Central Limit Theorems and Invariance Principles for Lorenz Attractors

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28 February, 2006

Abstract

We prove statistical limit laws for Hölder observations of the Lorenz attractor, and more generally for geometric Lorenz attractors. In particular, we prove the almost sure invariance principle (approximation by Brownian motion). Standard consequences of this result include the central limit theorem, the law of the iterated logarithm, and the functional versions of these results.

1 Introduction

The Lorenz equations

$$\dot{x} = 10(y - x), \quad \dot{y} = 28x - y - xz, \quad \dot{z} = xy - \tfrac{8}{3}z, \tag{1.1}$$

were introduced in 1963 by Lorenz [8], originally as a simplified nonlinear model for the weather, but more significantly to emphasise the presence of chaotic dynamics in simple-looking systems. The mathematical study of these equations began with the geometric Lorenz flows, introduced independently by Afraĭmovič, Bykov & Sil′nikov [1] and Guckenheimer & Williams [7, 20] as an abstraction of the numerically-observed features of solutions to (1.1). The geometric flows were shown to possess a "strange" attractor with sensitive dependence on initial conditions. Moreover, these attractors admit a "physical" (SRB) measure, namely an ergodic invariant probability measure μ with the property that time averages and space averages coincide for Lebesgue almost every solution starting close to the attractor. Further, the measure μ has a positive Lyapunov exponent.

The main result in this paper is the almost sure invariance principle (ASIP) for geometric Lorenz attractors.

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Theorem 1.1 (ASIP) *Let $T_t : \mathbb{R}^3 \to \mathbb{R}^3$ be a geometric Lorenz flow with SRB measure μ. Let $\psi : \mathbb{R}^3 \to \mathbb{R}$ be a Hölder continuous observable with $\int_{\mathbb{R}^3} \psi \, d\mu = 0$.*

Then (on a possibly enriched probability space) there is a Brownian motion $W(t)$ with variance $\sigma^2 \geq 0$, and there exists $\epsilon > 0$ such that μ-a.e.

$$\int_0^t \psi \circ T_s \, ds = W(t) + O(t^{\frac{1}{2}-\epsilon}) \quad \text{as } t \to \infty.$$

Remark 1.2 For any geometric Lorenz attractor, the ASIP is nondegenerate ($\sigma^2 > 0$) for typical Hölder observables ψ. Indeed, the observables for which $\sigma^2 = 0$ lie inside a closed subspace of infinite codimension, see §4.3.

There are a number of consequences of the ASIP [13]. These include the central limit theorem and law of the iterated logarithm.

Corollary 1.3 *Assume the set up of Theorem 1.1. Then*

(CLT) $\frac{1}{\sqrt{t}} \int_0^t \psi \circ T_s \, ds \to_d N(0, \sigma^2)$ *as* $t \to \infty$.

(LIL) $\limsup_{t\to\infty} \frac{1}{\sqrt{2t \log\log t}} \int_0^t \psi \circ T_s \, ds = \sigma$ *almost everywhere.* ∎

Recently, Tucker [17, 18] obtained a numerically-assisted proof that the Lorenz equations (1.1) indeed define a geometric Lorenz flow. In particular, the Lorenz equations possess a strange attractor with an SRB measure with positive Lyapunov exponent. Moreover, the attractor is robust, so nearby flows also possess a strange attractor with these properties.

Corollary 1.4 *The ASIP, and hence CLT and LIL, are valid for Hölder observables of the Lorenz equations* (1.1) *and all nearby flows.* ∎

Remark 1.5 In [9], it is shown that mixing is automatic for Lorenz attractors, but currently there are no results on the rate of decay of correlations. Oddly, the logarithmic singularity of h is is a crucial part of the proof of mixing in [9], yet is the main obstruction to proving statistical limit laws or rapid decay of correlations. The techniques in the current paper overcome the obstruction in the case of statistical limit laws but require further refinement for decay of correlations, see Remark 3.3(iv).

In the remainder of this section, we outline the structure of this paper and describe how the proof of Theorem 1.1 relates to techniques developed in the uniformly and nonuniformly hyperbolic settings.

For uniformly hyperbolic flows, the CLT and ASIP were proved by Ratner [14] and Denker & Philipp [5]. A key step is to view the flow as a suspension (or special flow) over a uniformly hyperbolic diffeomorphism with Hölder roof function h. Statistical limit laws proved in the simpler setting of diffeomorphisms then transfer to the suspension flow under certain conditions. The boundedness of the roof function was relaxed by Melbourne & Török [12].

Gouëzel [6] (resp. Melbourne & Nicol [11]) used the approach in [12] to prove the CLT and stable laws (resp. the ASIP) for classes of nonuniformly hyperbolic flows. The flow is again a suspension over a nonuniformly hyperbolic diffeomorphism with Hölder roof function h, but the diffeomorphism itself is now modelled as a discrete suspension (Young tower [21, 22]) over a uniformly hyperbolic diffeomorphism (induced map) with an unbounded integer-valued roof function r. Statistical laws for the flow are obtained by twice applying the suspension procedure, see [11].

Geometric Lorenz flows are suspension flows over a singular uniformly hyperbolic diffeomorphism P, where the roof function h has a logarithmic singularity. The diffeomorphism P, being singular, has poor distortion properties. It is well-known how to construct an inducing scheme to model P as a discrete suspension over a diffeomorphism P^r that inherits the uniform hyperbolicity and in addition has good distortion. The ASIP for P^r is then a consequence of [11], but the logarithmic singularity for h is a nontrivial obstacle to lifting the ASIP to the flow. In this paper, we construct a more complicated inducing scheme along the lines used by Benedicks & Carleson [2] and Benedicks & Young [3] for Hénon-like diffeomorphisms. The inducing scheme is constructed specifically in order to control the growth of h along trajectories and enables us to prove the ASIP for the underlying flow. (Nevertheless, the construction is considerably simpler than that required in the Hénon case.)

In §2, we collect some background material on geometric Lorenz attractors. In §3, we describe the inducing scheme constructed in this paper. In §4, we use the inducing scheme to prove Theorem 1.1. The remainder of the paper is concerned with the proof of the inducing scheme. The inducing scheme is explained in §5. For completeness, all proofs are given. However, some of the lengthier (though standard) arguments are included as appendices.

2 Background on geometric Lorenz attractors

In this section, we describe some of the structure associated with geometric Lorenz attractors. Let 0 be an equilibrium for a smooth (at least $C^{1+\epsilon}$) flow T_t on $\mathbb{R}^3$. Denote the corresponding vector field $Z : \mathbb{R}^3 \to \mathbb{R}^3$. We suppose that the eigenvalues of $(dZ)_0$ are real and satisfy

$$\lambda_{ss} < \lambda_s < 0 < \lambda_u \quad \text{and} \quad \lambda_u > |\lambda_s|. \tag{2.1}$$

Choose coordinates (x_1, x_2, x_3) so that $(dZ)_0 = \operatorname{diag}\{\lambda_u, \lambda_{ss}, \lambda_s\}$. We suppose that the flow T_t is $C^{1+\epsilon}$-linearisable in a neighbourhood of 0.

Remark 2.1 By Sternberg [16], the flow for the actual Lorenz equations is C^∞-linearisable near 0. Moreover, the flow for all nearby sufficiently smooth vector fields is C^2-linearisable (see for example [15]).

After rescaling, we may suppose that the flow is linearised in a neighbourhood of the unit cube. Define the cross-sections $X = \{(x_1, x_2, 1) : |x_1|, |x_2| \le 1\}$, $X' = \{(1, x_2, x_3) : |x_2|, |x_3| \le 1\}$. The Poincaré map $P : X \to X$ (where defined) decomposes into $P = P_2 \circ P_1$ where $P_1 : X \to X'$ and $P_2 : X' \to X$. Write $Px = T_{h(x)}x$ where $h : X \to \mathbb{R}^+$ is the first return time to X.

Proposition 2.2 *Let* $\beta = |\lambda_s|/\lambda_u \in (0, 1)$, $\beta' = |\lambda_{ss}|/\lambda_u > \beta$. *Then* $P_1(x_1, x_2, 1) = (1, x_1^{\beta'} x_2, x_1^{\beta})$, *and* $h(x) = -\lambda_u^{-1} \log |x_1| + h_0(x)$ *where* $h_0 \in C^\beta(X)$.

Proof Note that $h(x) = \tau_1(x) + \tau_2(P_1 x)$ where $\tau_1 : X \to \mathbb{R}^+$ and $\tau_2 : X' \to \mathbb{R}^+$ are the first-hit times for P_1 and P_2 respectively. A standard calculation using the linearised flow between X and X' yields the required formula for P_1 and shows that $\tau_1(x) = -\lambda_u^{-1} \log |x_1|$. Moreover, P_2 is a diffeomorphism so τ_2 is smooth. This combined with the formula for P_1 implies that $h_0 = \tau_2 \circ P_1 \in C^\beta(X)$. ∎

Definition 2.3 The flow T_t has a *stable foliation* if there is a compact neighbourhood $N \subset X$ satisfying $P(N - \{x_1 = 0\}) \subset X$ with a $C^{1+\epsilon}$ P-invariant foliation into *stable leaves* (including the "singular" leaf $\{x_1 = 0\}$), and a constant $\lambda_0 \in (0, 1)$ such that for all x, y in the same leaf and all $n \ge 1$,

$$|P^n x - P^n y| \le C\lambda_0^n. \tag{2.2}$$

Remark 2.4 Throughout this paper, C denotes a generic constant $C \ge 1$ that depends only on the geometric Lorenz flow T_t and which may vary from line to line.

We assume that T_t has a stable foliation. For notational convenience, suppose that each stable leaf intersects $X_1 = \{(x_1, 0, 1) : |x_1| \le 1\} \cong [-1, 1]$ in a single point. Let $\widehat{\ }: X \to X_1$ denote the $C^{1+\epsilon}$ projection along leaves. Quotienting along stable leaves, we obtain a $C^{1+\epsilon}$ one-dimensional map $T : X_1 \to X_1$ with a singularity at 0. In symbols, $Tx_1 = \{P(x_1, 0, 1)\}\widehat{\ }$.

Remark 2.5 We can view X_1 as a subset or quotient of X. The interpretation should be clear from the context. In later sections we often write $\bar{X}$ instead of X_1.

