Hidden Symmetries on Partially Unbounded Domains *

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To the memory of John David Crawford

Abstract

Systems of reaction-diffusion equations posed on bounded rectangular domains with Neumann boundary conditions often exhibit behavior that seems degenerate given the physical symmetries of the problem. It is now well-understood that Neumann boundary conditions lead to hidden symmetries that are responsible for subtle changes in the generic bifurcations of such systems.

In this article, we consider the analogous situation for partially unbounded domains such as the strip $\mathbb{R} \times [0, \pi]$. We show that hidden symmetries due to assuming Neumann boundary conditions have remarkable consequences for the validity of Ginzburg-Landau equations which govern the local bifurcations. A single Ginzburg-Landau equation (which is universal for general boundary conditions on $\mathbb{R} \times [0, \pi]$) no longer suffices in general. Instead, it is necessary to consider p coupled Ginzburg-Landau equations, where p is an arbitrary positive integer.

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1 Introduction

Consider a system of reaction-diffusion equations $u_t = \mathcal{F}(u)$ posed on a bounded domain $\Omega \subset \mathbb{R}^n$. Suppose that Neumann boundary conditions (NBC) are imposed on the boundary of Ω. The equations themselves are invariant under the Euclidean group $\mathbf{E}(n)$, so the physical symmetries of the equations on Ω are given by the compact subgroup $\Gamma \subset \mathbf{E}(n)$ that preserves the domain Ω.

For certain domains, it turns out that there are 'hidden symmetries' in $\mathbf{E}(n) - \Gamma$ that have important, though subtle, consequences for the generic bifurcations of the system of reaction-diffusion equations. The simplest case is $n = 1$, $\Omega = [0, \pi]$. The group of physical symmetries is $\Gamma = \mathbb{Z}_2$ generated by the reflection $x \mapsto \pi - x$. However, every solution that satisfies NBC on $[0, \pi]$ extends to a 2π-periodic solution on $\mathbb{R}$ so that the NBC problem embeds in a problem with full $\mathbf{E}(1)$ symmetry. The effects of hidden translations in $\mathbf{E}(1) - \mathbb{Z}_2$ are documented in Fujii *et al.* [11] and formalized in Armbruster and Dangelmayr [1]. These ideas were developed by Crawford, Golubitsky, Gomes, Knobloch and Stewart [8] and were expanded upon in a series of articles. See the recent review article of Gomes *et al.* [16], and references therein.

The effects of hidden symmetries are not restricted to reaction-diffusion equations. Experiments of Simonelli and Gollub [27] on the dynamics of parametrically excited surface waves in square containers ($\Omega = [0, \pi]^2$) exhibit apparently degenerate behavior that is not expected for problems with square symmetry ($\Gamma = \mathbb{D}_4$). In a series of papers, Crawford investigated the effects of hidden translations and hidden rotations for $\mathbf{E}(2)$-equivariant equations posed on square domains with NBC. In particular, experiments devised in Crawford [3, 4] and carried out in Crawford, Gollub and Lane [7] established that the 'degenerate' phenomena observed by Simonelli and Gollub [27] are a direct consequence of hidden translations. Remarkably, transitions in the surface wave experiment are controlled by symmetries that exist only in the mathematical models and not in physical space. See Crawford [5, 6] for a discussion of the theoretical (and potentially experimental) implications of hidden rotations.

In this paper, we consider systems of reaction-diffusion equations on partially unbounded domains Ω (that is, we suppose that some spatial domain variables are bounded, and some are unbounded). A concrete example is $\Omega = \mathbb{R} \times [0, \pi] \subset \mathbb{R}^2$, so $\Gamma = \mathbf{E}(1) \times \mathbb{Z}_2$. It is immediate that solutions satisfying NBC on Ω extend to solutions on $\mathbb{R}^2$ that are 2π-periodic in the x_2-direction. Hence, we expect that hidden translations and rotations in $\mathbf{E}(2) - \Gamma$ have subtle effects on the generic bifurcations of PDEs posed on Ω.

However, the purpose of this paper is to point out a rather dramatic implication of hidden symmetries for the Ginzburg-Landau theory of PDEs on partially unbounded domains. Again, consider the partially unbounded domain $\mathbb{R} \times [0, \pi]$. Since there is a single unbounded domain variable, generically certain bifurcations (steady-state bifurcations with nonzero critical wavenumber) are governed by a universal envelope equation, or *Ginzburg-Landau equation*, of the form

$$A_T = c_0 A_{XX} + c_1 A + c_2 |A|^2 A, \tag{1.1}$$

where A is a complex-valued amplitude function depending slowly on the unbounded space and time variables, and c_0, c_1, c_2 are real constants. See for example Newell [21] and also [9, 23]. For rigorous results, see [24, 19]. In particular, for general boundary conditions on $\mathbb{R} \times [0, \pi]$, it follows from [19, 20] that generically there is a rigorous justification of equation (1.1).

When NBC are imposed on $\mathbb{R} \times [0, \pi]$, we show that equation (1.1) is no longer universally valid. Indeed, p critical wavenumbers are excited simultaneously, where p is an arbitrary positive integer. In place of equation (1.1), we require a system of p coupled Ginzburg-Landau equations. (It can then be shown, as in [19, 20], that generically there is a rigorous justification of the coupled Ginzburg-Landau equations.)

