is a heteroclinic *cycle* rather than a heteroclinic *network*. See [2, 13] for results on networks.

Suppose that $\Sigma \subset \Gamma$ is a subgroup, and define its *fixed-point subspace*

$$\operatorname{Fix}\Sigma = \{x \in \mathbb{R}^n : \sigma x = x \text{ for all } \sigma \in \Sigma\}.$$

Any Γ-equivariant vector field on $\mathbb{R}^n$ maps $\operatorname{Fix}\Sigma$ into itself and hence $\operatorname{Fix}\Sigma$ is flow-invariant.

Definition 2.1 The cycle X is a *robust heteroclinic cycle* if for each $j = 1, \ldots, m$, there is a fixed-point subspace $P_j = \operatorname{Fix}\Sigma_j$ where $\Sigma_j \subset \Gamma$, such that

(i) ξ_{j+1} is a sink in P_j, and

(ii) $W^u(\xi_j) \subset P_j$.

Remark 2.2 (a) Conditions (i) and (ii) were called (H1) in [16]. The j'th heteroclinic connection X_j is a structurally stable saddle-sink connection from ξ_j to ξ_{j+1} inside of P_j. It follows that the cycle X persists under Γ-equivariant perturbations of the vector field. Note that there is no restriction on the dimension of P_j.
(b) In [17], we considered a more general class of heteroclinic cycles. Such cycles may have strong stability properties even when they are not asymptotically stable. However, any cycle in [17] that is asymptotically stable necessarily satisfies the definition given above.

Recall that if $x \in \mathbb{R}^n$, the *isotropy subgroup* of x is the subgroup $\Sigma_x \subset \Gamma$ defined by $\Sigma_x = \{\gamma \in \Gamma : \gamma x = x\}$. Without loss of generality, we may assume in our definition of robust heteroclinic cycle that the subgroups Σ_j are isotropy subgroups.

Eigenvalue data Our conditions for asymptotic stability depend on the magnitudes of the real parts of certain eigenvalues of the linearisation of the vector field f at each equilibrium. The geometry allows us to divide the eigenvalues of $(df)_{\xi_j}$ into four classes:

eigenvalue class	subspace
radial (r)	$L_j = P_{j-1} \cap P_j$
contracting (c)	$V_j(c) = P_{j-1} \ominus L_j$
expanding (e)	$V_j(e) = P_j \ominus L_j$
transverse (t)	$V_j(t) = (P_{j-1} + P_j)^\perp$.

Here, $P \ominus L$ denotes the orthogonal complement in P of the subspace L.

By construction, the radial, contracting and transverse eigenvalues have negative real part, and there is at least one expanding eigenvalue with positive real part. (We refer to all the eigenvalues corresponding to the subspace $V_j(e) = P_j \ominus L_j$ as expanding eigenvalues even though some of these may have negative real part.)

A basic result of [16] states that asymptotic stability of robust heteroclinic cycles is independent of the radial eigenvalues. Associated to the remaining eigenvalues, we define $c_j, e_j > 0$ and $t_j < 0$ as follows: Let $-c_j$ be the maximum real part of the contracting eigenvalues. Thus $c_j > 0$ corresponds to the weakest contracting eigenvalue at ξ_j. Let $e_j > 0$ to be the maximum real part of an eigenvalue of $(df)_{\xi_j}$; the strongest *expanding* eigenvalue. Finally, let t_j be the maximum real part of transverse eigenvalues; the weakest transverse eigenvalue. If $\mathbb{R}^n = P_{j-1} + P_j$, then set $t_j = -\infty$.

Isotypic decomposition Let $\Sigma \subset \Gamma$ be an isotropy subgroup. Recall that $\mathbb{R}^n$ can be written as a direct sum of Σ-irreducible subspaces $\mathbb{R}^n = V_0 \oplus \cdots \oplus V_p$. Some of the V_i may be Σ-isomorphic, that is they carry isomorphic representations of Σ. Group together the isomorphic representations to obtain the unique *isotypic decomposition* $\mathbb{R}^n = W_0 \oplus \cdots \oplus W_q$ where each *isotypic components* W_j is a direct sum of irreducible subspaces, and two irreducible subspaces are contained in the same W_j if and only if they are isomorphic. We may choose $W_0 = \mathrm{Fix}\,\Sigma$.

Since distinct isotypic components carry nonisomorphic representations of Σ, any linear map L commuting with the action of Σ satisfies $L(W_j) \subset W_j$. If $\xi_j \in \mathrm{Fix}\,\Sigma$, then the linearisation $(df)_{\xi_j}$ commutes with Σ. It follows that generically each generalised eigenspace corresponding to a nonzero eigenvalue lies in a single isotypic component of Σ.

Previous results on asymptotic stability We can now recall the main results in [16]. Suppose that X is a robust heteroclinic cycle as in Definition 2.1. Recall that the heteroclinic connections $W^u(\xi_j)$ lie in fixed-point subspaces $P_j = \mathrm{Fix}\,\Sigma_j$ of certain isotropy subgroups $\Sigma_j \subset \Gamma$. By construction, the eigenspaces corresponding to e_j and c_{j+1} lie inside $\mathrm{Fix}\,\Sigma_j$. The eigenspaces corresponding to c_j, t_j, e_{j+1} and t_{j+1} lie in $(\mathrm{Fix}\,\Sigma_j)^\perp$.

Theorem 2.3 ([16]) *Suppose that $\Gamma \subset \mathbf{O}(n)$ is a finite group and that $X \subset \mathbb{R}^n$ is a robust heteroclinic cycle.*

(a) If condition (1.1) *is satisfied, then X is asymptotically stable.*

(b) Suppose further that for each $j \geq 1$, (i) $\dim W^u(\xi_j) = 1$, and (ii) the eigenspaces corresponding to c_j, t_j, e_{j+1} and t_{j+1} lie in the same Σ_j-isotypic component.

Then, generically, condition (1.1) *is necessary and sufficient for asymptotic stability.*

Remark 2.4 (a) Conditions (i) and (ii) in part (b) correspond to (H3) and (H2) in [16]. They are automatically satisfied when $n = 3$.
(b) It is always the case that P_j is an isotypic component for Σ_j. Hence if each isotypic

decomposition of $\mathbb{R}^n$ under Σ_j consists of two isotypic components, then condition (ii) is valid. This is certainly the case if $\Sigma_j = \mathbb{Z}_2$ or $\Sigma_j = \mathbb{Z}_3$.

3 Simple robust heteroclinic cycles in $\mathbb{R}^4$

In this section, we concentrate on a class of 'simple' robust cycles in $\mathbb{R}^4$, where the heteroclinic connections are assumed to lie in two-dimensional fixed-point subspaces. In Subsection (a), we define the class of simple cycles and divide them into three types: A, B and C. The cycles of Type B and C are enumerated in Subsection (b). In Subsection (c), we give a local characterization of the three types.

(a) Cycles of Type A, B and C

Assume that $\Gamma \subset \mathbf{O}(4)$ is a finite group acting on $\mathbb{R}^4$ and that $X \subset \mathbb{R}^4$ is a robust heteroclinic cycle as defined in Definition 2.1. Thus, for each $j = 1, \ldots, m$, the heteroclinic connection from ξ_j to ξ_{j+1} is a saddle-sink connection in a fixed-point subspace $P_j = \operatorname{Fix} \Sigma_j$ where $\Sigma_j \subset \Gamma$ is an isotropy subgroup. Recall the notation $L_j = P_{j-1} \cap P_j$.

We say that X is a *simple robust heteroclinic cycle* if $X \subset \mathbb{R}^4 - \{0\}$, and

(i) $\dim P_j = 2$ for each j, and

(ii) X intersects each connected component of $L_j - \{0\}$ in at most one point.