Proposition 2.6 *(a)* $T'x_1 = |x_1|^{\beta-1} g(x_1)$ *where* $g \in C^{\beta\epsilon}(X_1)$, $g > 0$.

(b) $h = h_1 + h_2$ *where* $h_1(x) = -\lambda_u^{-1} \log |\widehat{x}|$ *and* $h_2 \in C^\epsilon(X)$.

(c) $h = \widetilde{h}_1 + \widetilde{h}_2$ *where* $\widetilde{h}_1(x) = \lambda_u^{-1}(1 - \beta)^{-1} \log T'\widehat{x}$ *and* $\widetilde{h}_2 \in C^{\beta\epsilon}(X)$.

Proof Part (c) is immediate from (a) and (b). By Proposition 2.2, $Tx_1 = \{P_2(1, 0, x_1^\beta)\}\widehat{\ }$. Part (a) follows since $\widehat{\ }$ and P_2 are $C^{1+\epsilon}$.

Note that $\widehat{x}$ has the same sign as x_1 and in particular, $\widehat{x} = 0$ if and only if $x_1 = 0$. Define $v(x) = \log |\widehat{x}| - \log |x_1|$ when $x_1 \ne 0$ and $v(x) = 0$ when $x_1 = 0$. We show that $v : X \to \mathbb{R}$ is Hölder.

Let $p(x) = \widehat{x}$ be the projection along stable leaves. Since p is $C^{1+\epsilon}$ and $p(0, x_2) \equiv 0$, we can write $p(x) = x_1 q(x)$ where q is C^ϵ and positive. Hence

$$v(x) = \log|p(x)| - \log|x_1| = \log(|x_1|q(x)) - \log|x_1| = \log q(x),$$

which is Hölder provided that $q(x) = \widehat{x}/x_1$ is bounded below.

Let $u_{\widehat{x}}$ be the graph of the stable leaf containing $\widehat{x}$. Then $x_1 = u_{\widehat{x}}(x_2) = u_{\widehat{x}}(x_2) - u_0(x_2)$ so $|x_1| \le C|\widehat{x}|$ (since the foliation is Lipschitz). Hence $q(x)$ is bounded below proving part (b). ∎

In addition, we assume that T is uniformly expanding: there are constants $\lambda_1 > 1$ and $c > 0$ such that for all $x_1 \in X_1$ and $n \ge 1$,

$$(T^n)'(x_1) \ge c\lambda_1^n. \tag{2.3}$$

For our purposes, a *geometric Lorenz flow* is a three-dimensional flow with an equilibrium satisfying the eigenvalue conditions (2.1), possessing a stable foliation as in Definition 2.3 with quotient map T satisfying (2.3).

Proposition 2.7 *Let $\gamma = \max\{\lambda_0, \lambda_1^{-1}\} \in (0,1)$. If $x, y \in Y$ satisfy $\operatorname{sgn} T^j\widehat{x} = \operatorname{sgn} T^j\widehat{y}$ for $j = 0, 1\dots, n$, then $|P^m\widehat{x} - P^m\widehat{y}| \le C\gamma^{n-m}$ for $m = 0, 1, \dots, n$.*

Proof By assumption, T^{n-j} restricts to a bijection on an interval containing $T^j\widehat{x}$ and $T^j\widehat{y}$. By (2.3), we have $2 \ge |T^{n-j}T^j\widehat{x} - T^{n-j}T^j\widehat{y}| \ge c\lambda_1^{n-j}|T^j\widehat{x} - T^j\widehat{y}|$. Hence

$$|T^j\widehat{x} - T^j\widehat{y}| \le 2c^{-1}\gamma^{n-j}. \tag{2.4}$$

We claim that

$$\|(dP^m)_x\| \le C\sum_{j=1}^m \gamma^{m-j}(T^j)'(\widehat{x}). \tag{2.5}$$

Then it follows from (2.4) and (2.5) that

$$|P^m\widehat{x} - P^m\widehat{y}| \le 2c^{-1}C\sum_{j=1}^m \gamma^{m-j}\gamma^{n-j} \le 2c^{-1}C(1-\gamma^2)^{-1}\gamma^{n-m}.$$

It remains to prove the claim. After a preliminary C^1 change of coordinates, we may assume that the stable leaves are vertical in X. Recall that $P = P_2 \circ P_1$ where P_1 is as given in Proposition 2.2. The assumption on the stable foliation means that

$$P_2(1, u_2, u_3) = (Q(u_3), R(u_2, u_3), 1),$$

where Q and R are smooth. Hence $Px = (Tx_1, Sx)$ where $S(x) = g(x_1^\beta, x_1^{\beta'}x_2)$ and g is C^1. We can write $(dP^m)_x = \begin{pmatrix} (T^m)'(x_1) & 0 \\ U_m(x) & V_m(x) \end{pmatrix}$. By definition of the stable foliation, $|V_m| \le C\lambda_0^m$. Also, $U_1(x) = \partial_{x_1}S(x)$ so by Proposition 2.6(a)

$$|U_1(x)| \le Cx_1^{\beta-1} \le CT'x_1.$$

Since $(dP^m)_x = (dP^{m-1})_{Px}(dP)_x$, we have the relation $U_m(x) = U_{m-1}(Px)T'(x_1) + V_{m-1}(Px)U_1(x)$. It follows by induction that

$$U_{m+1}(x) = U_1(P^m x)(T^m)'(x_1) + \sum_{j=0}^{m-1} V_{m-j}(P^{j+1}x)U_1(P^j x)(T^j)'(x_1),$$

and so $|U_m(x)| \le C\sum_{j=1}^m \lambda_1^{m-j}(T^j)'(x_1)$. We have now obtained estimates for all the terms in $(dP^m)_x$, and claim (2.5) is verified. ∎

3 Inducing scheme for Lorenz-like expanding maps

Let $\bar{X} = X_1 = [-1,1]$ with Lebesgue measure m. In §2, we recalled how to extract a one-dimensional singular expanding map $T : \bar{X} \to \bar{X}$ from a geometric Lorenz flow. In this section, we state a result yielding an induced map $F : \bar{Y} \to \bar{Y}$. The ASIP for "weighted Lipschitz" observables on the induced system then follows from [11].

Definition 3.1 A $C^{1+\epsilon}$ map $T : \bar{X} - \{0\} \to \bar{X}$ is called a *Lorenz-like expanding map* if T satisfies Proposition 2.6(a) and condition (2.3), and $T(0^+) = -1$, $T(0^-) = +1$, $T(1) \in (0,1)$, $T(-1) \in (-1,0)$.

The following result is proved in §5.

Theorem 3.2 *There exists a measurable subset $\bar{Y} = \bar{Y}^- \cup \bar{Y}^+ \subset \bar{X} - \{0\}$ with $m(\bar{Y}^\pm) > 0$ satisfying the following properties:*

(1) There is a countable measurable partition $\{\bar{Y}_j\}$ of $\bar{Y}$ consisting of subsets of $(-1,0)$ and $(0,1)$.

(2) There is a **return time function** *$r : \bar{Y} \to \mathbb{Z}^+$, constant on each $\bar{Y}_j$ such that the* **induced map** *$F(y) = T^{r(y)}(y)$ restricts to a bijection $F : \bar{Y}_j \to \bar{Y}^-$ or $F : \bar{Y}_j \to \bar{Y}^+$ for all j.*

(3) $\sum_{\ell=0}^{r(y)-1} |\log T'(T^\ell x) - \log T'(T^\ell y)| \le C|Fx - Fy|^{\epsilon\beta}$ for all $x, y \in \bar{Y}_j$.

(4) $m(r \ge n) \le C\rho^{n^\gamma}$ where $\rho, \gamma \in (0,1)$.

(5) $-\log|T^n y| \le Cn$ for all $y \in \bar{Y}$, $n \ge 1$.

Remark 3.3 (i) A consequence of condition (3) is that $|\log F'x - \log F'y| \le C|Fx - Fy|^{\epsilon\beta}$ for all $x, y \in \bar{Y}_j$, i.e. $F : \bar{Y} \to \bar{Y}$ has bounded distortion.
(ii) It is well-known that there exist inducing schemes satisfying conditions (1)–(4) for Lorenz-like expanding maps and which moreover achieve exponential decay in (4). We require the auxiliary condition (5), which is achieved at the cost of weakening the decay rate in (4) to a stretched exponential decay rate (for any $\gamma \in (0, \frac{1}{2})$).

(iii) The full strength of the estimates in (4) and (5) are not required for the ASIP. For example, if we assume the growth estimate in (5), then we can weaken (4) to the condition that $\sum_{n\ge1} n^{4+\epsilon}m(r=n)<\infty$ for some $\epsilon>0$.
(iv) Condition (5) implies uniform quadratic growth in $r(y)$ for $\sum_{\ell=0}^{r(y)-1}\log|T^\ell y|$. One approach [10] to proving rapid decay of correlations for flows relies crucially on a uniform linear growth rate for this quantity coupled with an exponential decay rate in condition (4). It seems likely that parameter exclusion arguments combined with the techniques in this paper and in [10] will yield rapid decay of correlations for at least a positive measure set of parametrised Lorenz attractors. This is the subject of work in progress.

Standard arguments guarantee that there is a unique ergodic F-invariant probability measure $\bar\mu$ on $\bar Y$ equivalent to $m|_{\bar Y}$. Moreover, $d\bar\mu/dm|_{\bar Y}$ is bounded.

Proposition 3.4 *$r\in L^p(\bar Y,\bar\mu)$ for all $1\le p<\infty$.*

Proof It suffices to show that $r\in L^p(\bar Y,m|_{\bar Y})$. This follows from Theorem 3.2(4). ∎

For $x,y\in\bar Y$, define the *separation time* $s(x,y)$ to be the least integer $n\ge0$ such that F^nx,F^ny lie in distinct partition elements $\bar Y_j$. Given $\theta\in(0,1)$ we define the metric $d_\theta(x,y)=\theta^{s(x,y)}$.

Lemma 3.5 *Let $\Phi:\bar Y\to\mathbb{R}$ be an observable with $\int_{\bar Y}\Phi\,d\bar\mu=0$. Suppose that there exists an integer $q\ge0$ such that*

(a) $|\Phi(y)|\le Cr(y)^q$, and

(b) $|\Phi(x)-\Phi(y)|\le Cr(y)^q d_\theta(x,y)$,

for all $x,y\in\bar Y_j$, $j\ge1$. Then Φ satisfies the ASIP on $(\bar Y,\bar\mu)$.