The need for p coupled equations can be seen as follows. The underlying equations have $\mathbf{E}(2)$ symmetry when posed on the whole of $\mathbb{R}^2$ and generically have a unique critical wavenumber $k_c \geq 0$. The critical eigenfunctions have the form $v_k e^{ikx}$ where $k \in \mathbb{R}^2$ satisfies $|k| = k_c$. Restricting to the NBC problem, we have the constraint that $k_2 \in \mathbb{Z}$. Summarizing, the critical eigenfunctions for the NBC problem are given by $v_k e^{ik_1 x_1} \cos k_2 x_2$ where $k_1 \in \mathbb{R}$, $k_2 \in \mathbb{Z}$ and $k_1^2 + k_2^2 = k_c^2$. Thus k_2 is constrained to finitely many, but arbitrarily many, values. Suppose that k_c is not an integer, and let $p \in \mathbb{Z}$ be such that $p - 1 < k_c < p$. Then $k_2 = 0, 1, \ldots, p - 1$ and so there are p critical wavenumbers k_1 for the unbounded spatial variable x_1; namely

$$k_c,\ \sqrt{k_c^2 - 1},\ \sqrt{k_c^2 - 4}, \cdots, \sqrt{k_c^2 - (p-1)^2}.$$

Note that the integer p depends on the aspect ratio k_c/π which compares the critical wavenumber for the planar PDE to the width of the strip Ω.

The remainder of this paper is organized as follows. In Section 2, we recall ideas of Crawford *et al.* [8] on hidden symmetries resulting from NBC on certain bounded domains. In Section 3, we consider the partially unbounded domain $\Omega = \mathbb{R} \times [0, \pi]$. Further extensions of these ideas are described in Section 4.

2 Boundary conditions as symmetry constraints

In this section, we review the main ideas described in Crawford *et al.* [8]. For the purposes of this paper, it is convenient to focus attention on the spatial domains $[0, \pi]$ (Fujii *et al.* [11], Armbruster and Dangelmayr [1]) and $[0, \pi]^2$ (Crawford *et al.* [3, 4, 5, 6, 7, 8]).

Elsewhere, hidden translation symmetries have been studied in detail for hyperrectangular domains by Gomes [14, 15] and Gomes and Stewart [17, 18]. A systematic analysis of hidden rotation symmetries for hyperrectangular domains can be found in Ashwin [2]. Field, Golubitsky and Stewart [10] study hidden symmetries on hemispherical domains, and present a general framework for hidden symmetries induced by Neumann and Dirichlet boundary conditions on a large class of spatial domains. These results have found a wide variety of applications, and we refer again to the review article by Gomes *et al.* [16] for further details and references.

Neumann boundary conditions on $[0, \pi]$

To fix ideas, we consider a nonlinear heat equation of the form

$$u_t = u_{xx} + f(u), \tag{2.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a smooth nonlinearity, and $u : \mathbb{R} \to \mathbb{R}$.

When equation (2.1) is restricted to the interval $[0, \pi]$ with identical boundary conditions at 0 and π, the physical symmetry group is $\Gamma = \mathbb{Z}_2$ generated by the reflection $x \mapsto \pi - x$.

However, suppose that we impose NBC so $u'(0) = u'(\pi) = 0$. Any C^1 function u satisfying NBC on $[0, \pi]$ can be extended to a C^1 function satisfying PBC on $[-\pi, \pi]$ by setting $u(x) = u(-x)$ for $x \in [-\pi, 0]$. Moreover, it follows from elliptic regularity that if u is a smooth solution to equation (2.1) on $[0, \pi]$ then the extension yields a smooth solution on $[-\pi, \pi]$. Since the solution on $[-\pi, \pi]$ satisfies PBC, there is a further extension to a 2π-periodic solution on $\mathbb{R}$. Note that not every 2π-periodic solution is obtained in this manner, since by construction the solution is invariant under the reflection $x \mapsto -x$.

Conversely, suppose that u is a 2π-periodic solution on $\mathbb{R}$ satisfying $u(-x) = u(x)$. Then $u'(0) = 0$. Moreover, $u(2\pi - x) = u(x)$ so that $u'(\pi) = 0$. Hence u restricts to a solution on $[0, \pi]$ satisfying NBC.

Whereas the NBC problem has only $\mathbb{Z}_2$ symmetry, the PBC problem has $\mathbf{O}(2)$ symmetry generated by translations modulo 2π and reflections. Moreover, solutions to the NBC problem are precisely those solutions to the PBC problem that satisfy the symmetry constraint $u(-x) = u(x)$. Let $\mathbb{D}_1$ denote the subgroup of $\mathbf{O}(2)$ generated by the reflection $x \mapsto -x$. We have shown that the NBC problem is the restriction of the PBC problem to the fixed-point subspace $\mathrm{Fix}\,\mathbb{D}_1$.

To understand the implications of this observation, it is convenient to revisit the notion of *hidden symmetries* [12].