Clearly, each L_j is one-dimensional. Moreover, the eigenvalues of $(df)_{\xi_j}$ are real. Indeed, there is a unique eigenvalue of each type: radial $-r_j$, contracting $-c_j$, expanding e_j, and transverse t_j. The corresponding eigenvectors span the subspaces L_j, $V_j(c) = P_{j-1} \ominus L_j$, $V_j(e) = P_j \ominus L_j$ and $V_j(t) = (P_{j-1} + P_j)^{\perp}$ respectively. Moreover,

$$\mathbb{R}^4 = L_j \oplus V_j(c) \oplus V_j(e) \oplus V_j(t), \tag{3.1}$$

is the isotypic decomposition of $\mathbb{R}^4$ under the isotropy subgroup T_j of points in $L_j - \{0\}$. (Since T_j contains Σ_{j-1} and Σ_j, and $L_j = \operatorname{Fix} T_j$, $P_{j-1} = \operatorname{Fix} \Sigma_{j-1}$, $P_j = \operatorname{Fix} \Sigma_j$.) Note that the orthogonality of $V_j(c)$ and $V_j(e)$ means that successive planes P_{j-1} and P_j intersect orthogonally (in the obvious sense).

An immediate consequence of decomposition (3.1) is that $T_j \subset \mathbb{Z}_2^3$ where in the coordinates (3.1), $\mathbb{Z}_2^3$ consists of diagonal matrices with entries $\{1, \pm 1, \pm 1, \pm 1\}$.

Proposition 3.1 *Either $T_j \cong \mathbb{Z}_2^2$ and $\Sigma_j \cong \mathbb{Z}_2$ for all j, or $T_j \cong \mathbb{Z}_2^3$ and $\Sigma_j \cong \mathbb{Z}_2^2$ for all j.*

Proof Fix j and write $T_j \subset \mathbb{Z}_2^3$. It is easy to see that the only possibilities compatible with the constraints $\dim \operatorname{Fix} T_j = 1$ and $\dim \operatorname{Fix} \Sigma_j = 2$ are either $T_j \cong \mathbb{Z}_2^2$ and $\Sigma_j \cong \mathbb{Z}_2$ or $T_j \cong \mathbb{Z}_2^3$ and $\Sigma_j \cong \mathbb{Z}_2^2$. The same observation applies to the inclusion $\Sigma_{j-1} \subset T_j$, yielding the required result. ■

Definition 3.2 (cf. [4]) Let $X \subset \mathbb{R}^4$ be a simple robust heteroclinic cycle.

X is of *Type A* if $\Sigma_j = \mathbb{Z}_2$ for all j.

X is of *Type B* if there is a fixed-point subspace Q with $\dim Q = 3$ such that $X \subset Q$.

X is of *Type C* if it is not of Type A nor of Type B.

Note that if the cycle is of Type B, then the fixed-point subspace Q contains P_j and hence corresponds to a proper subgroup of Σ_j, so that $\Sigma_j \neq \mathbb{Z}_2$. It follows that the three types in Definition 3.2 are mutually exclusive. Types B and C correspond to the second possibility in Proposition 3.1.

Remark 3.3 Condition (i) of Theorem 2.3(b) is automatically satisfied for simple cycles in $\mathbb{R}^4$. Condition (ii) corresponds to Type A. In particular, condition (1.1) is a necessary and sufficient condition for asymptotic stability for cycles of Type A.

If X is of Type B, then X lies in the reflection hyperplane Q and it follows from [16, Corollary 4.8] that the stability is determined by the stability within Q. Restricting to the three-dimensional fixed-point subspace Q, the hypotheses of Theorem 2.3(b) are automatically satisfied and (1.1) is optimal. As there are no transverse eigenvalues in Q ($t_j = -\infty$ for all j) condition (1.1) simplifies to $\prod_{j=1}^m c_j > \prod_{j=1}^m e_j$ for cycles of Type B.

Proposition 3.4 *Let $X \subset \mathbb{R}^4$ be a simple robust heteroclinic cycle. All one-dimensional fixed-point subspaces in P_j are conjugate to L_j or L_{j+1}.*

Proof Certainly, L_j and L_{j+1} are invariant lines in P_j. If $L_j = L_{j+1}$, then it follows from the definition of simple cycle that ξ_j and ξ_{j+1} lie in distinct components of $L_j - \{0\}$. Since the j'th heteroclinic connection consists of points of isotropy precisely Σ_j, it follows that L_j is the only invariant line in P_j.

Similar considerations show that if $L_j \neq L_{j+1}$, then these are 'adjacent' lines in P_j. Choose elements $\kappa \in T_j$, $\kappa' \in T_{j+1}$ acting as reflections on P_j with fixed-point subspace L_j and L_{j+1} respectively. Their action on P_j generates a dihedral group $\mathbb{D}_n \subset \mathbf{O}(2)$ with n equally-spaced invariant lines $\mathcal{L}$ all conjugate to L_j and L_{j+1}. Moreover, $P_j - \mathcal{L}$ consists of $2n$ connected components each of which is a fundamental domain for the action of $\mathbb{D}_n$. It follows that each successive pair of half-lines in $\mathcal{L}$ is connected by a heteroclinic connection consisting of points with isotropy Σ_j and hence there are no further invariant lines in P_j. ■

Corollary 3.5 *Let $X \subset \mathbb{R}^4$ be a simple robust heteroclinic cycle. Then X is of Type A if and only if there are no elements of Γ that act as reflections on $\mathbb{R}^4$.*

Proof First note that T_1 is generated by reflections if X is of Type B or C and contains no reflections if X is of Type A. In particular, if X is not of Type A, then Γ contains reflections.

Conversely, suppose that $\tau \in \Gamma$ is a reflection, and set $E = \mathrm{Fix}\,\tau \cap P_1$. Then E is a line or a plane. If E is a plane, then τ fixes all points in P_1 and hence $\tau \in \Sigma_1 \subset T_1$. Otherwise,

E is an invariant line in P_1 and it follows from Proposition 3.4 that E is conjugate to L_1 or L_2. Hence, we have shown that T_1 or T_2 contains a reflection and so X is not of Type A. ∎

Theorem 3.6 *Suppose that $X \subset \mathbb{R}^4$ is a simple robust cycle of Type B or Type C.*

If $-I \notin \Gamma$, then $L_j = L_{j+1}$ for all j.

If $-I \in \Gamma$, then L_j and L_{j+1} are orthogonal lines in P_j for all j.

In particular, L_j and L_{j+1} are the only one-dimensional fixed-point subspaces in P_j.

Proof We claim that for any fixed value of j, either $L_j = L_{j+1}$ or L_j is orthogonal to L_{j+1} in P_j. Suppose that L_j and L_{j+1} are distinct lines in P_j. As in the proof of Proposition 3.4, choose elements $\kappa \in T_j$, $\kappa' \in T_{j+1}$ acting as reflections on P_j with fixed-point subspace L_j and L_{j+1} respectively. Multiplying by elements in $\Sigma_j = \mathbb{Z}_2^2$, we can choose κ, κ' to be reflections in Γ (so they act trivially on $P_j^\perp$). With respect to the coordinates $P_j \oplus P_j^\perp$, we can write $\kappa\kappa' = R \oplus I_2$ where $R \in \mathbf{SO}(2)$. Since $L_j \neq L_{j+1}$ it follows that $R \neq I$ and so $P_j^\perp = \mathrm{Fix}(\kappa\kappa')$. Hence $P_j^\perp \cap P_{j-1}$ is an invariant subspace. But $P_j^\perp = V_j(c) \oplus V_j(t)$ in the decomposition (3.1), whereas $P_{j-1} = L_j \oplus V_j(c)$, so $P_j^\perp \cap P_{j-1}$ is a line in P_{j-1}. Combining these facts, we have shown that $P_j^\perp \cap P_{j-1}$ is an invariant line in P_{j-1} and hence, by Proposition 3.4, conjugate to either L_{j-1} or L_j. Moreover, $\kappa\kappa'$ fixes points in $P_j^\perp \cap P_{j-1}$ and so is conjugate to an element of T_{j-1} or T_j. But these are isomorphic to $\mathbb{Z}_2^3$ and it follows that $\kappa\kappa'$ has order two. That is $\kappa\kappa' = (-I_2) \oplus I_2$. This means (as in the proof of Proposition 3.4) that there are two invariant lines in P_j, completing the proof of the claim.