Proof By Proposition 3.4, $r^q\in L^{2+\delta}(\bar Y,\bar\mu)$. Now apply [11, Corollary 2.5]. ∎

4 ASIP for Lorenz flows

In this section we prove Theorem 1.1 (assuming Theorem 3.2 to be valid). In §4.1, we show how a Hölder observable ψ on the flow leads via a sequence of reductions to an observable $\widehat\Phi$ on the induced system $F:\bar Y\to\bar Y$ satisfying the hypotheses of Lemma 3.5 and hence satisfying the ASIP. In §4.2, we recall briefly how the absolutely continuous measure $\bar\mu$ on $\bar Y$ lifts to an SRB measure for the flow. In §4.3, we show that the ASIP for $\widehat\Phi$ lifts to the ASIP for ψ.

4.1 Estimates for observables

Let $\psi : \mathbb{R}^3 \to \mathbb{R}$ be a Hölder observable and define $\phi : X \to \mathbb{R}$, $\phi(x) = \int_0^{h(x)} \psi(T_t x)\, dt$.

Proposition 4.1 *Suppose that $\psi \in C^\eta(\mathbb{R}^3)$. Let $x, y \in X$ with $\operatorname{sgn} x_1 = \operatorname{sgn} y_1$. Then $|\phi(x)| \le |\psi|_\infty h(x)$ and $|\phi(x) - \phi(y)| \le C\|\psi\|_\eta \{h(x)|x-y|^{\eta\beta} + |h(x) - h(y)|\}$.*

Proof The first estimate is immediate. For the second estimate, we suppose that $x_1, y_1 > 0$. By Proposition 2.2, $h(x) = \tau_1(x) + h_0(x)$ where $\tau_1(x) = -\lambda_u^{-1} \log x_1$ and $h_0 \in C^\beta(X)$. Then $\phi = \phi_1 + \phi_2$, where

$$\phi_1(x) = \int_0^{\tau_1(x)} \psi(T_t x)\, dt, \qquad \phi_2(x) = \int_0^{h_0(x)} \psi(T_t P_1 x)\, dt.$$

It follows from Proposition 2.2 that $|P_1 x - P_1 y| \le 3|x-y|^\beta$. Since the flow is C^1, $|T_t P_1 x - T_t P_1 y| \le C|x-y|^\beta$ for all $0 \le t \le |h_0|_\infty$. Hence

$$\begin{aligned} |\phi_2(x) - \phi_2(y)| &\le |h_0(x) - h_0(y)||\psi|_\infty + |h_0|_\infty |\psi|_\eta |T_t P_1 x - T_t P_1 y|^\eta \\ &\le |h_0|_\beta |\psi|_\infty |x-y|^\beta + C|h_0|_\infty |\psi|_\eta |x-y|^{\eta\beta} \le C\|\psi\|_\eta |x-y|^{\eta\beta}. \end{aligned}$$

By a change of variables, $\phi_1(x) = \tau_1(x) \int_0^1 \psi(T_{t\tau_1(x)} x)\, dt$. Hence

$$|\phi_1(x) - \phi_1(y)| \le \tau_1(x) |\psi|_\eta \sup_{0 \le t \le 1} |T_{t\tau_1(x)} x - T_{t\tau_1(y)} y|^\eta + |\tau_1(x) - \tau_1(y)||\psi|_\infty.$$

Using the form of the linearised flow, we have

$$T_{t\tau_1(x)}(x_1, x_2, x_3) = (x_1^{1-t}, x_1^{t\beta'} x_2, x_1^{t\beta}), \quad 0 \le t \le 1,$$

and so $\sup_{0 \le t \le 1} |T_{t\tau_1(x)} x - T_{t\tau_1(y)} y| \le 4|x-y|^\beta$. The result follows. ∎

Let Y be the lift of $\bar{Y}$ to X with measurable partition $\{Y_j\}$, return time function $r : Y_j \to \mathbb{Z}^+$ constant on partition elements. For $x, y \in Y$, define the *separation time* $s(x, y)$ to be the least integer $n \ge 0$ such that $F^n \widehat{x}, F^n \widehat{y}$ lie in distinct partition elements $\bar{Y}_j$.

We model $P : X \to X$ by a tower $f : \Delta \to \Delta$, see Young [21, 22]. Here $\Delta = Y^r$ is a discrete suspension with partition $\Delta_{j,\ell} = Y_j \times \{\ell\}$ and $f(y, \ell) = f(y, \ell+1)$ for $\ell = 0, \dots, r(y) - 2$ and $f(y, r(y) - 1) = (Py, 0)$. The map $\pi : \Delta \to X$ given by $\pi(y, \ell) = P^\ell y$ satisfies $f \circ \pi = \pi \circ P$.

If $p = (y, \ell) \in \Delta$, we define $\widehat{p} = (\widehat{y}, \ell)$. Two points $p, q \in \Delta$ are said to lie in the same stable leaf if $\widehat{p} = \widehat{q}$. Quotienting along stable leaves, we obtain the quotient tower $\bar{\Delta}$. For $\widehat{p} = (\widehat{x}, \ell)$, $\widehat{q} = (\widehat{y}, \ell) \in \bar{\Delta}$, define $s(\widehat{p}, \widehat{q}) = s(\widehat{x}, \widehat{y})$. Given any $\theta \in (0, 1)$, we define a metric d_θ on $\bar{\Delta}$ by setting $d_\theta(\widehat{p}, \widehat{q}) = \theta^{s(\widehat{p}, \widehat{q})}$.

Define $\chi : \Delta \to \mathbb{R}$, $\chi(p) = \sum_{m=0}^\infty \phi(P^m \pi p) - \phi(P^m \pi \widehat{p})$.

Lemma 4.2 *$|\chi(p)| \le C\|\psi\|_\eta$ for all $p \in \Delta$.*

Proof We suppose without loss that $p=(y,0)\in Y$. (If $p=(y,\ell)$, then we can replace p by $(y,0)$ which gives an overestimate.) Then $P^m\pi p=P^m y$. By Proposition 2.6(b) and Theorem 3.2(5),

$$h(P^m y)=-\lambda_u^{-1}\log|\widehat{P^m y}|+h_2(P^m y)\le-\lambda_u^{-1}\log|T^m\widehat{y}|+|h_2|_\infty\le C(m+1),$$
$$|h(P^m y)-h(P^m\widehat{y})|=|h_2(P^m y)-h_2(P^m\widehat{y})|\le|h_2|_\epsilon|P^m y-P^m\widehat{y}|^\epsilon.$$

Hence, by condition (2.2) and Proposition 4.1,

$$\begin{aligned}|\chi(p)|&\le C\|\psi\|_\eta\sum_{m=0}^\infty(m+1)|P^m y-P^m\widehat{y}|^{\eta\beta}+|h_2|_\epsilon|P^m y-P^m\widehat{y}|^\epsilon\\&\le C\|\psi\|_\eta\sum_{m=0}^\infty(m+1)\lambda_0^{\alpha m}\le C\|\psi\|_\eta;\qquad\alpha=\min\{\epsilon,\eta\beta\}.\end{aligned}$$

∎

Now define $\widehat{\phi}=\phi\circ\pi+\chi\circ f-\chi:\Delta\to\mathbb{R}$. Note that $\widehat{\phi}:\Delta\to\mathbb{R}$ is a mean zero observation and that $\widehat{\phi}(p)=\sum_{m=0}^\infty\phi(P^m\pi\widehat{p})-\phi(P^m\pi\widehat{fp})$. In particular, $\widehat{\phi}(p)=\widehat{\phi}(\widehat{p})$, so $\widehat{\phi}$ depends only on future coordinates and can be viewed as an observable $\widehat{\phi}:\bar{\Delta}\to\mathbb{R}$. In the next result, we define $r:\Delta\to\mathbb{Z}^+$ by $r(y,\ell)=r(y)$.

Lemma 4.3 *There exists $\theta\in(0,1)$ such that $|\widehat{\phi}(p)-\widehat{\phi}(q)|\le C\|\psi\|_\eta r(p)d_\theta(\widehat{p},\widehat{q})$, for all $p,q\in\bar{\Delta}$.*

Proof We let $s(\widehat{p},\widehat{q})=2N$ and prove that $|\widehat{\phi}(p)-\widehat{\phi}(q)|\le C\|\psi\|_\eta r(p)N\gamma_1^N$ where $\gamma_1=\gamma^\alpha$, $\gamma=\max\{\lambda_0,\lambda_1^{-1}\}$, $\alpha=\min\{\epsilon,\eta\beta\}$. The result follows for any $\theta>\gamma_1^{1/2}$.

Write $|\widehat{\phi}(p)-\widehat{\phi}(q)|\le I+II+III+IV$, where

$$I=\sum_{m=0}^N|\phi(P^m\pi\widehat{p})-\phi(P^m\pi\widehat{q})|,\qquad II=\sum_{m=0}^{N-1}|\phi(P^m\pi\widehat{fp})-\phi(P^m\pi\widehat{fq})|,$$
$$III=\sum_{m=N}^\infty|\phi(P^m\pi f\widehat{p})-\phi(P^m\pi\widehat{fp})|,\quad IV=\sum_{m=N}^\infty|\phi(P^m\pi f\widehat{q})-\phi(P^m\pi\widehat{fq})|.$$

We give the details for terms I and III, the remaining terms being similar.

In term III, note that $f\widehat{p}$ and $\widehat{fp}$ lie in the same stable leaf. Moreover, $f\widehat{p}=\widehat{fp}$ (and so $III=0$) except possibly if $f\widehat{p},\widehat{fp}\in Y$. In this case, arguing as in the proof of Lemma 4.2, but with the sum starting at $m=N$ instead of $m=0$, we obtain $III\le C\|\psi\|_\eta N\gamma_1^N$.