Hidden symmetries Suppose quite generally that Γ is a compact Lie group acting linearly on $\mathbb{R}^m$ and that Σ is an isotropy subgroup of Γ with fixed-point subspace $\mathrm{Fix}\,\Sigma$. Any Γ-equivariant vector field $f : \mathbb{R}^m \to \mathbb{R}^m$ restricts to a vector field $f_\Sigma : \mathrm{Fix}\,\Sigma \to \mathrm{Fix}\,\Sigma$.

The largest subgroup of Γ that preserves $\mathrm{Fix}\,\Sigma$ is the normalizer $N(\Sigma)$. Quotienting out by Σ (which acts trivially on $\mathrm{Fix}\,\Sigma$) we obtain that f_Σ is a D_Σ-equivariant map where $D_\Sigma = N(\Sigma)/\Sigma$.

However, it is not always the case that f_Σ is a general D_Σ-equivariant vector field. That is, not every D_Σ-equivariant vector field extends smoothly to a Γ-equivariant vector field on $\mathbb{R}^m$. The simplest example of this is for the group $\Gamma = \mathbb{D}_5$ acting in its standard action on $\mathbb{R}^2 \cong \mathbb{C}$ (generated by $z \mapsto e^{2\pi i/5}z$, $z \mapsto \bar{z}$). Let Σ be the subgroup generated by $z \mapsto \bar{z}$, so $\mathrm{Fix}\,\Sigma \cong \mathbb{R}$. Note that $N(\Sigma) = \Sigma$ so that $D_\Sigma = \mathbf{1}$. In particular, $x \mapsto x^2$ is a D_Σ-equivariant vector field on $\mathrm{Fix}\,\Sigma$. But a calculation shows that there are no nontrivial $\mathbb{D}_5$-equivariant quadratic maps on $\mathbb{R}^2$ and hence there is no smooth equivariant map $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(x) = x^2$ for $x \in \mathrm{Fix}\,\Sigma$.

Returning to the equation (2.1), we have seen that the PBC problem has $\mathbf{O}(2)$ symmetry and that the NBC problem is the restriction of the PBC problem to $\mathrm{Fix}\,\mathbb{D}_1$ where $\mathbb{D}_1$ is generated by the reflection $x \mapsto -x$. We compute that $N(\mathbb{D}_1) = \mathbb{D}_2$ generated by $x \mapsto -x$ and $x \mapsto \pi - x$. Hence, $D_\Sigma = \mathbb{Z}_2$ generated by $x \mapsto \pi - x$ corresponding to the obvious symmetry of the NBC problem. Again, it is not the case that every $\mathbb{Z}_2$-equivariant vector field on $\mathrm{Fix}\,\mathbb{D}_1$ extends smoothly to an $\mathbf{O}(2)$-equivariant vector field. Hence, the NBC problem has hidden translation symmetry leading to the 'degeneracies' discussed in [1, 11].

Remark 2.1 The analysis of hidden symmetries for equation (2.1) applies to a much larger class of PDEs including systems of reaction-diffusion equations. Indeed, the reflection trick that embeds NBC in PBC goes through provided elements of $\mathbf{E}(1)$ act on functions u by transforming the spatial domain variables in the standard way:

$$x \mapsto x + b, \qquad x \mapsto b - x.$$

This is the case for equation (2.1) and for systems of reaction-diffusion equations in general, but it is easy to write down equations with reflections acting as $u(x) \mapsto Au(b - x)$ where A is a range symmetry (with $A^2 = I$). If A is nontrivial, the reflection trick fails. Hidden symmetries may still be significant at the level of critical eigenfunctions, but we do not address such issues here.

Neumann boundary conditions on $[0,\pi]^2$

Next, we consider the analogous situation in $\mathbb{R}^2$. For definiteness, we consider the equation

$$u_t = \Delta u + f(u), \tag{2.2}$$

where $f : \mathbb{R} \to \mathbb{R}$ is smooth, and Δ is the two dimensional Laplacian. Thus, u is a real-valued function of two variables x_1, x_2.

Suppose that equation (2.2) is posed on the square $[0,\pi]^2$ so that $\Gamma = \mathbb{D}_4$. Again, NBC on $[0,\pi]^2$ embeds in PBC on $[-\pi,\pi]^2$. The PBC problem has symmetry group $\mathbb{D}_4 \ltimes T^2$ where the 2-torus T^2 corresponds to translations mod 2π parallel to the sides of the square. (To make the correspondence with the one dimensional case, it is convenient to think of $\mathbf{O}(2)$ as $\mathbb{Z}_2 \ltimes T^1$.)

Thus, we may view the NBC problem as the restriction of a $\mathbb{D}_4 \ltimes T^2$-equivariant problem to the fixed-point subspace $\operatorname{Fix}\Sigma$, where $\Sigma = \mathbb{D}_1^2$ consists of the elements $(x_1, x_2) \mapsto (\pm x_1, \pm x_2)$. We compute that $N(\Sigma) = \mathbb{D}_4 \ltimes \mathbb{D}_1^2$ so that $D_\Sigma \cong \mathbb{D}_4$ as expected.