The proof of the claim shows also that if $L_j \neq L_{j+1}$ then $-I \in \Gamma$ (since $\kappa\kappa' = (-I_2) \oplus I_2$ and Σ_j contains $I_2 \oplus (-I_2)$). The converse is also true (otherwise the components of $L_j - \{0\}$ cannot be adjacent) so that the two possibilities in the claim are distinguished by whether or not $-I \in \Gamma$. It is immediate that the situation is identical for all j. ∎

Let R denote the normal subgroup of Γ generated by reflections. We have already seen (Corollary 3.5) that $R = \mathbf{1}$ if and only if X is of Type A.

Corollary 3.7 *Suppose that $X \subset \mathbb{R}^4$ is a simple robust cycle of Type B or Type C. Then either $R = \mathbb{Z}_2^3$ ($-I \notin \Gamma$) or $R = \mathbb{Z}_2^4$ ($-I \in \Gamma$).*

Proof The proof of Corollary 3.5 shows that reflections in Γ lie in the isotropy subgroups of invariant lines in P_1. By Theorem 3.6, L_1 and L_2 are the only invariant lines in P_1. Hence R is contained in the subgroup generated by T_1 and T_2. Moreover, each T_j is generated by reflections, so R is the group generated by T_1 and T_2. If $L_1 = L_2$ we have $R = T_1 = \mathbb{Z}_2^3$. Otherwise, L_1 is orthogonal to L_2 and we have $R = \mathbb{Z}_2^4$. ∎

(b) Enumeration of simple cycles of Types B and C

We now have sufficient information to list the possible cycles of Types B and C. It turns out that there are seven such cycles, denoted by $B_m^{\pm}$ and $C_m^{\pm}$ where the first letter denotes the type of the cycle, and the superscript $\pm$ indicates whether $-I \in \Gamma$ $(-)$ or $-I \notin \Gamma$ $(+)$. The subscript m indicates, as usual, the order of the cycle.

In this notation, there are four cycles of Type B and three cycles of Type C:

$$B_1^+, \quad B_2^+, \quad B_1^-, \quad B_3^-; \qquad C_1^-, \quad C_2^-, \quad C_4^-. \tag{3.2}$$

First, we show that these are the only possible cycles of Type B and C. Without loss, we may suppose that $P_1 = \{(x_1, x_2, 0, 0)\}$ and $P_2 = \{(0, x_2, x_3, 0)\}$. If $-I \notin \Gamma$, then $L = \{(0, x_2, 0, 0)\}$ is the only invariant line in P_1 and in P_2. The definition of simple cycle implies that there is a single equilibrium in each component of $L - \{0\}$. Label these equilibria ξ_1, ξ_2. There are connections from ξ_1 to ξ_2 in P_1 and from ξ_2 to ξ_1 in P_2. In particular, the cycle closes up after precisely two connections. Clearly, the cycle lies in the reflection hyperplane $\{x_4 = 0\}$ and hence is of Type B. The cycle is either 1-heteroclinic or 2-heteroclinic depending on whether or not ξ_1 is conjugate to ξ_2. These are the cycles B_1^+ and B_2^+.

If $-I \in \Gamma$, then it follows from Corollary 3.7 that we may choose coordinates so that each L_j is a coordinate axis and each P_j is a coordinate plane. Without loss, L_j is the x_j-axis for $j = 1, 2, 3$. There are now two possibilities: either (i) L_4 is the x_1-axis, or (ii) L_4 is the x_4-axis. In case (i), the cycle closes up after three connections (so the cycle is either 1-heteroclinic or 3-heteroclinic) and connects equilibria in the x_1, x_2 and x_3-axes. These cycles lie in the reflection hyperplane $\{x_4 = 0\}$ and have the form B_1^- and B_3^-. In case (ii), the cycle closes up after four connections (so the cycle is either 1, 2 or 4-heteroclinic) and connects equilibria in the x_1, x_2, x_3 and x_4-axes. Clearly, the cycle does not lie in a coordinate hyperplane. By Corollary 3.7, there are no other reflection hyperplanes, so we deduce that the cycle is of Type C and have the form C_1^-, C_2^- and C_4^-. It follows that the list (3.2) is complete.

Next, we show that each of the cycles can be realised for a finite group $\Gamma \subset \mathbf{O}(4)$.

The cycle B_2^+: Take $\Gamma = \mathbb{Z}_2^3$ consisting of the diagonal matrices with entries $\{1, \pm 1, \pm 1, \pm 1\}$. The fixed-point subspaces consist of the x_1-axis, the planes $\{(x_1, x_2, 0, 0)\}$, $\{x_1, 0, x_3, 0\}$, $\{x_1, 0, 0, x_4\}$, and the reflection hyperplanes $\{x_2 = 0\}$, $\{x_3 = 0\}$, $\{x_4 = 0\}$. We can arrange that there is a simple robust cycle connecting equilibria $\xi_1 = (1, 0, 0, 0)$ and $\xi_2 = (-1, 0, 0, 0)$ with $P_1 = \{x_1, x_2, 0, 0\}$ and $P_2 = \{x_1, 0, x_3, 0\}$ say. (We can choose P_1 and P_2 to be any pair of two-dimensional fixed-point subspaces.) Clearly, the cycle is of Type B and $-I \notin \Gamma$. Since the isotropy subgroup of ξ_1 and ξ_2 is the whole of Γ, there are no symmetries mapping ξ_1 to ξ_2 and so $m = 2$.

The cycle B_1^+: Take $\Gamma = \mathbb{Z}_2 \ltimes \mathbb{Z}_2^3$ where $\mathbb{Z}_2^3$ is as before and $\mathbb{Z}_2$ is generated by

$$(x_1, x_2, x_3, x_4) \mapsto (-x_1, x_3, x_2, x_4). \tag{3.3}$$

It is easily verified that the B_2^+ cycle is present as before, with the exception that the symmetry (3.3) interchanges ξ_1 and ξ_2 so that $m = 1$. This cycle is similar to [1].

The cycles B_3^- and C_4^-: Take $\Gamma = \mathbb{Z}_2^4$ consisting of the diagonal matrices with entries $\{\pm 1, \pm 1, \pm 1, \pm 1\}$. The fixed-point subspaces are precisely the coordinates subspaces. There is a number of cycles of Type B connecting equilibria in any three coordinate axes, and of Type C connecting equilibria in all four coordinate axes (in any order). These examples arise in quadruple Hopf bifurcation in systems without symmetry (generalising the triple Hopf bifurcation studied in [18]).

The cycle B_1^-: Take $\Gamma = \mathbb{Z}_3 \ltimes \mathbb{Z}_2^4$ where $\mathbb{Z}_3$ is generated by $(x_1, x_2, x_3, x_4) \mapsto (x_2, x_3, x_1, x_4)$. Let $\xi_1 = (1, 0, 0, 0)$, $\xi_2 = (0, 1, 0, 0)$, $\xi_3 = (0, 0, 1, 0)$. This cycle is similar to [10].

The cycle C_4^-: Take $\Gamma = \mathbb{Z}_4 \ltimes \mathbb{Z}_2^4$ where $\mathbb{Z}_4$ is generated by $(x_1, x_2, x_3, x_4) \mapsto (x_2, x_3, x_4, x_1)$. Let $\xi_1 = (1, 0, 0, 0)$, $\xi_2 = (0, 1, 0, 0)$, $\xi_3 = (0, 0, 1, 0)$, $\xi_4 = (0, 0, 0, 1)$. See [9].