Next we consider term I. Note that p and q do not separate during this part of the calculation since $s(p,q)=2N$. Writing $p=(x,\ell)$, $q=(y,\ell)$,

$$I=\sum_{m=0}^N|\phi(P^{m+\ell}\widehat{x})-\phi(P^{m+\ell}\widehat{y})|=\Big(\sum_{m=\ell}^{r(x)-1}+\sum_{m=r(x)}^{N+\ell}\Big)|\phi(P^m\widehat{x})-\phi(P^m\widehat{y})|$$

so that

$$|I| \le \sum_{k=\ell}^{r(x)-1} |\phi(P^k\widehat{x}) - \phi(P^k\widehat{y})| + \sum_{m=0}^{N} |\phi(P^m F\widehat{x}) - \phi(P^m F\widehat{y})|.$$

By Proposition 2.6(b,c), Theorem 3.2(5) and Proposition 4.1,

$$|\phi(P^k\widehat{x}) - \phi(P^k\widehat{y})| \le C\|\psi\|_\eta\big\{(k+1)|P^k\widehat{x} - P^k\widehat{y}|^\alpha + |\log T'(T^k\widehat{x}) - \log T'(T^k\widehat{y})|\big\}.$$

Note that $s(P^k\widehat{x}, P^k\widehat{y}) = 2N$ for $0 \le k \le r(x)-1$. Hence by Proposition 2.7 and Theorem 3.2(3), and noting that $r(x) - \ell \le N$,

$$\sum_{k=\ell}^{r(x)-1} |\phi(P^k\widehat{x}) - \phi(P^k\widehat{y})| \le C\|\psi\|_\eta\big\{ \sum_{k=\ell}^{r(x)-1} (k+1)\gamma_1^{2N} + \sum_{k=0}^{r(x)-1} |\log T'(T^k\widehat{x}) - \log T'(T^k\widehat{y})|\big\}$$
$$\le C\|\psi\|_\eta\{Nr(x)\gamma_1^{2N} + |F\widehat{x} - F\widehat{y}|^\alpha\} \le C\|\psi\|_\eta\{Nr(x)\gamma_1^{2N} + \gamma_1^{2N-1}\} \le C\|\psi\|_\eta Nr(x)\gamma_1^{2N}.$$

Since $s(P^m F\widehat{x}, P^m F\widehat{y}) \ge 2N - m - 1$, a similar argument shows that

$$\sum_{m=0}^{N} |\phi(P^m F\widehat{x}) - \phi(P^m F\widehat{y})| \le C\|\psi\|_\eta N\gamma_1^N,$$

completing the proof. ∎

Finally, define $\widehat{\Phi} : Y \to \mathbb{R}$ by setting $\widehat{\Phi}(y) = \sum_{\ell=0}^{r(y)-1} \widehat{\phi}(y,\ell)$. Again $\widehat{\Phi}$ depends only on future coordinates and can be viewed as an observable $\widehat{\Phi} : \bar{Y} \to \mathbb{R}$.

Corollary 4.4 *Let $x, y \in \bar{Y}_j$ for some j. Then*

(a) $|\widehat{\Phi}(y)| \le C\|\psi\|_\eta r(y)^2$.

(b) $|\widehat{\Phi}(x) - \widehat{\Phi}(y)| \le C\|\psi\|_\eta r(y)^2 d_\theta(x,y)$.

Proof We have $\widehat{\Phi}(y) = \sum_{\ell=0}^{r(y)-1} \phi(T^\ell y) + \chi(Fy) - \chi(y)$. By Proposition 2.6(b), Theorem 3.2(5), Proposition 4.1 and Lemma 4.2, $|\widehat{\Phi}(y)| \le C|\psi|_\infty \sum_{\ell=0}^{r(y)-1} \ell + C\|\psi\|_\eta$, proving part (a). Part (b) is immediate from Lemma 4.3. ∎

4.2 Measures

Recall from §3 that there is a unique ergodic F-invariant probability measure $\bar{\mu}$ on $\bar{Y}$ equivalent to Lebesgue measure m on $\bar{Y}$. Moreover the density $g = d\bar{\mu}/dm$ is bounded above. Standard techniques [19, Chapter 6.3] lead from $\bar{\mu}$ to an SRB measure μ for the underlying flow. We take a marginally different, but equivalent, route to the definition of μ, defining intermediate measures on Y, Δ and X.

μ_Y**:** Let $P^r : Y \to Y$ be the induced map for $P : X \to X$ (in the same way that $F = T^r : \bar{Y} \to \bar{Y}$ is the induced map for $T : \bar{X} \to \bar{X}$). Using Definition 2.3, a general construction (see [4, p. 22] or [19, Chapter 6.3]) leads to an ergodic P^r-invariant probability measure μ_Y on Y. Moreover, $\bar{\mu}$ is the push forward of μ_Y by the projection $Y \to \bar{Y}$ along stable leaves.

μ_Δ**:** Recall that Y is the base of the tower Δ. We define μ_Δ on $\Delta_{j,\ell} = Y_j \times \ell$ to be a copy of $\mu_Y|_{Y_j}$. Normalising by $\int_Y r\, d\mu_Y$ yields an ergodic f-invariant probability measure μ_Δ on Δ.

μ_X**:** Use the projection $\pi : \Delta \to X$ to define the push forward measure $\mu_X = \pi_* \mu_\Delta$. This defines an ergodic T-invariant probability measure on X.

μ**:** Define $\mu = \mu_X \times \text{Lebesgue} / \int_X h\, d\mu_X$ to obtain an ergodic T_t-invariant probability measure on the suspension $X^h \subset \mathbb{R}^3$.

It is immediate from the definitions that $\bar{\mu}$ is an SRB measure for $F : \bar{Y} \to \bar{Y}$. The SRB property is preserved throughout the steps described above, resulting in the SRB measure μ for the original flow T_t.

Proposition 4.5 *$h \in L^p(X, \mu_X)$ for all $1 \le p < \infty$.*

Proof Write $h = h_1 + h_2$ as in Proposition 2.6(b) with h_2 bounded. Compute that

$$\int_X h_1^p \, d\mu_X = \int_\Delta h_1^p \circ \pi \, d\mu_\Delta = \sum_{j,\ell} \int_{\Delta_{j,\ell}} h_1^p \circ \pi \, d\mu_\Delta = \sum_{j,\ell} \int_{\bar{Y}_j} h_1^p \circ T^\ell \, d\bar{\mu}$$

$$= \sum_{j,\ell} \int_{\bar{Y}_j} h_1^p \circ T^\ell \, g \, dm \le |g|_\infty \sum_j \int_{\bar{Y}_j} \Big(\sum_{\ell=0}^{r(y)-1} h_1(T^\ell y) \Big)^p \, dm.$$

By Theorem 3.2(5), $\sum_{\ell=0}^{r(y)-1} h_1(T^\ell y) \le C r(y)^2$ so $\int_X h_1^p \, d\mu_X \le C \sum_j \int_{\bar{Y}} r^{2p} dm.$ ∎

4.3 Proof of the ASIP

To prove the ASIP for ψ on (X^h, μ) we begin with the ASIP for $\widehat{\Phi}$ on $(\bar{Y}, \bar{\mu})$ and follow the route in §4.2 proving intermediate ASIPs on Y, Δ and X.

$\widehat{\Phi}$ **on** $\bar{Y}$**:** This follows from Lemma 3.5 and Corollary 4.4.

$\widehat{\Phi}$ **on** Y**:** Since the projection $Y \to \bar{Y}$ is measure-preserving.

$\widehat{\phi}$ **on** Δ**:** By [12] (see eg. [11, Corollary B.2]), this follows from the ASIP on Y for $\widehat{\Phi}$ and similarly for r (more precisely $r - \int_Y r\, d\mu_Y$), and the fact that $r \in L^{2+\delta}(Y, \mu_Y)$.

ϕ **on** X**:** Since $\phi \circ \pi = \widehat{\phi} + \chi - \chi \circ T$ where χ is bounded (Lemma 4.2), $\phi \circ \pi$ satisfies the ASIP on Δ. Now use the fact that $\pi : \Delta \to X$ is measure-preserving.

ψ on X^h: The roof function h (more precisely $h - \int_X h\, d\mu_X$) is a special case of ϕ (with $\psi \equiv 1$) and so h also satisfies the ASIP on (X, μ_X). By Proposition 4.5, $h \in L^{2+\delta}(X, \mu)$. The ASIP for ψ on (X^h, μ) follows by [12, Theorem 4.2].

Nondegeneracy We end this section by discussing the nondegeneracy criterion in Remark 1.2. Recall that d_θ defines a metric on $\bar{Y}$.

Proposition 4.6 *The ASIP for $\psi : X^h \to \mathbb{R}$ is degenerate if and only if there exists a Lipschitz function $w : \bar{Y} \to \mathbb{R}$ such that $\widehat{\Phi} = w \circ F - w$.*

Proof First notice that nondegeneracy is preserved throughout all the steps from $\bar{Y}$ to X^h (by [12] for the suspension steps and immediately for the other two steps). Hence it suffices to consider nondegeneracy at the level of $\widehat{\Phi}$ on $\bar{Y}$. Now apply Melbourne & Nicol [11, Corollary 2.3(c)]. ∎

We can view $F : \bar{Y} \to \bar{Y}$ as a shift on infinitely many symbols and it is clear that there are infinitely many periodic orbits for F. (It is less clear how these relate to the periodic orbits for the underlying flow, but that is of no consequence for this argument). If $y \in \bar{Y}$ is a periodic point of period k, define $\tau(y) = \sum_{j=0}^{k-1} \widehat{\Phi}(F^j y)$. It follows from Proposition 4.6 that if the ASIP is degenerate then $\tau(y) = 0$ for every periodic point y. Hence degeneracy occurs inside a closed subspace of infinite codimension as claimed in Remark 1.2.

5 Construction of the inducing scheme

In the remainder of the paper, we prove Theorem 3.2. For simplicity of notation, we drop the "bars". In particular, $T : X \to X$ is a Lorenz-like expanding map, $X \cong [-1, 1]$, as in Definition 3.1 and we construct an induced map $F : Y \to Y$ satisfying the properties listed in Theorem 3.2.

The structure of the proof is organised as follows. In §5.1, we define the set $Y = Y^- \cup Y^+ \subset X$ and establish some elementary properties of Y. In particular, the set Y is defined so that estimate (5) holds and we verify that $m(Y^\pm) > 0$. In §5.2, we construct a partition $\{Y_j\}$ with return time function $r : Y \to \mathbb{Z}^+$ satisfying conditions (1,2). The bounded distortion estimates (3) and tail estimates (4) are proved in the appendices.