The implications of embedding NBC in PBC are considered in Crawford, Gollub and Lane [7], with emphasis on the surface wave experiment in a square domain [27]. In the event of a steady-state bifurcation, the space of critical eigenfunctions should generically be an irreducible representation of the physical symmetry group $\Gamma = \mathbb{D}_4$; see [13]. However, certain irreducible representations of $\mathbb{D}_4 \ltimes T^2$ have the property that their restriction to $\operatorname{Fix}\mathbb{D}_1^2$ is not $\mathbb{D}_4$-irreducible. Some of these calculations are repeated below (the case $k_c = \sqrt{10}$). In particular, critical eigenspaces that are not $\mathbb{D}_4$-irreducible may (and do) occur in the surface wave experiment, even though this is degenerate behavior for a general problem with $\mathbb{D}_4$-symmetry. For a detailed analysis of the implications of hidden translations both at the linear and nonlinear level, we refer to [7]. Moreover, experiments devised in [3, 4] and carried out in [7] establish that certain observed phenomena in the surface wave experiment are a direct consequence of hidden translations.

There is a further embedding, where the $\mathbb{D}_4 \ltimes T^2$-equivariant PBC problem can be viewed as a restriction of an $\mathbf{E}(2)$-equivariant problem posed on the whole of $\mathbb{R}^2$. Note that assuming PBC means restricting the $\mathbf{E}(2)$-equivariant problem to $\operatorname{Fix}\mathcal{L}$ where $\mathcal{L} = (2\pi\mathbb{Z})^2$ is the two dimensional lattice generated by the translations $(x_1, x_2) \mapsto (x_1 + 2\pi, x_2)$, $(x_1, x_2) \mapsto (x_1, x_2 + 2\pi)$. Moreover $N(\mathcal{L}) = \mathbb{D}_4 \ltimes \mathbb{R}^2$ where $\mathbb{R}^2$ consists of all translations. Hence $D_\mathcal{L} \cong \mathbb{D}_4 \ltimes T^2$ where $T^2 = \mathbb{R}^2/\mathcal{L}$.

Thus, we have the hierarchy of embeddings

NBC	$\subset$	PBC	$\subset$	infinite plane
$\mathbb{D}_4$		$\mathbb{D}_4 \ltimes T^2$		$\mathbf{E}(2)$

The second embedding takes into account hidden rotations (and reflections) in $\mathbf{E}(2)$ that are not present in the PBC problem [5, 6]. We note that this second step of embedding the PBC problem in the full problem on the unbounded domain is not required in the one dimensional case. The difference is that in one dimension, the Euclidean group $\mathbf{E}(1)$ preserves the invariant subspace $\mathrm{Fix}\, 2\pi\mathbb{Z}$ corresponding to PBC, whereas in two dimensions, only the subgroup $\mathbb{D}_4 \ltimes \mathbb{R}^2$ preserves the subspace $\mathrm{Fix}\, \mathcal{L}$ corresponding to PBC. (Put differently, $\Sigma = 2\pi\mathbb{Z}$ is a normal subgroup of $\mathbf{E}(1)$, so there are no further hidden symmetries, whereas $\Sigma = \mathcal{L}$ is not a normal subgroup of $\mathbf{E}(2)$.)

As in Remark 2.1, the method of embedding NBC in PBC is valid for systems of reaction-diffusion equations, and more generally for systems of $\mathbf{E}(2)$-equivariant PDEs for which the action of $\mathbf{E}(2)$ on functions $u = u(x_1, x_2)$ is given purely in terms of the standard action of $\mathbf{E}(2)$ on the domain variables x_1, x_2. This includes the Swift-Hohenberg equation

$$u_t = -(\Delta + k_c^2)^2 u + f(u), \tag{2.3}$$

where $f : \mathbb{R} \to \mathbb{R}$ is smooth and $k_c > 0$.

Equation (2.3) has an important property that is shared by certain systems of reaction-diffusion equations but not by equation (2.1). Hence, for illustrative purposes, it is convenient to consider equation (2.3). Suppose that $f'(0) = 0$. If equation (2.3) is posed on the whole plane (so $u : \mathbb{R}^2 \to \mathbb{R}$), then the critical eigenfunctions corresponding to the linear term $-(\Delta + k_c^2)^2$ are given by $e^{ik\cdot x}$ where $|k| = k_c$. Thus equation (2.3) has nonzero *critical wavenumber* k_c (whereas for reaction-diffusion equations, it is necessary to consider systems to obtain a nonzero critical wavenumber).

We now repeat certain calculations, carried out in [3, 7], for the Swift-Hohenberg equation. Consider equation (2.3) posed on the domain $[-\pi, \pi]^2$ with PBC. The critical eigenspace is spanned by eigenfunctions $e^{ik\cdot x}$ with $|k| = k_c$ such that $k_1, k_2 \in \mathbb{Z}$. We consider in turn the cases $k_c = \sqrt{5}$ (which exhibits no degeneracies at the linear level), $k_c = \sqrt{10}$ (degeneracies due to hidden translations) and $k_c = 5$ (degeneracies due to hidden rotations).