The cycle C_2^-: Take $\Gamma = \mathbb{Z}_2 \ltimes \mathbb{Z}_2^4$ where $\mathbb{Z}_2$ is generated by $(x_1, x_2, x_3, x_4) \mapsto (x_3, x_4, x_1, x_2)$. Let $\xi_1 = (1, 0, 0, 0)$, $\xi_2 = (0, 1, 0, 0)$, $\xi_3 = (0, 0, 1, 0)$, $\xi_4 = (0, 0, 0, 1)$.

Remark 3.8 In the appendix, we show that there are no finite groups $\Gamma \subset \mathbf{O}(4)$ that admit simple cycles of Types B and C other than the six groups listed above. So we have a complete classification of the seven simple cycles of Types B and C and also a complete classification of the six group actions that admit these cycles.

Remark 3.9 The enumeration of Type A cycles is considerably more complicated, although we have some partial results. However, recently a complete and efficient classification of *homoclinic* cycles of Type A (with $m = 1$) has been obtained by Sottocornola [21, 23] using Galois-theoretic techniques. It is anticipated that jointly with Sottocornola and using the new techniques in [21, 22, 23] it will be possible also to completely classify the type A heteroclinic cycles in $\mathbb{R}^4$.

(c) Local characterization of Types A, B and C

The classification of simple cycles $X \subset \mathbb{R}^4$ into the Types A, B and C needs to be reformulated to obtain results on asymptotic stability. It turns out to be important to focus attention on certain local objects — the individual heteroclinic connections — rather than on the global cycle X. It is purely an artifact of the geometry of four-dimensional Lie group actions that the global characterization was possible.

Define the three-dimensional subspace $Q_j = P_{j-1} + P_j$. There is the question as to whether or not Q_j is a reflection hyperplane, that is, whether or not $Q_j = \operatorname{Fix} \tau_j$ for some reflection $\tau_j \in \Gamma$.

Definition 3.10 Let $X \subset \mathbb{R}^4$ be a simple robust heteroclinic cycle with heteroclinic connections $X_j = W^u(\xi_j)$, $j = 1, \ldots, m$.

The connection X_j is of *Type A* if Q_j is not a reflection hyperplane.

The connection X_j is of *Type B* if Q_j is a reflection hyperplane and $P_{j+1} \subset Q_j$.

The connection X_j is of *Type C* if Q_j is a reflection hyperplane and $P_{j+1} \not\subset Q_j$.

Theorem 3.11 *Let $X \subset \mathbb{R}^4$ be a simple robust heteroclinic cycle.*

The cycle is of Type A if and only if each connection is of Type A.

The cycle is of Type B if and only if each connection is of Type B.

The cycle is of Type C if and only if each connection is of Type C.

Proof If X is of Type A, then there are no reflection hyperplanes by Corollary 3.5, so each connection is of Type A. On the other hand, if X is of Type B or Type C, then $\Sigma_{j-1} \cap \Sigma_j = \mathbb{Z}_2$ and $Q_j = \mathrm{Fix}(\Sigma_{j-1} \cap \Sigma_j)$ and so each connection is of Type B or Type C.

The Type A statement follows immediately. The remaining statements follow from the classification of cycles of Types B and C in Subsection (b). ∎

When X is of Type B or C, the isotropy subgroup $\Sigma_j \cong \mathbb{Z}_2^2$ is generated by reflections. Hence there are two reflection hyperplanes containing P_j: the subspace Q_j and a second subspace which we denote by R_j. When X is of Type B, Q_j contains the directions c_j and e_{j+1} (in fact $X \subset Q_j$) and R_j contains the directions t_j and t_{j+1}. When X is of Type C, Q_j contains the directions c_j and t_{j+1} and R_j contains the directions t_j and e_{j+1}.

4 Asymptotic stability of simple cycles in $\mathbb{R}^4$

In this section, we complete the analysis of asymptotic stability for simple cycles $X \subset \mathbb{R}^4$. By Remark 3.3, it remains to compute asymptotic stability only for cycles of Type C. This is done in Subsection (c) below. First, in Subsections (a) and (b), we recall material on Poincaré maps [16] and transition matrices [9, 11, 12].

(a) Poincaré maps

General material on Poincaré maps for heteroclinic cycles can be found in [16]. For convenience, we recall this material in the specific setting of simple cycles in $\mathbb{R}^4$.

Let $X \subset \mathbb{R}^4$ be a simple robust heteroclinic cycle for the equivariant vector field $f : \mathbb{R}^4 \to \mathbb{R}^4$. We begin by linearizing f in a neighborhood of each equilibrium $\xi_1, \dots, \xi_m$. In Section 2, we used the geometry of the heteroclinic cycle to define the radial, contracting, expanding and transverse eigenvalues of the linearisations $(df)_{\xi_j}$. Since the cycle is simple, there is a unique eigenvalue of each type. In the region of linearised flow, we introduce local coordinates (u, v, w, z) around ξ_j corresponding to these four directions.

We may assume that the unit cube $\{|u|, |v|, |w|, |z| \le 1\}$ lies within the region of linearised flow. The connection leaving ξ_j lies in the subspace $\{u = v = z = 0\}$ and so we define the cross-section

$$H_j^{(out)} = \{(u, v, w, z) : |u|, |v|, |z| \le 1,\ w = 1\}.$$

The connection approaching ξ_j lies in the subspace P_{j-1} which is coordinatised locally by u and v. We define the cross-section

$$H_j^{(in)} = \{(u, v, w, z) : |u|^2 + |v|^2 = 1,\ |w|, |z| \leq 1\}.$$

Now define the *first hit maps* $\phi_j : H_j^{(in)} \to H_j^{(out)}$, and the *connecting diffeomorphisms* $\psi_j : H_j^{(out)} \to H_{j+1}^{(in)}$. Then define $g_j = \psi_j \circ \phi_j : H_j^{(in)} \to H_{j+1}^{(in)}$. Finally, define the Poincaré map $g = g_m \circ \cdots \circ g_1 : H_1^{(in)} \to H_1^{(in)}$.

As shown in [16], it is only the w and z-components of each map g_j that are significant. We recall the computation. The first hit map ϕ_j has the form

$$\phi_j(u, v, w, z) = (uw^{r_j/e_j}, vw^{c_j/e_j}, 1, zw^{-t_j/e_j}).$$

The cycle intersects $H_i^{(in)}$ at some point $(u_0, v_0, 0, 0)$ where $u_0^2 + v_0^2 = 1$. Generically, $u_0, v_0 \neq 0$ and at lowest order, $\phi_j(u, v, w, z) = (u_0 w^{r_j/e_j}, v_0 w^{c_j/e_j}, 1, zw^{-t_j/e_j})$. It follows that when computing ψ_{j-1}, it is only the w and z components that are significant.

