Maximal isotropy subgroups for absolutely irreducible representations of compact Lie groups *

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Abstract

We answer some longstanding questions concerning absolutely irreducible representations of compact Lie groups. Such representations provide the natural setting for steady-state equivariant bifurcation theory [19]. We prove the existence of maximal isotropy subgroups for which there are no branches of equilibria or relative equilibria. Also, we obtain examples of complex and quaternionic maximal isotropy subgroups. A consequence of this is the existence of primary branches of nontrivial relative equilibria (rotating waves).

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1 Introduction

Suppose that Γ is a compact Lie group acting on $\mathbb{R}^n$. A central issue in equivariant bifurcation theory [19] and in physical theories of spontaneous symmetry breaking [25] is the problem of determining the symmetries of branches of equilibria for Γ-equivariant vector fields on $\mathbb{R}^n$. These symmetries correspond to isotropy subgroups of the group Γ. (More precisely, branches consist of group orbits of equilibria and associated to each branch is a conjugacy class of isotropy subgroups.) A natural question to ask is which isotropy subgroups of Γ arise generically as symmetries of these equilibria. Early computations suggested the answer to be precisely the maximal isotropy subgroups. This led to various forms of the Maximal Isotropy Subgroup Conjecture (MISC) [17], [25] (see also [15] for a precise statement of the MISC).

There are by now several counterexamples to the MISC: branches of equilibria with submaximal isotropy subgroup appear in [2, 4, 16, 21, 22, 23, 27]. Moreover, the examples of [16] show that failure of the MISC is quite common and not at all exceptional.

In the other direction, examples of maximal isotropy subgroups for which there is generically no branching of equilibria have proven more elusive. This paper contains the first such examples. (In the variational context it is well-known that there exist generically branches of equilibria corresponding to all maximal isotropy subgroups [6, 26, 28].) In addition, we construct the first examples of primary steady-state bifurcation to nontrivial relative equilibria, rotating waves, arising as a consequence of the existence of complex and quaternionic maximal isotropy subgroups. (See [3, 5] for an ultimately unsuccessful attempt to find primary branches of rotating waves in steady-state bifurcations with $\mathbf{O}(3)$-symmetry.) It is interesting to note that quaternionic maximal isotropy subgroups yield branches of Hopf fibrations: invariant 3-spheres foliated by rotating waves.

The remainder of this paper is structured in the following way. In Section 2, we describe the relationship between isotropy subgroups for absolutely irreducible representations and the corresponding steady-state equivariant bifurcation problems. Also, we recall the definition of complex and quaternionic maximal isotropy subgroups and explore the implications for bifurcation to relative equilibria. In particular, Theorem 2.4 gives a complete description of the possibilities for branching of relative equilibria associated to maximal isotropy subgroups. Then Section 3 consists of explicit examples of the phenomena described in this introduction. In particular, we give examples of each of the following.

(a) A complex maximal isotropy subgroup, and primary bifurcation to rotating waves.

(b) A quaternionic maximal isotropy subgroup, and primary bifurcation to Hopf fibrations.

(c) A (real) maximal isotropy subgroup for which there are no branches of relative equilibria (though there is a branch of periodic solutions).

2 Maximal isotropy subgroups and equivariant bifurcation theory

The examples in this paper resolve certain open problems concerning on the one hand absolutely irreducible representations of compact Lie groups and their maximal isotropy subgroups, and on the other hand equivariant steady-state bifurcation theory and the symmetry types of bifurcating equilibria and relative equilibria. In this section, we recall these notions and the interrelationships between them.

We suppose throughout that Γ is a compact Lie group acting linearly on $\mathbb{R}^n$. Often we shall assume that the action is absolutely irreducible, that is the linear maps commuting with the action of Γ are scalar multiples of the identity. (We note that absolutely irreducible representations are irreducible.) As shown in [19], absolutely irreducible representations provide the natural setting for equivariant steady-state bifurcation theory.