$k_c = \sqrt{5}$ The critical eigenfunctions are $e^{ik\cdot x}$ where $k = (\pm 2, \pm 1)$, $(\pm 1, \pm 2)$, so that there is an eight dimensional critical subspace spanned by

$$\begin{gathered}\cos 2x_1 \cos x_2,\ \cos 2x_1 \sin x_2,\ \sin 2x_1 \cos x_2,\ \sin 2x_1 \sin x_2 \\ \cos x_1 \cos 2x_2,\ \cos x_1 \sin 2x_2,\ \sin x_1 \cos 2x_2,\ \sin x_1 \sin 2x_2\end{gathered}$$

This is an irreducible representation of $\mathbb{D}_4 \ltimes T^2$. Restricting to NBC on $[0, \pi]^2$ (so first and third order normal derivatives vanish on the boundary, or equivalently PBC together with the symmetry constraint $u(\pm x_1, \pm x_2) = u(x_1, x_2)$) we obtain the two dimensional subspace spanned by

$$v_1 = \cos 2x_1 \cos x_2, \qquad v_2 = \cos x_1 \cos 2x_2.$$

Observe that $\mathbb{D}_4$ is generated by the symmetries $(x_1, x_2) \mapsto (x_2, x_1)$, $(x_1, x_2) \mapsto (\pi - x_1, x_2)$. The induced action on the eigenfunctions v_1, v_2 is given by

$$(v_1, v_2) \mapsto (v_2, v_1), \quad (v_1, v_2) \mapsto (v_1, -v_2),$$

which is an irreducible action of $\mathbb{D}_4$ (in accordance with [13]).

$k_c = \sqrt{10}$ The critical eigenfunctions are $e^{ik\cdot x}$ where $k = (\pm 3, \pm 1), (\pm 1, \pm 3)$. Again, we obtain an eight dimensional irreducible representation of $\mathbb{D}_4 \ltimes T^2$, and restricting to NBC yields the two dimensional subspace spanned by

$$v_1 = \cos 3x_1 \cos x_2, \qquad v_2 = \cos x_1 \cos 3x_2.$$

However, the action of $\mathbb{D}_4$ on these eigenfunctions is now given by

$$(v_1, v_2) \mapsto (v_2, v_1), \quad (v_1, v_2) \mapsto (-v_1, -v_2),$$

which is no longer an irreducible action (the subspaces spanned by $v_1 \pm v_2$ are $\mathbb{D}_4$-invariant).

In this case, the 'degeneracy' is a direct consequence of the hidden translations $x_1 \mapsto x_1 + \pi$ and $x_2 \mapsto x_2 + \pi$; see [3, 4, 7].

$k_c = 5$ The critical eigenfunctions are $e^{ik\cdot x}$, where $k = (\pm 4, \pm 3)$, $(\pm 3, \pm 4)$, $(\pm 5, 0)$, $(0, \pm 5)$. The resulting twelve dimensional subspace is no longer an irreducible representation of $\mathbb{D}_4 \ltimes T^2$. This behavior for the PBC problem is degenerate for a $\mathbb{D}_4 \ltimes T^2$-equivariant problem and hence is a consequence of hidden rotations (in particular, the rotation that maps the vector $(3, 4)$ onto the vector $(5, 0)$); see [5, 6]. Restricting to NBC yields a four dimensional critical eigenspace as a result of hidden rotations.

Continuing in this way, we see that there are values of k_c that yield critical eigenspaces of arbitrarily high (but finite) dimension for the NBC problem with degeneracies due to the combined effects of hidden translations and rotations. This phenomenon occurs robustly in codimension one bifurcations for $\mathbf{E}(2)$-equivariant systems of PDEs posed on $\Omega = [0, \pi]^2$ with NBC. However, 'most' values of k_c yield (both $\mathbb{D}_4$-irreducible and reducible) critical eigenspaces of dimension two for the NBC problem).

3 Neumann boundary conditions on $\mathbb{R} \times [0, \pi]$

Suppose that equations of the form (2.2) or (2.3) are posed on the infinite strip $\mathbb{R} \times [0, \pi]$, with physical symmetry $\Gamma = \mathbf{E}(1) \times \mathbb{Z}_2$. Again, the NBC problem embeds in a PBC problem on $\mathbb{R} \times [-\pi, \pi]$ with symmetry $\mathbf{E}(1) \times \mathbf{O}(2)$. This in turn embeds in a problem on $\mathbb{R}^2$ with symmetry group $\mathbf{E}(2)$. As in the case of the bounded domain $\Omega = [0, \pi]^2$ considered in the previous section, the embedding of NBC in PBC implies hidden translation symmetry, and the embedding of PBC in the planar problem implies hidden rotation symmetry. We now highlight the new implications for Ginzburg-Landau theory that are not present for NBC on bounded domains.

To emphasize the bifurcation theory aspects, we incorporate a bifurcation parameter $\lambda \in \mathbb{R}$ into the Swift-Hohenberg equation (2.3). Thus, we consider the equation

$$u_t = -(\Delta + k_c^2)^2 u + \lambda u + f(u), \tag{3.1}$$

where $k_c > 0$, $f : \mathbb{R} \to \mathbb{R}$ is smooth and $f'(0) = 0$. Note that k_c is the critical wavenumber for the PDE posed on the infinite plane $\mathbb{R}^2$. That is, when $\lambda = 0$, the critical eigenfunctions have the form e^{ikx}, where $|k| = k_c$. (In other words, there is a steady-state bifurcation at $\lambda = 0$ with nonzero critical wavenumber k_c [20].)