Next, we consider the connecting diffeomorphisms ψ_j. Recall that ψ_j is Σ_j-equivariant and hence $\psi_j\left(P_j \cap H_j^{(out)}\right) \subset P_j \cap H_{j+1}^{(in)}$. In other words, the u and w-coordinates near $H_j^{(out)}$ are mapped onto the u and v-coordinates near $H_j^{(in)}$. It follows that $\psi_j^w(u, 0, w, 0) = \psi_j^z(u, 0, w, 0) = 0$. Hence, at lowest order

$$\psi_j^w(u, v, w, z) = \alpha_{j1} v + \alpha_{j2} z, \quad \psi_j^z(u, v, w, z) = \alpha_{j3} v + \alpha_{j4} z,$$

where $\alpha_{jk} \in \mathbb{R}$. Incorporating v_0 into the constants α_{jk} the w and z-components of g_j have at lowest order the form

$$g_j^w(u, v, w, z) = \alpha_{j1} w^{c_j/e_j} + \alpha_{j2} z w^{-t_j/e_j}, \qquad g_j^z(u, v, w, z) = \alpha_{j3} w^{c_j/e_j} + \alpha_{j4} z w^{-t_j/e_j}.$$

These expressions are independent of u and v. Hence we often view g_j as a map $g_j : \mathbb{R}^2 \to \mathbb{R}^2$ and write

$$g_j(w, z) = (\alpha_{j1} w^{c_j/e_j} + \alpha_{j2} z w^{-t_j/e_j},\ \alpha_{j3} w^{c_j/e_j} + \alpha_{j4} z w^{-t_j/e_j}). \tag{4.1}$$

Since ψ_j is a diffeomorphism, $\alpha_{j1}\alpha_{j4} - \alpha_{j2}\alpha_{j3} \neq 0$. By [16, §4.4], ψ_j can be considered as a general diffeomorphism that commutes with the action of Σ_j.

Proposition 4.1 *(i) If ξ_j is of Type A, there are no further restrictions on the α_{jk}. For example, generically $\alpha_{jk} \neq 0$.*

(ii) If ξ_j is of Type B, $\alpha_{j2} = \alpha_{j3} = 0$ (and $\alpha_{j1}, \alpha_{j4} \neq 0$). Moreover,

$$g_j(w, z) = (\alpha_{j1} w^{c_j/e_j} + o(w^{c_j/e_j}), \alpha_{j4} z w^{-t_j/e_j} + o(z w^{-t_j/e_j})).$$

(iii) If ξ_j is of Type C, $\alpha_{j1} = \alpha_{j4} = 0$ (and $\alpha_{j2}, \alpha_{j3} \neq 0$). Moreover,

$$g_j(w,z) = (\alpha_{j2} z w^{-t_j/e_j} + o(z w^{-t_j/e_j}), \alpha_{j3} w^{c_j/e_j} + o(w^{c_j/e_j})).$$

Proof When ξ_j is of Type A, the subspace $P_j^\perp$ is an isotypic component for the action of Σ_j and hence any matrix commutes with Σ_j. Thus there are no restrictions on the linearisation of ψ_j (except for invertibility) proving part (i).

For cycles of Types B and C, $P_j^\perp$ breaks up into two one-dimensional isotypic components; hence there are additional restrictions on ψ_j at linear order. In fact, there are restrictions at all orders due to the presence of the two three-dimensional invariant subspaces Q_j and R_j. If X is of Type B, the v-coordinate near $H_j^{(out)}$ is mapped to the w-coordinate near $H_{j+1}^{(in)}$. Similarly, ψ_j maps z to z. It follows that $\psi_j^w(0,0,0,z) = \psi_j^z(0,v,0,0) = 0$. In particular, $\alpha_{j2} = \alpha_{j3} = 0$. If ξ_j is of Type C, then ψ maps v to z and z to w so that $\alpha_{j1} = \alpha_{j4} = 0$. ■

(b) Transition matrices

Next we introduce the transition matrices of [9, 11, 12] and obtain a compact notation for the maps $g_j : H_j^{(in)} \to H_{j+1}^{(in)}$ whenever X is of Type B or Type C. Let Y denote the set of mappings $h : \mathbb{R}^2 \to \mathbb{R}^2$ that have at lowest order the form $h(w,z) = (Ew^a z^b, Fw^c z^d)$ for some constants $a,b,c,d \geq 0$ and nonzero constants E, F. Observe that Y is closed under composition of maps. We define the *transition matrix* of h to be the 2×2 matrix with nonnegative entries $M(h) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. It is easily verified that if $h_1, h_2 \in Y$, then $M(h_2 \circ h_1) = M(h_2)M(h_1)$. We have the following corollary to Proposition 4.1.

Corollary 4.2 *If X is of Type B, then the transition matrix $M_j = M_j(g_j)$ is given by $M_j = \begin{pmatrix} c_j/e_j & 0 \\ -t_j/e_j & 1 \end{pmatrix}$. If X is of Type C, then $M_j = \begin{pmatrix} -t_j/e_j & 1 \\ c_j/e_j & 0 \end{pmatrix}$.*

If X is of Type B or Type C then $g_j \in Y$ for each j. Moreover, $g \in Y$ and we can compute the transition matrix $M = M(g) = M_m \cdots M_2 M_1$ for g. In particular, at lowest order $g(w,z) = (Ew^a z^b, Fw^c z^d)$ where E and F are nonzero constants and the exponents a,b,c,d are the entries of the matrix $M(g)$.

When X is of Type B, $M = \begin{pmatrix} \rho & 0 \\ * & 1 \end{pmatrix}$, where $\rho = \prod_{j=1}^m c_j/e_j$ and $*$ is some positive number. The form of M is less clear for a cycle X of Type C, but it is evident that the entries are nonnegative and strictly positive with the possible exception of the bottom-right entry (which is zero when $m = 1$). The determinant of M is given by $(-1)^m \rho$.

(c) Asymptotic stability of simple cycles of Type C

In this subsection we obtain necessary and sufficient conditions for cycles of Type C to be asymptotically stable.

Theorem 4.3 *Suppose that $X \subset \mathbb{R}^4$ is a simple robust heteroclinic cycle of Type C. Then generically X is asymptotically stable if and only if*

$$\operatorname{tr} M > \min(2, 1 + \det M), \tag{4.2}$$

where $M = M_m \cdots M_2 M_1$ and $M_j = \begin{pmatrix} -t_j/e_j & 1 \\ c_j/e_j & 0 \end{pmatrix}$.

Proof (cf. [9, Remark 5.4]) Let $a_k, b_k, c_k, d_k \geq 0$ denote the entries of M^k. Then at lowest order, $g^k(w,z) = (E_k w^{a_k} z^{b_k}, F_k w^{c_k} z^{d_k})$, where $E_k, F_k \neq 0$ are constants. It follows that if the row sums $a_k + b_k$ and $c_k + d_k$ both diverge to infinity, then the cycle is asymptotically stable. Conversely, if the row sums converge to zero, then the cycle is unstable.

Note that the off-diagonal entries of M are nonzero. It follows from the Perron-Frobenius theory of irreducible nonnegative matrices that M has real eigenvalues $\lambda_\pm$ with $\lambda_+ > |\lambda_-|$ and that the eigenvector v_+ corresponding to λ_+ has strictly positive entries.

If $\lambda_+ < 1$, then M is a contraction and $M^k \to 0$ as $k \to \infty$. It follows that the cycle is unstable. Conversely, if $\lambda_+ > 1$ then $|M^k v_+| \to \infty$. Since both components of v_+ are nonzero, it follows that both row sums $a_k + b_k$, $c_k + d_k$ diverge to infinity and the cycle is asymptotically stable.

The eigenvalues of the M are given by $\lambda_\pm = \frac{1}{2}\{\operatorname{tr} M \pm \sqrt{(\operatorname{tr} M)^2 - 4 \det M}\}$. It is easily verified that the condition $\lambda_+ > 1$ is equivalent to condition (4.2). ■

In the case of homoclinic cycles ($m = 1$), condition (4.2) simplifies to $c_1 - t_1 > e_1$. This is the condition Field & Swift [9] derived for the cycle C_1^-.