Preliminaries Fix $\delta \in (0, 1)$ with $d_\delta = \log \delta^{-1} \in \mathbb{N}$ and let $U = (-\delta, \delta)$. For each $d \geq d_\delta$, define $I_d = [e^{-(d+1)}, e^{-d})$ and subdivide each I_d into d^2 identical subintervals $I_{d,m}$, $1 \leq m \leq d^2$. Set $I_{-d} = -I_d$, $I_{-d,m} = -I_{d,m}$. Define the interval partition of U $\mathcal{I}_U = \{I_{d,m} : |d| \geq d_\delta, \, m = 1, \ldots, d^2\}$. Then $\mathcal{I} = \mathcal{I}_U \cup \{0\} \cup \{\pm[\delta, 1]\}$ is an interval partition of X.

Given $I_{d,m} \in \mathcal{I}_U$, let $I^{\pm}_{d,m}$ denote the elements of $\mathcal{I}$ that are adjacent to $I_{d,m}$ on the right and left. Define $\widehat{I}_{d,m} = I^{-}_{d,m} \cup I_{d,m} \cup I^{+}_{d,m}$. Then $\widehat{\mathcal{I}} = \{\widehat{I}_{d,m} : |d| \geq d_\delta,\ m = 1, \dots, d^2\} \cup \{0\}$ is a cover of X.

5.1 Definition of Y

Choose δ small enough that $TU \cap U = \emptyset$. Let $\Omega_0 = \Omega_0^- \cup \Omega_0^+$ where $\Omega_0^{\pm} = I_{\pm d_\delta,1}$ are the outermost elements of $\mathcal{I}_U$. Following [3, Section 3.2], we fix $\alpha_1 > 0$ and define inductively $\Omega_0 \supset \Omega_1 \supset \cdots$ as follows. Let ω be a connected component of Ω_{n-1}. We delete from ω the interval $T^{-n}(-e^{-\alpha_1 n}, e^{-\alpha_1 n})$. Further, if ω' is a component of what remains of ω and $T^n\omega'$ does not cover an element of $\mathcal{I}_U$ then we delete ω' as well. (In particular, the deleted subset lies inside $T^{-n}(-e^{-(\alpha_1-1)n}, e^{-(\alpha_1-1)n})$.) Define $Y = \bigcap_{n\geq 1} \Omega_n$. Also, set $\Omega_n^{\pm} = \Omega_n \cap \Omega_0^{\pm}$ and $Y^{\pm} = Y \cap \Omega_0^{\pm}$.

Proposition 5.1 *Estimate (5) of Theorem 3.2 is valid, and* $\sum_{\ell=1}^n \log T'(T^\ell y) \leq Cn^2$ *for all* $y \in \Omega_n$, $n \geq 1$.

Proof The validity of estimate (5) is immediate from the definitions. The second statement follows from Proposition 2.6(a). ∎

Corollary 5.2 *There exists* $\lambda_2 > 1$ *such that* $c\lambda_1^n \leq (T^n)'(y) \leq \lambda_2^{n^2}$ *for all* $y \in \Omega_n$, $n \geq 1$. ∎

Proposition 5.3 *For all* $n \geq 1$, $m(\Omega_{n-1}^{\pm} - \Omega_n^{\pm}) \leq c^{-1}e^{-(\alpha_1-2)n}$.

Proof Let ω be a component of $\Omega_{n-1}^{\pm}$ and let $\omega' \subset \omega$ be the piece that will be deleted in the n'th step. Then $T^n\omega' \subset (-e^{-(\alpha_1-1)n}, e^{-(\alpha_1-1)n})$ and so by (2.3), $m(\omega') \leq c^{-1}m(T^n\omega') \leq 2c^{-1}e^{-(\alpha_1-1)n}$. Also $\Omega_{n-1}^{\pm}$ consists of at most 2^{n-1} components ω. ∎

Corollary 5.4 *For* α_1 *sufficiently large,* $m(Y^{\pm}) > 0$.

Proof Let $\alpha = \alpha_1 - 2$. By Proposition 5.3,

$$m(\Omega_0^{\pm} - Y^{\pm}) = \sum_{n\geq 1} m(\Omega_{n-1}^{\pm} - \Omega_n^{\pm}) \leq c^{-1}(e^\alpha - 1)^{-1}.$$

Choose α so that $c^{-1}(e^\alpha - 1)^{-1} \leq \frac{1}{2}m(\Omega_0^{\pm})$. Then $m(Y^{\pm}) \geq \frac{1}{2}m(\Omega_0^{\pm}) > 0$. ∎

We record the following elementary properties of Y for subsequent use.

Proposition 5.5 *Suppose that* $x \in \Omega_{n-1}$ *and* $T^n x \in Y$. *Then* $x \in Y$.

Proof We must show that $|T^k x| \ge e^{-\alpha_1 k}$ for all $k \ge 1$. The inequality holds for $1 \le k \le n-1$ since $x \in \Omega_{n-1}$ and holds for $k = n$ since $T^n x \in Y \subset (\delta, 1]$. If $k > n$, then $|T^k x| = |T^{k-n}(T^n x)| \ge e^{-\alpha_1(k-n)} \ge e^{-\alpha_1 k}$ since $T^n x \in Y \subset \Omega_{k-n}$. ∎

Note that Y takes the form of a Cantor set. Let $\mathcal{G}$ denote the set of connected components of $\Omega_0 - Y$ (i.e. the gaps of the Cantor set) and define $\mathcal{G}^\pm = \mathcal{G}|Y^\pm$.

Proposition 5.6 *For α_2 sufficiently large, $\sum_{\gamma \in \mathcal{G}: m(\gamma) \le \epsilon} m(\gamma) \le C \exp\{-\alpha_2 (\log \epsilon^{-1})^{1/2}\}$ for all $\epsilon > 0$.*

Proof Let $U_n = (-e^{-n\alpha_1}, e^{-n\alpha_1})$ and let ω be a connected component of Ω_{n-1}. If $T^n\omega$ covers U_n, then this creates a gap $\omega \cap T^{-n}U_n \subset \Omega_{n-1} - \Omega_n$. Since $\omega \subset \Omega_{n-1}$, we have $(T^{n-1})'|\omega \le \lambda_2^{(n-1)^2}$ by Corollary 5.2. Also, $TU \cap U = \emptyset$ and so T' is bounded on $T^{-1}U$. Hence $(T^n)'$ is bounded by $C\lambda_2^{n^2}$ on $\omega \cap T^{-n}U$ so that $m(\omega \cap T^{-n}U_n) \ge 2C^{-1}\lambda_2^{n^2} e^{-n\alpha_1} \ge \lambda_3^{-n^2}$ for some $\lambda_3 > 1$.

If $T^n\omega$ only partially covers U_n, then we can adjoin the partial gap to a previously created gap, thus ensuring that all gaps created at time n have measure at least $\lambda_3^{-n^2}$.

Choose n_0 so that $\lambda_3^{-n_0^2} \sim \epsilon$. Then $\sum_{\gamma \in \mathcal{G}: m(\gamma) \le \epsilon} m(\gamma) \le \sum_{n > n_0} m(\Omega_{n-1} - \Omega_n) \le Ce^{-(\alpha_1 - 2)n_0}$ by Proposition 5.3. The result follows. ∎

5.2 Markov structure for Y

We now define a new nested sequence of sets $\widetilde{\Omega}_n \subset \Omega_n$. We define also a sequence of interval partitions $\mathcal{P}_n$ of $\widetilde{\Omega}_n$ that are nested in the sense that $\mathcal{P}_n$ is a refinement of the restriction of $\mathcal{P}_{n-1}$ to $\widetilde{\Omega}_n$. Simultaneously, we define the return time function r.

Definition 5.7 An interval $\omega \subset \Omega_{n-1}$ makes a *regular return (to $\Omega_0^\pm$ at time n)* if either $T^n\omega \supset \Omega_0^-$ or $T^n\omega \supset \Omega_0^+$.

Set $\widetilde{\Omega}_0 = \Omega_0$ and $\mathcal{P}_0 = \{\Omega_0^\pm\}$. Assume inductively that $\widetilde{\Omega}_{n-1}$ and $\mathcal{P}_{n-1}$ are defined. Let $\omega \in \mathcal{P}_{n-1}$.

- If ω does *not* make a regular return, then (i) Put $\omega' = \omega \cap \Omega_n$ into $\widetilde{\Omega}_n$. (ii) Define $\mathcal{P}_n|\omega' = (T^{-n}\mathcal{I})|\omega'$.
- If ω *does* make a regular return to $\Omega_0^\pm$, then (i) Put $\omega' = (\omega - T^{-n}Y^\pm) \cap \Omega_n$ into $\widetilde{\Omega}_n$. (ii) Define $\mathcal{P}_n|\omega' = (T^{-n}\mathcal{I} \vee T^{-n}\mathcal{G}^\pm)|\omega'$. (iii) Define $r = n$ on $\omega \cap T^{-n}Y^\pm$. (If a regular return occurs simultaneously to Ω_0^- and Ω_0^+ then ignore the superscript $\pm$'s.)
- Modify the definition of $\mathcal{P}_n$: Any end subintervals of $T^n\omega'$ that only partially cover an $I_{d,m}$ are adjoined to the adjacent subinterval.

The properties of $\mathcal{P}_n$ that will be used in this paper can be summarised as follows.

Proposition 5.8 *Let $\omega \in \mathcal{P}_{n-1}$. Then*

(a) $T^\ell\omega$ is covered by an element of $\widehat{\mathcal{I}}$, for each $\ell = 0\dots,n-1$.

(b) If ω does not make a regular return at time n, then either (i) $\omega \cap \Omega_n \in \mathcal{P}_n$ (and is covered by an element of $\widehat{\mathcal{I}}$) or (ii) $\omega \cap \Omega_n = \bigcup \omega^{(d,m)}$ where where each $\omega^{(d,m)}$ lies in $\mathcal{P}_n$ and satisfies $T^n\omega^{(d,m)} \approx I_{d,m}$ for some $I_{d,m} \in \mathcal{I}_U$.