Maximal isotropy subgroups

A standard reference for the material in this subsection is [19]. If $x \in \mathbb{R}^n$, the *isotropy subgroup* Σ_x is defined to be the subgroup

$$\Sigma_x = \{\gamma \in \Gamma,\ \gamma x = x\}.$$

Observe that $\Sigma_{\gamma x} = \gamma \Sigma_x \gamma^{-1}$ so isotropy subgroups of points lying on a Γ-group orbit form a conjugacy class in Γ. If $\Sigma \subset \Gamma$, we define the *fixed-point subspace* of Σ,

$$\mathrm{Fix}(\Sigma) = \{x \in \mathbb{R}^n,\ \sigma x = x \text{ for all } \sigma \in \Sigma\}.$$

In the remainder of this subsection we suppose that $\mathrm{Fix}(\Gamma) = \{0\}$. (This condition is automatically satisfied if Γ acts irreducibly and nontrivially.) Let $N(\Sigma)$ denote the normalizer of Σ in Γ. Then $N(\Sigma)$ acts on $\mathrm{Fix}(\Sigma)$ (indeed $N(\Sigma)$ is the largest subgroup of Γ with this property). The quotient group $D_\Sigma = N(\Sigma)/\Sigma$ acts faithfully on $\mathrm{Fix}(\Sigma)$. Moreover, if Σ is a maximal isotropy subgroup of Γ then D_Σ acts fixed-point freely. Let D_Σ^0 denote the connected component of the identity in D_Σ. The following is a consequence of the classification of fixed-point free actions, see for example [1, Theorem III.8.5].

Proposition 2.1 ([17]) *Suppose that* $\mathrm{Fix}(\Gamma) = \{0\}$ *and that* Σ *is a maximal isotropy subgroup of* Γ*. Then* D_Σ^0 *is isomorphic to* 1*,* S^1 *or* $\mathbf{SU}(2)$*.*

A maximal isotropy subgroup is said to be *real* if $D_\Sigma^0 \cong 1$, *complex* if $D_\Sigma^0 \cong S^1$ and *quaternionic* if $D_\Sigma^0 \cong \mathbf{SU}(2)$. It is clear that $\dim \mathrm{Fix}(\Sigma)$ is even if Σ is complex and that $\dim \mathrm{Fix}(\Sigma) \equiv 0 \bmod 4$ if Σ is quaternionic. It is easy to construct examples of complex and quaternionic maximal isotropy subgroups but the examples in this paper are the first for which the action of Γ is absolutely irreducible.

Relative equilibria

Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth (C^∞) Γ-equivariant vector field, so $f(\gamma x) = \gamma f(x)$ for all $x \in \mathbb{R}^n$, $\gamma \in \Gamma$. By equivariance, the equilibria for f lie on group orbits. When Γ is not finite, it is natural to generalize the notion of equilibrium and to define a *relative equilibrium* to be a flow-invariant group orbit. We define the isotropy subgroup of a relative equilibrium X to be the isotropy subgroup of points in X (this is well-defined up to conjugacy). Recall that the rank, $\mathrm{rk}\, G$, of a compact Lie group G is the dimension of a maximal torus in G.

Theorem 2.2 *[8, 10, 20] Suppose that* X *is a relative equilibrium with isotropy subgroup* Σ *for a* Γ*-equivariant vector field. Then* X *is foliated by flow-invariant* k*-tori where* $k = \mathrm{rk}\, D_\Sigma$*. Generically, the flow on each* k*-torus is transitive.*

Remark 2.3 (a) This result can be extended to parametrized families of equivariant vector fields, [12] (see also [20]).
(b) If $\mathrm{Fix}(\Gamma) = \{0\}$ and Σ is a real maximal isotropy subgroup, then $\mathrm{rk}\, D_\Sigma = 0$ and relative equilibria with isotropy Σ consist simply of equilibria.