Next, consider the cycle C_2^- so $m = 2$. Define $C_j = c_j/e_j$ and $T_j = t_j/e_j$. Then $\det M = C_1 C_2$ and $\operatorname{tr} M = C_1 + C_2 + T_1 T_1$. Hence the condition for stability is

$$C_1 + C_2 + T_1 T_1 > \min\{2, 1 + C_1 C_2\}.$$

Similarly, the condition for stability of the cycle C_4^- is

$$C_1C_3 + C_2C_4 + T_1T_2C_3 + T_2T_3C_4 + T_3T_4C_1 + T_4T_1C_2 + T_1T_2T_3T_4 > \min\{2, 1 + C_1C_2C_3C_4\}.$$

5 Higher-dimensional robust heteroclinic cycles

In this section, the aim is to define a large class of higher-dimensional robust heteroclinic cycles for which optimal asymptotic stability results are available. One result in this direction was already obtained in [16], see Theorem 2.3(b). Indeed the isotypic decomposition condition in Theorem 2.3(b) serves as a definition of *Type A* cycle in higher dimensions. Here, we are concerned with higher-dimensional analogues for cycles of Type B and Type C. A major difference from Section 3(c) is that the heteroclinic connections need not all be of the same type. Roughly, a cycle is Type B if all connections are of Type B, and a cycle is of Type C if at least one connection is of Type C and the remaining connections are of Type B.

Technical difficulties arise due to the fact that the contracting, expanding and transverse eigenvalues are not necessarily unique (or real) in higher-dimensions. In addition, we now allow the symmetry group Γ to be any compact Lie group, and generalise from equilibria ξ_j to *relative equilibria*. These technical difficulties are dealt with just as they were in [16], but they complicate the definitions.

Background on continuous symmetry groups Let $\Gamma \subset \mathbf{O}(n)$ be a compact Lie group and suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth Γ-equivariant vector field. Let $X \subset \mathbb{R}^n$ be a robust heteroclinic cycle connecting hyperbolic relative equilibria ξ_j with connections in fixed-point subspaces $P_j = \operatorname{Fix}\Sigma_j$. Using results of [7, 14], we can speak of the real parts of eigenvalues at each ξ_j (see [16, Section 3]). The eigenvalues are again divided into radial, contracting, expanding and transverse eigenvalues, and we define $c_j, e_j > 0$, $t_j < 0$ (possibly $t_j = -\infty$) in exactly the same way as we did in Section 2. With these definitions, Theorem 2.3 holds for continuous symmetry groups. Moreover, condition (i) can be weakened to $\dim W^u(\xi_j) = \dim N(\Sigma_j)/\Sigma_j + 1$ where $N(\Sigma_j)$ is the normaliser of Σ_j in Γ. See [16, Theorem 3.1].

(a) Generalisation of cycles of Type B to higher dimensions

There are two natural and distinct ways to generalise the definition of Type B cycles. The first, adopted in [16], is to require that X lies in a proper fixed-point subspace $Q \subset \mathbb{R}^n$ and to relate the stability of X in $\mathbb{R}^n$ to the stability of X in Q. The second method is to adopt a local approach along the lines of Subsection 3(c). This is what we do now.

Recall that $V_j(t) = (P_{j-1} \oplus P_j)^\perp$ denotes the sum of the generalised eigenspaces corresponding to transverse eigenvalues at ξ_j.

Definition 5.1 Suppose that $X \subset \mathbb{R}^n$ is a robust heteroclinic cycle. The cycle is of *Type B* if for each j, there is a fixed-point subspace R_j such that

$$R_j = P_j \oplus V_j(t) = P_j \oplus V_{j+1}(t).$$

Theorem 5.2 *A sufficient condition for asymptotic stability of the Type B cycle $X \subset \mathbb{R}^n$ is that*

$$\textstyle\prod_{j=1}^m c_j > \prod_{j=1}^m e_j. \tag{5.1}$$

Proof For simplicity, we assume that the linearisations at each relative equilibrium are semisimple. Then by a standard argument (cf. [16, p. 135]) there is a constant $K > 0$ such that at lowest order, $g_j = \psi_j \circ \phi_j$ satisfies

$$|g_j^w(y)| \le K(|w|^{c_j/e_j} + |w|^{-t_j/e_j}|z|), \qquad |g_j^z(y)| \le K(|w|^{c_j/e_j} + |w|^{-t_j/e_j}|z|),$$

for all $y = (u, v, w, z) \in H_j^{(in)}$ near the heteroclinic cycle. (If the linearisations are not semisimple, then an $\epsilon > 0$ is introduced into these estimates, but as in [16, p. 135] ϵ can be chosen sufficiently small that the results are not affected.)

Restricting to the subspace R_j, we find that $\psi_j^w(u, 0, w, z) = 0$, so that $\psi_j^w(u, v, w, z) = A(y)v + o(v)$. Hence, at lowest order, $|g_j^w(y)| \leq K_j|w|^{c_j/e_j}$. Hence the Poincaré map $g = g_m \circ \cdots \circ g_1$ satisfies $|g^w(y)| \leq K|w|^\rho$ where $\rho = \prod_{j=1}^m c_j / \prod_{j=1}^m e_j > 1$. Asymptotic stability follows easily from the contraction of the w-coordinates. (See [16, Theorem 4.7] for details.) ■

Theorem 5.3 *Let $X \subset \mathbb{R}^n$ be a robust heteroclinic cycle of Type B, with subspaces R_j as in Definition 5.1. Suppose that for each j*

(i) $\dim W^u(\xi_j) = \dim N(\Sigma_j)/\Sigma_j + 1$, *and*

(ii) There exists a Σ_j-isotypic component $\widetilde{Q}_j$ such that the eigenvectors corresponding to c_j and e_{j+1} lie in $\widetilde{Q}_j$.

Then, generically, condition (5.1) *is necessary and sufficient for asymptotic stability.*

Proof It suffices to show that the lowest order term w^ρ (now expanding since $\rho < 1$) in the proof of Theorem 5.2 is present in the w-component of the Poincaré map — at least most of the time. Condition (ii) guarantees that there is no algebraic obstruction to the presence of such terms, and the remainder of the proof is similar to (and slightly simpler than) the proof in [16, Section 5.3]. ■

(b) Generalisation of cycles of Type C to higher dimensions

Recall that $V_j(c)$, $V_j(e)$ and $V_j(t)$ denote the sums of the generalised eigenspaces corresponding to contracting, expanding and transverse eigenvalues at ξ_j.

Definition 5.4 Suppose that $X \subset \mathbb{R}^n$ is a robust heteroclinic cycle.

The j'th connection is of *Type B* if there are fixed-point subspaces Q_j, R_j such that

$$Q_j = P_j \oplus V_j(c) = P_j \oplus V_{j+1}(e) \text{ and } R_j = P_j \oplus V_j(t) = P_j \oplus V_{j+1}(t).$$

The j'th connection is of *Type C* if there are fixed-point subspaces Q_j, R_j such that

$$Q_j = P_j \oplus V_j(c) = P_j \oplus V_{j+1}(t) \text{ and } R_j = P_j \oplus V_j(t) = P_j \oplus V_{j+1}(e).$$

Definition 5.5 A robust heteroclinic cycle $X \subset \mathbb{R}^n$ is of *Type C* if each connection is of Type B or Type C, and at least one connection is of Type C.

Remark 5.6 Clearly, if each connection is of Type B, then the cycle is of Type B, but our definition of Type B cycle is less restrictive.

Suppose that $X \subset \mathbb{R}^n$ is a robust heteroclinic cycle of Type C. Depending on whether the j'th connection is of Type B or C, we define the transition matrix M_j to be

$$M_j = \begin{pmatrix} c_j/e_j & 0 \\ -t_j/e_j & 1 \end{pmatrix} \quad \text{or} \quad M_j = \begin{pmatrix} -t_j/e_j & 1 \\ c_j/e_j & 0 \end{pmatrix},$$

respectively. Form the product $M = M_m \cdots M_2 M_1$.