Now consider equation (3.1) posed on the strip $\Omega = \mathbb{R} \times [0, \pi]$ subject to NBC. There is a one-to-one correspondence between solutions satisfying NBC on $\mathbb{R} \times [0, \pi]$ and solutions on the whole of $\mathbb{R}^2$ satisfying the symmetry constraints

$$u(x_1, -x_2) = u(x_1, x_2), \quad u(x_1, x_2 + 2\pi) = u(x_1, x_2).$$

It follows that the critical eigenfunctions at $\lambda = 0$ are given by $e^{ik_1x_1} \cos k_2 x_2$ where $k_1 \in \mathbb{R}$, $k_2 \in \mathbb{Z}$, and $k_1^2 + k_2^2 = k_c^2$. Note that there is a finite but arbitrarily large number of possible values for k_2 (depending on the value of k_c^2). Indeed, the number p of nonnegative values of k_2 is given by $[k_c] + 1$ where $[k_c]$ is the integer part of k_c.

If $k_c < 1$, then $k_2 = 0$ and the critical wave functions are $e^{ik_1x_1}$ where $k_1 = \pm k_c$. As in [19], we can reduce to a single Ginzburg-Landau equation on the line (see equation (1.1)). However, if $1 < k_c < 2$, then $k_2 = 0$ or $k_2 = 1$ and the critical wave functions are $e^{ik_1x_1}$, $k_1 = \pm k_c$, and $e^{ik_1x_1} \cos x_2$, $k_1 = \pm\sqrt{k_c^2 - 1}$. Accordingly, we make the Ginzburg-Landau-type ansatz

$$\begin{aligned} &u(x_1, x_2, t) = \epsilon A(X, T) e^{ik_c x_1} + \epsilon B(X, T) e^{i\sqrt{k_c^2 - 1}\, x_1} \cos x_2 + \text{c.c.} \\ &X = \epsilon x_1, \quad T = \epsilon^2 t, \quad \lambda = \epsilon^2, \end{aligned}$$

where A and B are complex-valued amplitude functions depending slowly on the unbounded space and time variables. In this way, we obtain a pair of coupled Ginzburg-Landau equations. For generic values of $k_c \in (1, 2)$, the coupling terms are the obvious ones and the Ginzburg-Landau equations have the form

$$\begin{aligned} A_T &= 4k_c^2 A_{XX} + A + c_1|A|^2A + c_2|B|^2A \\ B_T &= (4k_c^2-1)B_{XX} + B + d_1|A|^2B + d_2|B|^2B, \end{aligned}$$

where c_1, c_2, d_1, d_2 are real constants that depend on the quadratic and cubic terms in f. For example, if $f(u) = \alpha u^3 + O(u^4)$, then $c_1 = d_2 = 3\alpha$, $c_2 = d_1 = 6\alpha$. If f has a quadratic term, then the expressions for the constants are considerably more complicated.

In general, suppose that p is a positive integer and that $p-1 < k_c < p$. Then $k_2 = 0, 1, \ldots, p-1$ and the critical eigenfunctions are given by

$$e^{\pm i\sqrt{k_c^2-j^2}\,x_1}\cos jx_2, \;\; j = 0, 1, \ldots, p-1.$$

The ansatz

$$\begin{aligned} u(x_1,x_2,t) &= \epsilon A_1(X,T)e^{ik_cx_1} + \epsilon A_2(X,T)e^{i\sqrt{k_c^2-1}\,x_1}\cos x_2 + \cdots \\ &\quad + \epsilon A_p(X,T)e^{i\sqrt{k_c^2-(p-1)^2}\,x_1}\cos(p-1)x_2 + \text{c.c.} \end{aligned}$$

leads to a system of p coupled Ginzburg-Landau equations of the form

$$\begin{aligned} A_{1,T} &= 4k_c^2 A_{1,XX} + A_1 + c_{11}|A_1|^2A_1 + \cdots + c_{1p}|A_p|^2A_1 \\ A_{2,T} &= 4(k_c^2-1)A_{2,XX} + A_2 + c_{21}|A_1|^2A_2 + \cdots + c_{2p}|A_p|^2A_2 \\ &\;\vdots \\ A_{p,T} &= 4(k_c^2-(p-1)^2)A_{p,XX} + A_p + c_{p1}|A_1|^2A_p + \cdots + c_{pp}|A_p|^2A_p. \end{aligned}$$

If $f(u) = \alpha u^3 + O(u^4)$, then we have $c_{ij} = 3\alpha$ if $i = j$ and 6α if $i \neq j$.

Note that the size p of the reduced system increases monotonically with k_c and hence is often large (in contrast to the case $\Omega = [0,\pi]^2$ discussed in Section 2 where large systems of reduced equations occur robustly but relatively rarely.)

It follows from the methods in Melbourne [20] that the phenomena we have described for the Swift-Hohenberg equation (3.1) hold universally for systems of reaction-diffusion equations (and more generally for systems of $\mathbf{E}(2)$-equivariant PDEs as in Remark 2.1) posed on $\mathbb{R} \times [0,\pi]$ with NBC. Evidently, the size p of the reduced system is independent of the size of the original system (the reduced system may be much larger than the original system) and is governed by the (nonuniversal!) value of the critical wavenumber k_c — or more precisely the aspect ratio k_c/π comparing the critical wavenumber of the PDE on $\mathbb{R}^2$ and the width of the strip Ω.