However, if Σ is a complex or quaternionic maximal isotropy subgroup, then $\mathrm{rk}\, D_\Sigma = 1$ and it follows that generically relative equilibria with isotropy Σ are foliated by periodic solutions. Such relative equilibria are called *rotating waves*. Moreover, in the quaternionic case the relative equilibria are foliated by invariant 3-spheres which are themselves foliated by periodic solutions (of identical period). Recall that the Hopf fibration is the foliation of S^3 by circles, the fibration being parametrized by S^2. Accordingly, we shall refer to the relative equilibria arising from quaternionic maximal isotropy subgroups as 'Hopf fibrations'.

Steady-state bifurcations

Suppose now that Γ acts absolutely irreducibly (and nontrivially) on $\mathbb{R}^n$ and that $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a smooth one-parameter family of Γ-equivariant vector fields. Denote the bifurcation parameter by $\lambda \in \mathbb{R}$. It follows from absolute irreducibility that $f(0, \lambda) \equiv 0$, so there is a fully symmetric 'trivial' solution $x = 0$. In addition, $(d_x f)_{0,\lambda} = c(\lambda)I$ where $c : \mathbb{R} \to \mathbb{R}$ is smooth. If $c(0) = 0$ we say that f is a *steady-state bifurcation problem with* Γ*-symmetry*, or Γ*-bifurcation problem* for short [19].

Adapting the terminology of [15] we shall say that an isotropy subgroup $\Sigma \subset \Gamma$ is *(generically) symmetry breaking* if Γ-bifurcation problems possess generically a branch of relative equilibria with isotropy Σ bifurcating from the trivial solution as λ passes through zero. A complete summary of the symmetry breaking properties of maximal isotropy subgroups is as follows:

Theorem 2.4 *Suppose that Γ is a compact Lie group acting absolutely irreducibly on $\mathbb{R}^n$ and that Σ is a maximal isotropy subgroup.*

(i) If $\mathrm{Fix}(\Sigma)$ is of odd dimension then Σ is symmetry breaking.

(ii) If Σ is complex or quaternionic then Σ is symmetry breaking.

(iii) If Σ is real and $\mathrm{Fix}(\Sigma)$ is of even dimension then Σ may be, but need not be, symmetry breaking.

The branches of relative equilibria with isotropy Σ consist of equilibria in cases (i) and (iii) and consist of rotating waves or Hopf fibrations in case (ii).

Proof The proof of case (i) may be found in [7]. (Strictly speaking, for $\dim \mathrm{Fix}(\Sigma) > 1$ it is necessary to couple the result of [7] with general theory [15] in order to obtain *branches* of equilibria.) When $\dim \mathrm{Fix}(\Sigma) = 1$, the result is called the equivariant branching lemma [19].

The proof of case (ii) is a slight variation of a technique used in [13] to obtain a theorem of [9] on equivariant Hopf bifurcation. See also [14, Section 11] and [14, Example 4.3.10] for the simpler case when $\mathrm{Fix}(\Sigma)$ is of minimal dimension. For completeness we sketch the proof for Σ complex. The analysis of the quaternionic case is similar. Let $m = \dim_{\mathbb{C}} \mathrm{Fix}(\Sigma)$.

Suppose that f is a Γ-bifurcation problem on $\mathbb{R}^n$ and for $a \in \mathbb{R}$ define $f^a(x, \lambda) = f(x, \lambda) - a|x|^2 x$. It follows from [14, Theorem 4.4.5] (see also [12, 13]) that generically the branching of relative equilibria for f^a is independent of a. Consider the restricted vector field $g^a = f^a|_{\mathrm{Fix}(\Sigma)}$. Since $D^0_\Sigma = S^1$, g^a is (at least) S^1-equivariant. The linear term of g^a is a scalar multiple of the identity (since this was the case for f^a) and there are no quadratic terms since S^1 acts freely. Hence the invariant sphere theorem [12] applies for $a > 0$ large enough. In particular, we may assume that there is a branch of flow-invariant S^1-invariant $(2m-1)$-spheres bifurcating from the trivial solution.