(c) If ω makes a regular return to $\Omega_0^\pm$ at time n, then

$$(\omega - T^{-n}Y^\pm) \cap \Omega_n = (\bigcup \omega^{(d,m)}) \cup \omega_Y \cup \widetilde{\omega},$$

where each $\omega^{(d,m)}$ lies in $\mathcal{P}_n$ and satisfies $T^n\omega^{(d,m)} \approx I_{d,m}$; ω_Y is a union of elements of $\mathcal{P}_n$ and $T^n\omega_Y$ is a union of elements of $\mathcal{G}$; $\widetilde{\omega}$ is the union of finitely many (at most 2) elements of $\mathcal{P}_n$. ∎

Remark 5.9 We refer to the situation of Proposition 5.8(b)(ii) as an *essential return*, and say that ω_i has *essential return depth d*. Note that $T^n\omega \approx I_{d,m}$ means that $I_{d,m} \subset T^n\omega \subset \widehat{I}_{d,m} \cap U$. It is then immediate from the definitions and Proposition 2.6(a) that $m(T^n\omega) \le Ce^{-d}d^{-2}$ and $m(T^{n+1}\omega) \ge C^{-1}e^{-\beta d}d^{-2}$.

Remark 5.10 It follows from the definitions and Proposition 5.5 that

$$\{x \in X : r(x) = n\} = Y \cap (\widetilde{\Omega}_{n-1} - \widetilde{\Omega}_n).$$

We can extend r to a function $r : Y \to \mathbb{Z}^+ \cup \{\infty\}$ by setting $r = \infty$ on $\bigcap_{n\ge0} \widetilde{\Omega}_n$.

Definition 5.11 Define

$$\{Y_j\} = \bigvee_{n\ge1} \{\omega \cap Y : \omega \in \mathcal{P}_{n-1} \text{ makes a regular return at time } n\}.$$

Remark 5.12 Note that the Y_j are distinct since points are discarded once they make a good return. Moreover r is constant on elements Y_j. We can make $\{Y_j\}$ into a partition of Y by adjoining the set $\bigcap_{n\ge1} \widetilde{\Omega}_n$ (where $r = \infty$). However, it turns out in Appendix B that $m(r = \infty) = 0$.

Corollary 5.13 *If $r|Y_j = n$, then $T^nY_j = Y^-$ or $T^nY_j = Y^+$.*

Proof By definition, $Y_j = \omega \cap Y$ where $\omega \in \mathcal{P}_{n-1}$ makes a regular return at time n. Let $y \in Y^\pm$. By definition of regular returns, there exists $x \in \omega$ such that $T^nx = y$. Since $x \in \omega \subset \widetilde{\Omega}_{n-1} \subset \Omega_{n-1}$, it follows from Proposition 5.5 that $x \in Y$. Hence $x \in \omega \cap Y$ and $T^nx = y$. ∎

A Appendix: Bounded distortion estimates

In this appendix, we prove the bounded distortion estimate in Theorem 3.2(3).

Define the cover $\widehat{\mathcal{I}}$ of X starting from the subset $U = (-\delta, \delta) \subset X$ as in §5. Define $\widehat{I}_d$ to be the union of all elements $J \in \widehat{\mathcal{I}}$ such that $J \cap (-I_d \cup I_d) \neq \emptyset$. Then

$$m(J) \leq C \operatorname{dist}(\widehat{I}_d, 0)/d^2, \tag{A.1}$$

for all $J \in \widehat{\mathcal{I}}$, $J \subset \widehat{I}_d$, $|d| \geq d_\delta$. (Note that C is roughly order δ^{-1}.)

Proposition A.1 *Let $\alpha = \beta\epsilon$ be the Hölder exponent in Proposition 2.6(a). If $\operatorname{sgn} x = \operatorname{sgn} y$, then $|\log T'x - \log T'y| \leq C(|x-y|^\alpha + |x-y|/\min\{|x|,|y|\})$.*

Proof By Proposition 2.6(a), $\log T'x = f + (\beta - 1)\log|x|$ where f is C^α. Moreover, for $0 < y < x < 1$, $\log x - \log y = \log(1 + x/y - 1) \leq x/y - 1 = (x-y)/y$. ∎

Theorem A.2 *Let $\omega \subset X$ be an interval and let $n \geq 1$. Suppose that for $\ell = 0, \ldots, n-1$ there exists $J_\ell \in \widehat{\mathcal{I}}$ such that $T^\ell\omega \subset J_\ell$. Then*

(a) $\sum_{\ell=0}^{n-1} |T^\ell x - T^\ell y|^\alpha \leq C$ and $\sum_{\ell=0}^{n-1} |T^\ell x - T^\ell y|/\min\{|T^\ell x|, |T^\ell y|\} \leq C$ for all $x, y \in \omega$.

(b) $C^{-1}\dfrac{m(T^k\omega')}{m(T^k\omega)} \leq \dfrac{m(T^\ell\omega')}{m(T^\ell\omega)} \leq C\dfrac{m(T^k\omega')}{m(T^k\omega)}$ for all $0 \leq k, \ell \leq n$ and all $\omega' \subset \omega$.

Proof For each $d \geq 1$, let $L(d)$ be the largest ℓ so that $J_\ell \subset \widehat{I}_d$. Then by (2.3) and (A.1), for all ℓ with $J_\ell \subset \widehat{I}_d$,

$$m(T^\ell\omega) \leq (c\lambda_1^{L(d)-\ell})^{-1} m(T^{L(d)}\omega) \leq (c\lambda_1^{L(d)-\ell})^{-1} m(J_{L(d)}) \leq C(\lambda_1^{L(d)-\ell})^{-1} \operatorname{dist}(\widehat{I}_d, 0)/d^2.$$

Hence $\sum_{J_\ell \subset \widehat{I}_d} m(T^\ell\omega) \leq C \operatorname{dist}(\widehat{I}_d, 0)/d^2$, and so

$$\sum_{\ell=0}^{n-1} |T^\ell x - T^\ell y|/\min\{|T^\ell x|, |T^\ell y|\} \leq \sum_{d\geq 1}\sum_{J_\ell \subset \widehat{I}_d} m(T^\ell\omega)/\operatorname{dist}(\widehat{I}_d, 0) \leq C\sum_{d\geq 1} 1/d^2.$$

A simpler argument yields

$$\sum_{\ell=0}^{n-1} |T^\ell x - T^\ell y|^\alpha \leq \sum_{d\geq 1}\sum_{J_\ell \subset \widehat{I}_d} m(T^\ell\omega)^\alpha \leq C\sum_{d\geq 1} m(\widehat{I}_d)^\alpha \leq C\sum_{d\geq 1} e^{-d\alpha},$$

completing the proof of part (a).

Let $x, y \in \omega$ and $\ell \le n$. By Proposition A.1 and part (a),

$$\begin{aligned}\log \frac{(T^\ell)'(x)}{(T^\ell)'(y)} &= \sum_{j=0}^{\ell-1} \log T'(T^j x) - \log T'(T^j y) \\ &\le C\big(\sum_{j=0}^{\ell-1} |T^j x - T^j y|^\alpha + \sum_{j=0}^{\ell-1} |T^j x - T^j y| / \min\{|T^j x|, |T^j y|\}\big) \le C,\end{aligned}$$

and so $(T^\ell)'(x)/(T^\ell)'(y) \le C$. Hence for $\omega' \subset \omega$ and $k, \ell \le n$, it follows from the mean value theorem that there exist $\xi_1, \ldots, \xi_4 \in \omega$ such that

$$\frac{m(T^\ell \omega') m(T^k \omega)}{m(T^k \omega') m(T^\ell \omega)} = \frac{(T^\ell)'(\xi_1)(T^k)'(\xi_3)}{(T^k)'(\xi_2)(T^\ell)'(\xi_4)} \le C,$$

proving part (b). ∎

Corollary A.3 *Estimate (3) of Theorem 3.2 is valid.*

Proof Let $n = r(x) = r(y)$, so $T^n Y_j = Y^\pm$. By construction, Y_j lies inside an interval $\omega \in \mathcal{P}_{n-1}$ and hence by Proposition 5.8(a), $T^\ell \omega$ is covered by an element of $\widehat{\mathcal{I}}$ for $0 \le \ell \le n-1$. By Theorem A.2(b),

$$m(T^\ell[x,y]) \le C m(T^n[x,y]) m(T^\ell \omega)/m(T^n \omega) \le C m(Y^\pm)^{-1} m(T^n[x,y]) m(T^\ell \omega),$$

for all $x, y \in \omega$ and all $\ell \le n$. By Proposition A.1,

$$\begin{aligned}|\log T'(T^\ell x) - \log T'(T^\ell y)| &\le C\big\{m(T^\ell[x,y])^\alpha + m(T^\ell[x,y])/\operatorname{dist}(T^\ell \omega, 0)\big\} \\ &\le C m(T^n[x,y])^\alpha \big\{m(T^\ell \omega)^\alpha + m(T^\ell \omega)/\operatorname{dist}(T^\ell \omega, 0)\big\}.\end{aligned}$$

Hence

$$\sum_{\ell=0}^{n-1} |\log T'(T^\ell x) - \log T'(T^\ell y)| \le C m(T^n[x,y])^\alpha \sum_{\ell=0}^{n-1} \{m(T^\ell \omega)^\alpha + m(T^\ell \omega)/\operatorname{dist}(T^\ell \omega, 0)\}.$$

Applying Theorem A.2(a) yields the required result. ∎

In Appendix B, we require the following additional bounded distortion estimate.

Lemma A.4 *Let $\omega \subset X$ be an interval and let $n \ge 1$. Suppose that for $\ell = 0, \ldots, n-1$ there exists $J_\ell \in \widehat{\mathcal{I}}$ such that $T^\ell \omega \subset J_\ell$. Suppose in addition that $T^{n-1}\omega \subset U$. Then there is a constant C_0 independent of δ such that*

$$C_0^{-1} \frac{m(\omega')}{m(\omega)} \le \frac{m(T^n \omega')}{m(T^n \omega)} \le C_0 \frac{m(\omega')}{m(\omega)}.$$

Proof We follow the proof of Theorem A.2. The only constant that depends on δ occurs in the estimate for $m(T^\ell\omega)$ which uses (A.1). The issue is that $\widehat{I}_{d,m}$ is significantly larger than $I_{d,m}$ when $I_{d,m}$ is an outermost element of $\mathcal{I}_U$. In particular, we can take $C = C_0$ independent of δ in (A.1) provided $|d| \geq d_\delta + 1$. Hence, for $T^\ell\omega \subset \widehat{I}_d$ with $|d| \geq d_\delta + 1$, we obtain $m(T^\ell\omega) \leq C_0(\lambda_1^{L(d)-\ell})^{-1}\operatorname{dist}(\widehat{I}_d, 0)/d^2$. It remains to find a similar estimate when $|d| = d_\delta$. Write $T^{n-1}\omega \subset \widehat{I}_{d_*,m_*} \cap U$. Then $m(T^\ell\omega) \leq (c\lambda_1^{n-1-\ell})^{-1}m(I_{d_*,m_*} \cap U) \leq C_0(\lambda_1^{n-1-\ell})^{-1}\operatorname{dist}(\widehat{I}_{d_*}, 0)/d_*^2 \leq C_0(\lambda_1^{n-1-\ell})^{-1}\operatorname{dist}(\widehat{I}_{d_\delta}, 0)/d_\delta^2$. ∎

B Appendix: Tail estimates

In this appendix, we show that $m(r > n)$ decays at a stretched exponential rate provided $U = (-\delta, \delta)$ is chosen small enough, proving Theorem 3.2(4). We follow the argument used in [3] for Hénon maps, and [21] for piecewise expanding maps, where the partition elements are Cantor sets.