Theorem 5.7 *A sufficient condition for asymptotic stability of the Type C cycle $X \subset \mathbb{R}^n$ is that*

$$\operatorname{tr} M > \min(2, 1 + \det M). \tag{5.2}$$

Proof We begin as in the proof of Theorem 5.2. Suppose that the j'th connection is of Type B. Restricting to the subspace R_j as before, we find that $\psi_j^w(u,0,w,z) = 0$, so that $\psi_j^w(u,v,w,z) = A(y)v + o(v)$. Restricting to the subspace Q_j, we have that $\psi_j^z(u,v,w,z) = D(y)z + o(z)$. Hence, at lowest order

$$|g_j^w(y)| \le K|w|^{c_j/e_j}, \qquad |g_j^z(y)| \le K|w|^{-t_j/e_j}|z|.$$

It follows that g_j is dominated by a map with transition matrix $M_j = \begin{pmatrix} c_j/e_j & 0 \\ -t_j/e_j & 1 \end{pmatrix}$.

Similarly, if the j'th connection is of Type C, then

$$|g_j^w(y)| \le K|w|^{-t_j/e_j}|z|, \qquad |g_j^z(y)| \le K|w|^{c_j/e_j},$$

so that g_j is dominated by a map with transition matrix $M_j = \begin{pmatrix} -t_j/e_j & 1 \\ c_j/e_j & 0 \end{pmatrix}$.

The Poincaré map $g = g_m \circ \cdots \circ g_1$ is dominated by a map with transition matrix $M = M_m \cdots M_2 M_1$. Since at least one connection is of Type C, the off-diagonal entries of M are nonzero, so the proof of Theorem 4.3 applies. ■

Theorem 5.8 *Let $X \subset \mathbb{R}^n$ be a robust heteroclinic cycle of Type C, with subspaces Q_j, R_j as in Definition 5.4. Suppose that for each j*

(i) $\dim W^u(\xi_j) = \dim N(\Sigma_j)/\Sigma_j + 1$, *and*

(ii) There exist Σ_j-isotypic components $\widetilde{Q}_j \subset Q_j$ and $\widetilde{R}_j \subset R_j$ such that either (Type B connection) the eigenvectors for c_j, e_{j+1} lie in $\widetilde{Q}_j$ and those for t_j, t_{j+1} lie in $\widetilde{R}_j$, or (Type C connection) the eigenvectors for c_j, t_{j+1} lie in $\widetilde{Q}_j$ and those for t_j, e_{j+1} lie in $\widetilde{R}_j$.

Then, generically, condition (5.2) *is necessary and sufficient for asymptotic stability.*

Proof In the proof of Theorem 5.7, we showed that the Poincaré map g is dominated by a map $\widetilde{g} \in Y$ with transition matrix M. It follows from the proof of Theorem 4.3 that generically condition (5.2) is necessary and sufficient for asymptotic stability of the origin for the map $\widetilde{g}$. Hence, it remains to show that the terms in $\widetilde{g}$ are present in g — at least most of the time. Condition (ii) guarantees that there is no algebraic obstruction to the presence of such terms, and the remainder of the proof is again similar to, and slightly simpler than, the proof in [16, Section 5.3]. ■

6 Mode interactions with $\mathbf{O}(2)$ symmetry

Codimension two mode interactions in systems with $\mathbf{O}(2)$ symmetry provide a rich supply of robust heteroclinic cycles between equilibria and/or periodic solutions [1, 20, 19]. as shown in [16], many of these turn out to be 'Type A cycles' for which the condition (1.1) is optimal by Theorem 2.3(b). However, certain cycles that occur in the Hopf/Hopf mode-interaction [19] do not fall into this category, and we investigate their stability in the section. It turns out that all but one of these cycles is of Type C.

Recall that codimension two Hopf/Hopf bifurcation occurs when a steady-state loses stability by having two pairs of complex eigenvalues of the linearised equation simultaneously pass through the imaginary axis, at $\pm\omega_1 i$, $\pm\omega_2 i$, where ω_1/ω_2 is irrational. Irreducible representations of $\mathbf{O}(2)$ are either one or two-dimensional, and the eigenvalues of the linearised equation generically have multiplicity one or two. It follows that the center manifold for the Hopf/Hopf mode interaction is generically of dimension four, six or eight.

It turns out that robust heteroclinic cycles occur only when all eigenvalues are double, and there is an eight-dimensional center manifold $\mathbb{R}^8 \cong \mathbb{C}^4$ with the symmetry group $\mathbf{O}(2) \times T^2$. The T^2-symmetry is a normal form symmetry, arising from the simultaneous Hopf bifurcations and is present through arbitrarily high order. We can choose coordinates $z = (z_1, z_2, z_3, z_4)$ so that the action of $\mathbf{O}(2) \times T^2$ is as follows:

$$\begin{aligned} \phi \cdot z &= (e^{i\ell\phi} z_1, e^{-i\ell\phi} z_2, e^{im\phi} z_3, e^{-im\phi} z_4), \quad \phi \in \mathbf{SO}(2), \\ (\psi_1, \psi_2) \cdot z &= (e^{i\psi_1} z_1, e^{i\psi_1} z_2, e^{i\psi_2} z_3, e^{i\psi_2} z_4), \quad (\psi_1, \psi_2) \in T^2, \\ \kappa \cdot z &= (z_2, z_1, z_4, z_3), \end{aligned}$$

where ℓ and m are positive coprime integers and $\ell \le m$. The robust heteroclinic cycles that arise when $\ell = m = 1$ all satisfy the hypotheses of Theorem 2.3(b) so that condition (1.1) is optimal, see [16]. Hence we concentrate on the case $\ell < m$.

Following [19], we define the subgroups $S(k, \ell, m) = \{(k\theta, \ell\theta, m\theta) \in \mathbf{SO}(2) \times T^2 : \theta \in S^1\}$. There is a robust heteroclinic cycle connecting rotating waves with isotropy subgroups

(1) and (4). The relevant isotropy subgroups together with their fixed-point subspaces are:

	Isotropy subgroup	Fixed-point subspace
(1)	$S(0,0,1) \times S(1,-\ell,0)$	$(z_1,0,0,0)$
(4)	$S(0,1,0) \times S(1,0,m)$	$(0,0,0,z_4)$
(7)	$S(1,\ell,m)$	$(0,z_2,0,z_4)$
(8)	$S(1,\ell,-m)$	$(0,z_2,z_3,0)$

Each of the rotating wave solutions has a single zero eigenvalue (due to the continuous symmetry) and one radial eigenvalue. The remaining eigenvalues (contracting, expanding and transverse) are of multiplicity two due to continuous symmetries that preserve the relevant fixed-point subspaces. It is immediate that condition (i) in Theorem 5.8 is satisfied.

The case $1 < \ell < m$ We show that generically the cycle is asymptotically stable if and only if

$$C_1 + C_2 + T_1 T_2 > \min(2, 1 + C_1 C_2), \tag{6.1}$$

where $C_j = c_j/e_j$ and $T_j = t_j/e_j$. (This is the same condition as for the cycle C_2^- in $\mathbb{R}^4$.)

The isotropy subgroup (7) $= S(1,l,m)$ acts as $\theta \cdot z = (e^{2il\theta}z_1, z_2, e^{2im\theta}z_3, z_4)$, with four-dimensional fixed-point subspace $\{(0,z_2,0,z_4)\}$. The remaining isotypic components are two-dimensional, and we obtain the isotypic decomposition

$$\mathbb{R}^4 = \{(0,z_2,0,z_4)\} \oplus \{(z_1,0,0,0)\} \oplus \{(0,0,z_3,0)\}.$$

The isotypic decomposition under (8) is similar.

Taking $\theta = 2\pi/\ell$ and $\theta = 2\pi/m$, we find that the six-dimensional subspaces $\{(z_1,z_2,0,z_4)\}$ and $\{(0,z_2,z_3,z_4)\}$ are fixed-point subspaces. Similarly, $\{(z_1,0,z_3,z_4)\}$ and $\{(z_1,z_2,z_3,0)\}$ are fixed-point subspaces.