The orbit space for the free action of S^1 induced on these invariant spheres is complex projective space $\mathbb{P}^{m-1}(\mathbb{C})$ and has Euler characteristic $m > 0$. It follows that for sufficiently small $\lambda > 0$, the vector field induced on the orbit space has at least one zero. Consequently f^a, and in particular f, has a relative equilibrium with isotropy Σ for each λ. It follows that Σ is symmetry breaking. (In the notation of [12, Appendix], we have shown that $A^\star_\Sigma$ is the hyperplane $t_1 = 0$.)

Case (iii) is proved by exhibiting real maximal isotropy subgroups with the required properties. Examples where there is symmetry breaking are provided by [11, 23, 24]. Examples where there is no symmetry breaking appear in Section 3 of this paper.

The last statement of the theorem follows from Remark 2.3. ■

All the situations described in Theorem 2.4 can occur. The fact that $\dim \mathrm{Fix}(\Sigma)$ may be even or odd is well-known, indeed $\dim \mathrm{Fix}(\Sigma)$ may be any positive integer. In Section 3, we give examples of Σ complex and quaternionic in addition to an example where Σ is real and non-symmetry breaking as promised in the proof of the theorem.

3 Examples

A complex maximal isotropy subgroup

We exhibit a compact Lie group Γ acting absolutely irreducibly on $\mathbb{R}^6$ with a complex maximal isotropy subgroup. The group Γ is a semi-direct product of the symmetric group S_3 with the 3-torus T^3.

Identify $\mathbb{R}^6$ with $\mathbb{C}^3$. The action of $\theta = (\theta_1, \theta_2, \theta_3) \in T^3$ on $z = (z_1, z_2, z_3) \in \mathbb{C}^3$ is given by

$$\theta z = (e^{i\theta_1} z_1, e^{i\theta_2} z_2, e^{i\theta_3} z_3).$$

Take as generators of S_3 elements ρ, κ acting as

$$\rho z = (z_2, z_3, z_1), \qquad \kappa z = (\overline{z}_1, \overline{z}_3, \overline{z}_2).$$

It is readily verified that this indeed defines an action of Γ as a semi-direct product of S_3 and T^3, with T^3 a normal subgroup of Γ.

Proposition 3.1 *The group* Γ *acts absolutely irreducibly on* $\mathbb{C}^3$.

Proof Observe that $\mathbb{C}^3$ decomposes into a sum of three two-dimensional nonisomorphic irreducible subspaces under the action of T^3. These subspaces are permuted cyclically by ρ so that the action of Γ is irreducible. Moreover, the subgroup generated by T^3 and κ acts like $\mathbf{O}(2)$ on one of these subspaces, so the overall action is absolutely irreducible. ■

Let T_1 denote the subgroup of T^3 isomorphic to S^1 defined by setting $\theta_2 = \theta_3 = 0$. Define Σ to be the semi-direct product of the subgroup generated by κ with T_1.

Proposition 3.2 *The subgroup* Σ *is a complex maximal isotropy subgroup of* Γ *and* $\dim \mathrm{Fix}(\Sigma) = 2$.

Proof It is easily checked that $\mathrm{Fix}(\Sigma) = (0, z, \overline{z}) \cong \mathbb{C}$ and that Σ is the isotropy subgroup of all points in $\mathrm{Fix}(\Sigma) - \{0\}$. In particular, Σ is a maximal isotropy subgroup. The circle $\tilde{T} = \{(0, \theta_2, -\theta_2)\} \subset T^3$ acts fixed-point freely on $\mathrm{Fix}(\Sigma)$ and hence Σ is complex. ■

Corollary 3.3 *Generically there is a branch of rotating waves with isotropy Σ in Γ-equivariant bifurcation problems.*

Propositions 3.1 and 3.2 and Corollary 3.3 can be verified directly by performing routine computations. We list all the isotropy subgroups of Γ in Table 1. We also explicitly compute the branch of rotating waves. To do this we write down the Γ-bifurcation problem $f : \mathbb{R}^6 \times \mathbb{R} \to \mathbb{R}^6$ through third order. Proposition 3.1 follows from the computation of the linear terms.