Theorem B.1 *For any $\gamma \in (0, \frac{1}{2})$, there exists $\delta > 0$ and $\rho \in (0, 1)$ such that $m(y \in Y : r(y) \geq n) \leq C\rho^{n^\gamma}$.*

We continue to let $C \geq 1$ denote a generic constant depending on the the map $T : X \to X$ and the construction of $F : Y \to Y$ (with the exception of C_0 in Lemma A.4). In addition, we let $\rho \in (0, 1)$ denote a generic constant depending on T and F.

First, for each $x \in X$ we define a (possibly finite) sequence of *regular returns* $0 = \tau_0(x) < \tau_1(x) < \tau_2(x) < \cdots$. Assuming $\tau_{k-1}(x)$ is defined, let $\tau_k(x)$ be the smallest $j > \tau_{k-1}(x)$ such that $x \in \widetilde{\Omega}_{j-1}$ and the interval $\omega \in \mathcal{P}_{j-1}$ containing x makes a regular return to $\Omega_0^\pm$ at time j (if such a j exists). In particular, for $y \in Y$ the sequence $\{\tau_k(x)\}$ is finite and bounded above by $r(y)$.

To prove Theorem B.1, we show that for $y \in Y$, (i) regular returns occur often enough and quickly enough up to time $r(y)$, and (ii) sufficiently many points land in Y on each regular return. Point (ii) is dealt with easily. Let $\Theta_k = \{y \in Y : \tau_k(y) \text{ is defined}\}$.

Proposition B.2 *$m\{y \in \Theta_k : r(y) > \tau_k(y)\} \leq \rho^k$.*

Proof Note that

$$\Theta_k \subset \{y \in \Theta_{k-1} : r(y) > \tau_{k-1}(y)\}. \tag{B.1}$$

Now Θ_k is a disjoint union of subsets $\omega \cap Y$ where each ω lies in $\mathcal{P}_{j-1}$ for some j and $\tau_k|\omega = j$. (In particular, $T^j\omega \supset \Omega_0^\pm$.) By Theorem A.2(b),

$$\frac{m(\omega \cap T^{-j}Y)}{m(\omega)} \geq C^{-1}\frac{m(T^j\omega \cap Y)}{m(T^j\omega)} \geq \tfrac{1}{2}C^{-1}m(Y^\pm) \geq C^{-1} > 0.$$

Hence $m\{y \in \Theta_k : r(y) = \tau_k(y)\} \geq C^{-1}m(\Theta_k)$ and so $m\{y \in \Theta_k : r(y) > \tau_k(y)\} \leq \rho m(\Theta_k)$. The result follows inductively using (B.1). ∎

For point (i) we require the following preliminary lemma.

Lemma B.3 *There exist $\delta > 0$, such that the following holds for all $j \geq 0$, $n \geq 1$. Suppose that $\omega \in \mathcal{P}_{j-1}$ makes a regular return at time j and let $\omega' = (\omega - T^{-j}Y) \cap \Omega_j$. Then*

$$m\{x \in T^j\omega' : T^{-j}x \text{ has no regular returns between } j \text{ and } j+n+1\} \leq C\rho^{\sqrt{n}}.$$

We postpone the proof of Lemma B.3 to the end of the appendix.

Lemma B.4 *(a) For a.e. $y \in Y$, if $y \in \Theta_k$ and $r(y) > \tau_k(y)$, then $y \in \Theta_{k+1}$.*
(b) For any $0 < \gamma' < \gamma < \frac{1}{2}$, there exists $\epsilon > 0$ such that

$$m(\{x \in \widetilde{\Omega}_n \cap \Theta_N : \tau_N(x) > n\}) \leq \rho^{n^\gamma} \text{ for all } N \leq \epsilon n^{\gamma'}.$$

Proof By Lemma B.3, for each j, k,

$$m\{y \in \Theta_k : \tau_k(y) = j \text{ and there are no further regular returns before time } n\} \leq C\rho^{\sqrt{n}},$$

and so $m\{y \in \Theta_k : \tau_k(y) = j \text{ is the final regular return}\} = 0$. Hence for each k, $m\{y \in \Theta_k : \tau_k(y) \text{ is the final regular return}\} = 0$, proving part (a).

To prove part (b), let $1 \leq n_1 < \cdots < n_\ell \leq n$ be fixed for the moment. For $k \leq n$, define $A_k = A_k(n_1, \ldots, n_\ell)$ to be the set

$$\{x \in \widetilde{\Omega}_k : \text{the regular returns of } x \text{ up to time } k \text{ are exactly those } n_i\text{'s with } n_i \leq k\}.$$

(i.e. $\tau_i(x) = n_i$ for all $n_i \leq k$ and if $n_i > k$, then either $\tau_i(x) > k$ or $\tau_i(x)$ is undefined.)

Applying Lemma B.3 to $\omega = \Omega_0$ (with $j = 0$, $n = n_1 - 1$), we obtain $m(A_{n_1-1}) \leq C\rho^{\sqrt{n_1-1}}$. Now A_{n_1-1} is a union of intervals $\omega \in \mathcal{P}_{n_1-1}$. Discard those ω that fail to make a regular return at time n_1 and replace those that do by $\omega' = (\omega - T^{-n_1}Y) \cap \Omega_{n_1}$. Then A_{n_1} is the union of these ω'. By Theorem A.2(b) and Lemma B.3 (with $j = n_1$, $n = n_2 - n_1 - 1$),

$$\frac{m(\omega \cap A_{n_2-1})}{m(\omega)} \leq C\frac{m(T^{n_1}(\omega' \cap A_{n_2-1}))}{m(T^{n_1}\omega)} \leq C\frac{\rho^{\sqrt{n_2-n_1-1}}}{m(\Omega_0^{\pm})}.$$

Hence $m(A_{n_2-1})/m(A_{n_1-1}) \leq C\rho^{\sqrt{n_2-n_1-1}}$. Proceeding inductively,

$$m(A_n) = \frac{m(A_n)}{m(A_{n_\ell-1})}\frac{m(A_{n_\ell-1})}{m(A_{n_{\ell-1}-1})}\cdots\frac{m(A_{n_2-1})}{m(A_{n_1-1})}m(A_{n_1-1}) \leq C^\ell\rho^S \leq C^\ell\rho^{\sqrt{n}},$$

where

$$\begin{aligned}S &= (n_1-1)^{1/2} + (n_2-n_1-1)^{1/2} + \cdots + (n_\ell - n_{\ell-1}-1)^{1/2} + (n-n_\ell-1)^{1/2} \\ &\geq \{(n_1-1)+(n_2-n_1-1)+\cdots+(n-n_\ell-1)\}^{1/2} \geq (n-\ell-1)^{1/2} \\ &\geq n^{1/2} - (\ell+1)^{1/2}.\end{aligned}$$

Choose $\epsilon > 0$ so small that $C^\epsilon \rho = \rho' < 1$. Then for $\ell \leq \epsilon n^\gamma$ we have $m(A_n) \leq (\rho')^{n^{\gamma'}}$ and so for $N \leq \epsilon n^{\gamma'}$,

$$\begin{aligned}m\{x \in \widetilde{\Omega}_n \cap \Theta_N : \tau_N(x) > n\} &= \sum_{\ell=0}^{N-1} \sum_{1 \leq n_1 < \cdots < n_\ell \leq n} m(A_n(n_1,\ldots,n_\ell) \cap \Theta_N) \\ &\leq \sum_{\ell=0}^{N-1} \binom{n}{\ell} (\rho')^{n^{\gamma'}} \leq N \binom{n}{N} (\rho')^{n^{\gamma'}} \leq (\rho'')^{n^\gamma}.\end{aligned}$$

∎

Proof of Theorem B.1 Define $N = [\epsilon n^{\gamma'}]$ as in Lemma B.4(b). By Lemma B.4(a), for almost every $y \in Y$, either $r(y) > \tau_N(y)$, or $r(y) = \tau_k(y)$ for some $k \leq N$. It follows that

$$\{y \in Y : r(y) > n\} \subset \{y \in \Theta_N : r(y) > \tau_N(y)\} \cup \bigcup_{k \leq N} \{y \in \widetilde{\Omega}_n \cap \Theta_k : \tau_k(y) > n\}.$$

Hence the theorem follows from Proposition B.2 and Lemma B.4(b). ∎

Proof of Lemma B.3

Let $\omega \in \mathcal{P}_{j-1}$ be as in the statement of Lemma B.3. By Proposition 5.8(c), $(\omega - T^{-n}Y) \cap \Omega_n = \omega' = (\bigcup \omega^{(d,m)}) \cup \omega_Y \cup \widetilde{\omega}$ where: each $\omega^{(d,m)}$ lies in $\mathcal{P}_j$ and satisfies $T^j\omega^{(d,m)} \approx I_{d,m}$; ω_Y is a union of elements of $\mathcal{P}_j$ and $T^j\omega_Y$ is a union of elements of $\mathcal{G}$; $\widetilde{\omega}$ is the union of finitely many elements of $\mathcal{P}_j$. We obtain estimates in the following order: $\omega^{(d,m)}$ in Lemma B.5; $\bigcup \omega^{(d,m)}$ in Corollary B.6; $\widetilde{\omega}$ in Corollary B.7; ω_Y in Lemma B.8.