Remark 3.1 When $p-1 < k_c < p$, we have computed that the critical eigenspace is spanned by eigenfunctions

$$e^{\pm i\sqrt{k_c^2-j^2}\,x_1}\cos jx_2,\ j=0,1,\ldots,p-1.$$

We note that this subspace is irreducible for the physical symmetry group $\Gamma = \mathbf{E}(1)\times\mathbb{Z}_2$ if and only if $p=1$. In general, we have a direct sum of p nonisomorphic irreducible representations of $\mathbf{E}(1)\times\mathbb{Z}_2$. As in Section 2, this linear degeneracy is a consequence of hidden rotations in $\mathbf{E}(2)$.

4 Other partially unbounded domains

The results in Section 3 generalize in an obvious way to higher dimensional partially unbounded domains Ω. In this section, we consider the domains $\Omega = \mathbb{R}\times[0,\pi]^2$ and $\Omega = \mathbb{R}^m\times[0,\pi]$, $m\geq 2$.

Neumann boundary conditions on $\mathbb{R}\times[0,\pi]^2$

This domain combines the features of the domains $[0,\pi]^2$ (Section 2) and $\mathbb{R}\times[0,\pi]$ (Section 3). Consider the Swift-Hohenberg equation (2.3), where Δ is now the three dimensional Laplacian. As usual, NBC on $\mathbb{R}\times[0,\pi]^2$ (with physical symmetry group $\Gamma=\mathbf{E}(1)\times\mathbb{D}_4$) extends to PBC on $\mathbb{R}\times[-\pi,\pi]^2$ (with symmetry group $\mathbf{E}(1)\times(\mathbb{D}_4\ltimes T^2)$) which extends to the whole of $\mathbb{R}^3$ (with symmetry group $\mathbf{E}(3)$). The extended problem on $\mathbb{R}^3$ has critical eigenfunctions $e^{ik\cdot x}$ where $|k|=k_c$. Restricting to NBC, the critical eigenfunctions are

$$e^{ik_1x_1}\cos k_2x_2\cos k_3x_3,\quad k_1\in\mathbb{R},\ k_2,k_3\in\mathbb{Z},\ k_1^2+k_2^2+k_3^2=k_c^2.$$

The number of excited modes is given by

$$p=\#\{k_2,k_3=0,1,\ldots\mid k_2^2+k_3^2<k_c^2\},$$

leading generically as in Section 3 to a system of p coupled Ginzburg-Landau equations with critical wavenumbers $\sqrt{k_c^2-k_2^2-k_3^2}$.

Remark 4.1 As in Section 3, the positive integer p is arbitrarily large, and is monotonically increasing in k_c. However, p is no longer arbitrary and the increments in p are not uniform. For example, if $0<k_c<1$ the critical eigenfunctions have $(k_2,k_3)=(0,0)$ so that $p=1$, whereas if $1<k_c<\sqrt{2}$ we have $(k_2,k_3)=(1,0),(0,1)$

so that $p = 3$. Hence the value $p = 2$ is not admissible. The first few admissible values of p are

$$p = 1, 3, 4, 6, 8, 9, 11, 13, 15, 17, 19, 20, 22, 26, \ldots$$

It is easily seen that most of the increments are by 1 or 2, but an increment of 4 (from $p = 22$ to $p = 26$) occurs as k_c passes through 5 (since the modes $(k_2, k_3) = (3,4), (4,3), (5,0), (0,5)$ are excited simultaneously). Indeed, the largest increments correspond to those values of k_c that resulted in high dimensional critical eigenspaces for the case $\Omega = [0,\pi]^2$ discussed in Section 2. Arbitrarily large increments occur for appropriate choices of k_c. In contrast to Section 2, the degeneracy at $k_c = 5$ say is retained for all $k_c > 5$ due to monotonicity of p.

Neumann boundary conditions on $\mathbb{R}^m \times [0,\pi]$, $m \geq 2$

This case is similar to the case $\mathbb{R} \times [0,1]$ in Section 3, the main difference being that Ginzburg-Landau theory is less well understood when there are two or more unbounded spatial variables; see [9, 22].

Proceeding as in Section 3, we find that the critical eigenfunctions for NBC are given by

$$e^{i(k_1 x_1 + \cdots + k_{m-1} x_{m-1})} \cos k_m x_m,$$

where $k_1^2 + \cdots + k_m^2 = k_c^2$ and $k_m \in \mathbb{Z}$. So $k_m = 0, 1, \ldots, p-1$, where $p = [k_c]$ is an arbitrary positive integer. When $m = 2$, restricting to certain roll-like solutions and proceeding formally as in [23, 26] leads to a universal system of p coupled Newell-Whitehead-Segel equations. (Rigorous results in this direction can be found in Schneider [25]). For all m, a rigorous reduction that incorporates more general solutions [20] leads to a universal system of p coupled equations of Swift-Hohenberg type.