We now verify the stability condition (6.1). Since the action of κ interchanges z_1 with z_2 and z_3 with z_4, it is clear that rotating wave (1) has representatives in the z_1- and z_2-axes, and rotating wave (4) has representatives in the z_3- and z_4-axes. We may suppose that

$$P_m = \{(z_1,0,0,z_4)\}, \quad P_1 = \{(z_1,0,z_3,0)\}, \quad P_2 = \{(0,z_2,z_3,0)\}, \quad P_3 = \{(0,z_2,0,z_4)\},$$

respectively. We claim that the connections in P_1 and P_2 are of Type C and that condition (ii) in Theorem 5.8 is satisfied (we have already checked condition (i)). The result follows.

We give the details for the connection in P_1. We have the following identifications:

ξ_1	c_1	z_4	t_1	z_2
ξ_2	e_2	z_2	t_2	z_4

The fixed-point subspaces $Q_1 = \{(z_1,0,z_3,z_4)\}$ and $R_1 = \{(z_1,z_2,z_3,0)\}$ satisfy the criteria in Definition 5.4 for the connection to be of Type C. Moreover, $\widetilde{Q}_1 = \{(0,0,0,z_4)\}$ and $\widetilde{R}_1 = \{(0,z_2,0,0)\}$ are isotypic components for Σ_1 (which is conjugate to (7)) and contain c_1, t_2 and t_1, e_2 respectively, so that condition (ii) in Theorem 5.8 is satisfied. This completes the proof.

The case $\ell = 1$, $m > 1$ When $\ell = 1$, the subspaces $\{z_1, z_2, z_3, 0\}$, $\{z_1, z_2, 0, z_4\}$ are not fixed-point subspaces. (The other subspaces $\{(z_1, 0, z_3, z_4)\}$ and $\{(0, z_2, z_3, z_4)\}$ are fixed-point subspaces since $m \geq 2$.) This means that certain restrictions in the lowest order terms of the Poincaré map $g = g_2 \circ g_1$ are not present to all orders and certain 'nonlinear' terms must be included.

Due to the flow-invariant subspaces Q_1 and R_2, ψ_1^v has a factor of z and ψ_2^z has a factor of v. There are no flow-invariant subspaces Q_2 and R_1, but the isotypic decomposition of Σ_j (which is unchanged from the case $1 < \ell < m$) means that the restrictions on the linear terms are still present. Hence, there is no z term in ψ_1^z and no v term in ψ_2^v. Nevertheless, terms of the form z^m and v^m appear at high order and the connecting diffeomorphisms are given by

$$\psi_1(v,z) = (\alpha_{12}z + o(z), \alpha_{13}v + \alpha_{14}z^m + o(v, z^m)),$$
$$\psi_2(v,z) = (\alpha_{21}v^m + \alpha_{22}z + o(v^m, z), \alpha_{23}v + o(v)),$$

where generically $\alpha_{ij} \neq 0$. Hence the maps $g_j = \psi_j \circ \phi_j$ are given at lowest order by

$$g_1(w,z) = (\beta_{12}w^{-T_1}z, \beta_{13}w^{C_1} + \beta_{14}w^{-T_1 m}z^m), \quad g_2(w,z) = (\beta_{21}w^{C_2 m} + \beta_{22}w^{-T_2}z, \beta_{23}w^{C_2}),$$

where $C_j = c_j/e_j$, $T_j = t_j/e_j$, and generically $\beta_{ij} \neq 0$.

We conclude that the Poincaré map $g = g_2 \circ g_1$ is given at lowest order by

$$g(w,z) = (\gamma_1 w^{-T_1 C_2 m} z^{C_2 m} + \gamma_2 w^{T_1 T_2 + C_1} z^{-T_2} + \gamma_3 w^{T_1 T_2 - T_1 m} z^{-T_2 + m}, \delta w^{-T_1 C_2} z^{C_2}),$$

where generically $\gamma_j, \delta \neq 0$.

The stability results described in this paper do not apply to a Poincaré map of this form. Hence, we postpone discussions of stability of this heteroclinic cycle, and related classes of cycles, to future work currently in progress.

A Groups that admit simple cycles of Type B and C in $\mathbb{R}^4$

In this appendix, we verify that the finite subgroups of $\mathbf{O}(4)$ listed in Section 3(b) are the only ones that admit simple robust heteroclinic cycles of Types B and C.

Theorem A.1 *Suppose that $R = \mathbb{Z}_2^3$. Then $\Gamma = \mathbb{Z}_2^3$ or $\Gamma = \mathbb{Z}_2 \ltimes \mathbb{Z}_2^3$.*

Proof Represent $R = T_1$ as the set of matrices with diagonal elements $\{1, \pm 1, \pm 1, \pm 1\}$. We search for finite groups $\Gamma \subset \mathbf{O}(4)$ such that R is a normal subgroup of Γ and is the isotropy subgroup of the point $(1,0,0,0)$.

The signature of elements of $\mathbb{Z}_2^3$ is preserved under conjugation by orthogonal matrices. Hence the normality condition implies that every element of Γ commutes with the diagonal matrix with entries $\{1, -1, -1, -1\}$. Such elements have the form $\{\pm 1, A\} = \begin{pmatrix} \pm 1 & 0 \\ 0 & A \end{pmatrix}$

where $A \in \mathbf{O}(3)$. Moreover A lies in the normaliser $N_{\mathbf{O}(3)}(\mathbb{Z}_2^3) = S_3 \ltimes \mathbb{Z}_2^3$ of $\mathbb{Z}_2^3$ inside of $\mathbf{O}(3)$. Hence $N_{\mathbf{O}(4)}(R) \cong (\mathbb{Z}_2 \oplus S_3) \ltimes \mathbb{Z}_2^3$. It follows that $\Gamma = \Delta \ltimes \mathbb{Z}_2^3$ where $\Delta \subset \mathbb{Z}_2 \oplus S_3$.

Since $R = \mathbb{Z}_2^3 \subset \Gamma$ is the isotropy subgroup of $(1,0,0,0)$, it follows that $\Delta \cap S_3 = \mathbf{1}$. Hence $\Delta = \mathbf{1}$ or $\Delta \cong \mathbb{Z}_2$. In the latter case, the nontrivial element of Δ is $\{-1, A\}$ where $A \subset S_3$ is a transposition. The three possible order 16 subgroups $\Gamma = \mathbf{O}(4)$ obtained in this manner are conjugate inside of $\mathbf{O}(4)$. ∎

Theorem A.2 *Suppose that* $R = \mathbb{Z}_2^4$. *Then* $\Gamma = \mathbb{Z}_p \ltimes \mathbb{Z}_2^4$ *where* $p = 1, 2, 3, 4$.

Proof We have $\mathbb{Z}_2^4 \subset \Gamma \subset N(\mathbb{Z}_2^4) = S_4 \ltimes \mathbb{Z}_2^4$. Let $\Delta = \Gamma \cap S_4$ so that $\Gamma = \Delta \ltimes \mathbb{Z}_2^4$. We have the possibilities $\Delta = S_4, A_4, \mathbb{D}_4, \mathbb{D}_3, \mathbb{D}_2, \mathbb{Z}_4, \mathbb{Z}_3, \mathbb{Z}_2, 1$. We claim that Δ contains at most one element of order two. This rules out all but the cyclic subgroups of S_4 as required.

First note that Δ contains no reflections, hence no transpositions. The only remaining elements of order two in S_4 are $(12)(34)$, $(13)(24)$ and $(14)(23)$.

Since $-I \in \Gamma$, each of the coordinate axes are invariant in the plane $P_1 = \{(x_1, x_2, 0, 0)\}$. Moreover, by Proposition 3.4, these are the only invariant lines in P_1. It follows that $(12)(34) \notin \Gamma$. Finally, $(12)(34)$ is the product of $(13)(24)$ and $(14)(23)$, proving the claim. ∎

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