Lemma B.5 *There exists $\delta, c_1 > 0$ such that the following holds. Suppose that $\omega_0 = \omega^{(d_0,m_0)} \in \mathcal{P}_j$ with $\gamma = T^j\omega_0 \approx I_{d_0,m_0}$ where $d_0 \leq c_1 n$. Define*

$$\gamma_n = \{x \in \gamma : T^{-j}x \textit{ has no regular returns between } j \textit{ and } j+n+1\}.$$

Then $m(\gamma_n) \leq C\rho^n m(\gamma)$.

Proof Note that $d_\delta = \log \delta^{-1}$ is to be taken as large as required in this proof.

If ω is any interval and $0 \notin T^j\omega$ for $0 \leq j \leq k$, then $1 \geq m(T^k\omega) \geq c\lambda_1^k m(\omega)$ by (2.3) so that $k \leq -(\log c + \log m(\omega))/\log \lambda_1$. In particular, $m(I_{d,m}) = e^{-d}/d^2$, so if k is the time between an essential return of depth d and the next essential return, then for $d \geq d_\delta$ large enough, $k \leq 2d/\log \lambda_1$.

Let $\mathcal{Q} = \{\omega \in \mathcal{P}_{j+n} : T^j\omega \subset \gamma_n\}$. Then $T^j\mathcal{Q}$ is a partition of γ_n. To each $\omega \in \mathcal{Q}$, associate the itinerary $\omega_0 \supset \omega_1 \supset \cdots \supset \omega_n = \omega$, where $\omega_\ell \in \mathcal{P}_{j+\ell}$, $\ell = 0, \ldots, n$. Let

$$j = j + \eta_0 < j + \eta_1 < \cdots < j + \eta_s \leq j + n$$

be the essential returns between times j and $j+n$ for ω. Let d_i, $0 \leq i \leq s$, be the associated depths, so $T^{j+\eta_i}\omega_{\eta_i} \approx I_{\pm d_i, m_i}$ for $i = 0, \ldots, s$. In particular, $\eta_{i+1} - \eta_i \leq 2d_i/\log \lambda_1$ and hence $d_0 + d_1 + \cdots + d_s \geq \frac{1}{2} n \log \lambda_1$. For each $R \geq \frac{1}{2} n \log \lambda_1 - d_0$, define $\mathcal{Q}_R = \{\omega \in \mathcal{Q} : d_1 + \cdots + d_s = R\}$. We claim that for any $\beta_1 \in (0, 1-\beta)$, we can choose $d_\delta > 0$ sufficiently large that

(a) $\#\mathcal{Q}_R \leq e^{5R/d_\delta^{1/2}}$.

(b) $m(T^j\omega) \leq e^{-\beta_1 R} e^{2d_0} m(\gamma)$ for all $\omega \in \mathcal{Q}_R$.

It follows that for any $\beta_2 < \beta_1$, we can choose d_δ large enough that $\sum_{\omega \in \mathcal{Q}_R} m(T^j\omega) \leq e^{-\beta_2 R} e^{2d_0} m(\gamma)$. Hence,

$$\begin{aligned} m(\gamma_n) &= \sum_{R \geq \frac{1}{2} n \log \lambda_1 - d_0} \sum_{\omega \in \mathcal{Q}_R} m(T^j\omega) \leq e^{2d_0} m(\gamma) \sum_{R \geq \frac{1}{2} n \log \lambda_1 - d_0} e^{-\beta_2 R} \\ &= C e^{-\frac{1}{2} n \beta_2 \log \lambda_1} e^{(\beta_2 + 2)d_0} m(\gamma) \leq C e^{-n(\frac{1}{2}\beta_2 \log \lambda_1 - c_1(\beta_2+2))} m(\gamma). \end{aligned}$$

The result follows for any $c_1 < \frac{1}{2}\beta_2 \log \lambda_1/(\beta_2 + 2)$.

It remains to prove the claims. To prove (a), note that by Proposition 5.8(b) once a sequence of partition elements $I_{d_k, m_k} \in \mathcal{I}_U$ ($k = 1, \ldots, s$) is specified, there is at most one element $\omega \in \mathcal{P}_{j+n}$ that has precisely this sequence of essential returns, and no regular returns, from time j to $j+n+1$. A standard combinatorial estimate ([3, p. 35]) shows that for δ small, there are at most $e^{5R/d_\delta^{1/2}}$ different ways of choosing the sequence (d_k, m_k) such that $R = d_1 + \cdots + d_s$ (given the constraints $d_k \geq d_\delta$ and $m_k \leq d_k^2$).

To prove claim (b), write

$$m(T^j\omega) = \frac{m(T^j\omega)}{m(T^j\omega_{\eta_s})} \cdots \frac{m(T^j\omega_{\eta_2})}{m(T^j\omega_{\eta_1})} \frac{m(T^j\omega_{\eta_1})}{m(T^j\omega_{\eta_0})} m(\gamma) \leq 1 \cdot \prod_{k=1}^{s} \frac{m(T^j\omega_{\eta_k})}{m(T^j\omega_{\eta_{k-1}})} \cdot m(\gamma).$$

By condition (2.3), Remark 5.9 and Lemma A.4

$$\frac{m(T^j\omega_{\eta_k})}{m(T^j\omega_{\eta_{k-1}})} \leq C_0 \frac{m(T^{j+\eta_{k-1}+1}\omega_{\eta_k})}{m(T^{j+\eta_{k-1}+1}\omega_{\eta_{k-1}})} \leq C_1 \frac{m(T^{j+\eta_k}\omega_{\eta_k})}{m(T^{j+\eta_{k-1}+1}\omega_{\eta_{k-1}})} \leq C_2 \frac{e^{-d_k} d_k^{-2}}{e^{-\beta d_{k-1}} d_{k-1}^{-2}},$$

and C_2 is independent of δ. Hence

$$m(T^j\gamma) \le \prod_{k=1}^{s} C_2 \frac{e^{-d_k}d_k^{-2}}{e^{-\beta d_{k-1}}d_{k-1}^{-2}} m(\gamma) = C_2^s e^{-d_k} d_k^{-2} e^{-(1-\beta)(d_{k-1}+\cdots+d_1)} e^{\beta d_0} d_0^2 m(\gamma)$$
$$\le C_2^s e^{-(1-\beta)R} e^{2d_0} m(\gamma).$$

Finally, we note that $s < R/d_\delta$ (since $d_k \ge d_\delta$ for all k) so the claim follows. ∎

Corollary B.6 *Suppose that $\gamma = \bigcup_i T^j\omega_i$ where $\omega_i \in \mathcal{P}_j$ and $T^j\omega_i \approx I_{d_i,\ell_i}$ for each i. Define γ_n as in Lemma B.5. Then $m(\gamma_n) \le C\rho^n$.*

Proof Clearly $m(\gamma_n \cap (-e^{-c_1 n}, e^{-c_1 n})) \le 2e^{-c_1 n}$ and by Lemma B.5, $m(\gamma_n - (-e^{-c_1 n}, e^{-c_1 n})) \le \sum_i C\rho^n m(T^j\omega_i) \le C\rho^n$. Hence $m(\gamma_n) \le C\rho^n + 2e^{-c_1 n}$. ∎

Corollary B.7 *Suppose that $\gamma = T^j\omega$ where $\omega \in \mathcal{P}_j$. Define γ_n as in Lemma B.5. Then $m(\gamma_n) \le C\rho^n$.*

Proof We may suppose without loss that $m(\gamma) \ge c^{-1}e^{-\frac{1}{3}n\log\lambda_1}$. By the same argument used at the beginning of the proof of Lemma B.5, it follows from (2.3) that γ makes an essential return at time $j_0 < n/2$. Let $\gamma' = T^{j_0}\gamma_n$ so γ' satisfies the hypothesis of Corollary B.6. Moreover, points $x \in \gamma'$ have preimages $T^{-(j+j_0)}x$ (that have no regular returns between j and $j+n+1$ and hence certainly no regular returns between $j+j_0$ and $j+j_0+(n-j_0)+1$. In particular, $T^{-(j+j_0)}x$ has no regular returns between $j+j_0$ and $j+j_0+n/2$. By (2.3) and Corollary B.6, $m(\gamma) \le c^{-1}m(\gamma') \le C\rho^{n/2}$. ∎

Lemma B.8 *Suppose that $\omega \subset \widetilde{\Omega}_j$ is a union of elements of $\mathcal{P}_j$ and $\gamma = T^j\omega$ is a union of elements of $\mathcal{G}$. Define γ_n as in Lemma B.5. Then $m(\gamma_n) \le C\rho^{\sqrt{n}}$.*

Proof Choose $\widetilde{\rho} \in (\rho, 1)$ where ρ is as in Corollary B.7. Let $\mathcal{G}' = \{\gamma \in \mathcal{G} : m(\gamma) \le \widetilde{\rho}^n\}$ and $\mathcal{G}'' = \mathcal{G} - \mathcal{G}'$. Let $\gamma = \gamma' \cup \gamma''$ be the corresponding decomposition of γ. By Proposition 5.6, $m(\gamma') \le Ce^{-\alpha_3\sqrt{n}}$ where $\alpha_3 = \alpha_2(-\log\widetilde{\rho})^{1/2}$.

It remains to estimate $m(\gamma_n'')$. Let M denote the cardinality of $\mathcal{G}''$. Note that $M\widetilde{\rho}^n \le \sum_{\widetilde{\gamma}\in\mathcal{G}''} m(\widetilde{\gamma}) \le 2$, so that $M \le 2\widetilde{\rho}^{-n}$. If $\widetilde{\gamma} \subset \gamma$, then $\widetilde{\gamma} = T^j\widetilde{\omega}$ where $\widetilde{\omega} \in \mathcal{P}_j$ so, by Corollary B.7, $m(\widetilde{\gamma}_n) \le C\rho^n$. Hence

$$m(\gamma_n'') = \sum_{\widetilde{\gamma}\in\mathcal{G}'',\widetilde{\gamma}\subset\gamma} m(\widetilde{\gamma}_n) \le MC\rho^n \le C(\rho\widetilde{\rho}^{-1})^n.$$

∎

Acknowledgements This research was supported in part by EPSRC Grant GR/S11862/01 (MH and IM) and by a Leverhulme Research Fellowship (IM). We are grateful to Stefano Luzzatto for helpful suggestions and advice throughout, and to the Isaac Newton Institute for their hospitality during part of this research. IM is greatly indebted to the University of Houston for technological support.

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