References

[1] D. Armbruster and G. Dangelmayr. Coupled stationary bifurcations in nonflux boundary value problems. *Math. Proc. Camb. Phil. Soc.* **101** (1987), 167–192.

[2] P. Ashwin. High corank steady-state mode interactions on a rectangle. *Bifurcation and Symmetry* (E. Allgower et al., ed.), ISNM **104**, Birkhäuser, Basel, 1992, pp. 23–33.

[3] J. D. Crawford. Normal forms for driven surface waves: boundary conditions, symmetry, and genericity. *Physica D* **52** (1991), 697–739.

[4] J. D. Crawford. Surface waves in non-square containers with square symmetry. *Phys. Rev. Lett.* **67** (1991), 441–444.

[5] J. D. Crawford. $D_4\dot{+}T^2$ mode interactions and hidden rotational symmetry. *Nonlinearity* **7** (1994), 697–739.

[6] J. D. Crawford. D_4-symmetric maps with hidden Euclidean symmetry. *Pattern Formation: Symmetry Methods and Applications* (J. Chadam et al., ed.), Fields Institute Communications **5**, Amer. Math. Soc., 1996, pp. 93–124.

[7] J. D. Crawford, J. P. Gollub, and D. Lane. Hidden symmetries of parametrically forced waves. *Nonlinearity* **6** (1993), 119–164.

[8] J. D. Crawford, M. Golubitsky, M. G. M. Gomes, E. Knobloch, and I. N. Stewart. Boundary conditions as symmetry constraints. *Singularity Theory and its Applications, Part II* (R. M. Roberts and I. N. Stewart, eds.), Lecture Notes in Math. **1463**, Springer, New York, 1991, pp. 63–79.

[9] M. C. Cross and P. C. Hohenberg. Pattern formation outside of equilibrium. *Rev. of Mod. Phys.* **65** (1993), 851–1112.

[10] M. J. Field, M. Golubitsky, and I. N. Stewart. Bifurcations on hemispheres. *J. Nonlinear Sci.* **1** (1991), 201–223.

[11] H. Fujii, M. Mimura, and Y. Nishiura. A picture of the global bifurcation diagram in ecological interacting and diffusing systems. *Physica D* **5** (1982), 1–42.

[12] M. Golubitsky, J. E. Marsden, and D. G. Schaeffer. Bifurcation problems with hidden symmetries. *Partial Differential Equations and Dynamical Systems* (W. E. Fitzgibbon III, ed.), Research Notes in Math., Pitman, San Francisco, 1984, pp. 181–210.

[13] M. Golubitsky, I. N. Stewart, and D. Schaeffer. *Singularities and Groups in Bifurcation Theory, Vol. II.* Appl. Math. Sci. **69**, Springer, New York, 1988.

[14] M. G. M. Gomes. Steady-state mode interactions in rectangular domains. M. Sc. Thesis, University of Warwick, 1989.

[15] M. G. M. Gomes. Symmetries in bifurcation theory: the appropriate context. Ph. D. Thesis, University of Warwick, 1992.

[16] M. G. M. Gomes, I. S Labouriau, and E. M. Pinho. Spatial hidden symmetries in pattern formation. *Pattern Formation in Continuous and Coupled Systems* (M. Golubitsky, D. Luss, and S. Strogatz, eds.), IMA Volumes in Math. and its Applications **115**, Springer, New York, 1999, pp. 83–99.

[17] M. G. M. Gomes and I. N. Stewart. Steady PDEs on generalized rectangles: a change of genericity in mode interactions. *Nonlinearity* **7** (1994), 253–272.

[18] M. G. M. Gomes and I. N. Stewart. Hopf bifurcations on generalized rectangles with Neumann boundary conditions. *Dynamics, Bifurcations, Symmetry* (P. Chossat, ed.), Kluwer, The Netherlands, 1994, pp. 139–158.

[19] I. Melbourne. Derivation of the time-dependent Ginzburg-Landau equation on the line. *J. Nonlinear Sci.* **8** (1998), 1–15.

[20] I. Melbourne. Steady state bifurcation with Euclidean symmetry. *Trans. Amer. Math. Soc.* **391** (1999), 1575–1603.

[21] A. C. Newell. The dynamics and analysis of patterns. *Lectures in the Sciences of Complexity*, **1**, Addison Wesley, New York, 1989, pp. 107–173.

[22] A. C. Newell, T. Passot, and J. Lega. Order parameter equations for patterns. *Annu. Rev. Fluid. Mech.* **25** (1993), 399–453.

[23] A. C. Newell and J. A. Whitehead. Finite bandwidth, finite amplitude convection. *J. Fluid Mech.* **38** (1969), 279–303.

[24] G. Schneider. Error estimates for the Ginzburg-Landau approximation. *ZAMP* **45** (1994), 433–457.

[25] G. Schneider. Validity and limitation of the Newell-Whitehead equation. *Math. Nachr.* **176** (1995), 249–263.

[26] L. A. Segel. Distant side-walls cause slow amplitude modulation of cellular convection. *J. Fluid Mech.* **38** (1969), 203–224.

[27] F. Simonelli and J. P. Gollub. Surface wave mode interactions: effects of symmetry and degeneracy. *J. Fluid Mech.* **199** (1989), 471–494.