Generators for Σ	$\mathrm{Fix}(\Sigma)$	D_Σ
Γ	$(0,0,0)$	1
κ, $(0,\theta_2,\theta_3)$	$(x,0,0)$	$\mathbb{Z}_2$
S_3	(x,x,x)	$\mathbb{Z}_2$
κ, $(\theta_1,0,0)$	$(0,z,\overline{z})$	S^1
κ	$(x,z,\overline{z})$	$S^1 \times \mathbb{Z}_2$
$(\theta_1,0,0)$	$(0,z_1,z_2)$	$T^2 \times \mathbb{Z}_2$
1	(z_1,z_2,z_3)	Γ

Table 1: Isotropy subgroups $\Sigma \subset \Gamma = S_3 \dot{+} T^3$, together with $\mathrm{Fix}(\Sigma)$ and D_Σ ($x \in \mathbb{R}$, $z_1, z_2, z_3 \in \mathbb{C}$)

Suppose that $f : \mathbb{C}^3 \to \mathbb{C}^3$ is a Γ-equivariant vector field, in components $f = (f_1, f_2, f_3)$. Equivariance with respect to ρ implies that $f_2(z) = f_1(\rho z)$, $f_3(z) = f_1(\rho^2 z)$, so it suffices to write down f_1. By T^3-equivariance, $f_1(z) = z_1 g(|z_1|^2, |z_2|^2, |z_3|^2)$ where $g : \mathbb{R}^3 \to \mathbb{C}$ is an arbitrary smooth map. Let g_0 denote the truncation of g at leading order, $g_0 = \mu + \alpha|z_1|^2 + \beta|z_2|^2 + \gamma|z_3|^2$, where $\mu, \alpha, \beta, \gamma \in \mathbb{C}$. The restrictions from κ-equivariance are that $\mu, \alpha \in \mathbb{R}$, $\gamma = \overline{\beta}$. We may identify μ with the bifurcation parameter so that

$$g_0 = \lambda + a|z_1|^2 + (b + ic)|z_2|^2 + (b - ic)|z_3|^2,$$

where $a, b, c \in \mathbb{R}$.

Next we compute branches of rotating waves for the ODE $\dot{z} = f(z, \lambda)$. We shall work formally; the existence of branches can be established rigorously by a standard application of the implicit function theorem, and follows in any case from Theorem 2.4. Write $z_j(t) = r_j e^{i\omega_j t}$, $j = 1, 2, 3$, where $r_j, \omega_j \in \mathbb{R}$,

$r_j \geq 0$. Substituting this ansatz into the second equation $\dot{z}_2 = f_2(z, \lambda)$ we obtain

$$0 = r_2 \Re g(r_2^2, r_3^2, r_1^2), \qquad \omega_2 r_2 = r_2 \Im g(r_2^2, r_3^2, r_1^2).$$

At lowest order, these equations become

$$0 = r_2(\lambda + a r_2^2 + b(r_3^2 + r_1^2)), \qquad \omega_2 r_2 = c r_2 (r_3^2 - r_1^2).$$

To compute branches of relative equilibria with isotropy Σ we set $r_1 = 0$, $r_2 = r_3 > 0$ and $\omega_3 = -\omega_2$. It is sufficient to solve the equation $\dot{z}_2 = f_2(z, \lambda)$ which reduces to

$$0 = \lambda + (a + b) r_2^2, \qquad \omega_2 = c r_2^2,$$

at lowest order. In particular, provided $a + b \neq 0$, $c \neq 0$, we have a branch of rotating waves in $\mathrm{Fix}(\Sigma)$. Moreover the rotating waves are normally hyperbolic in $\mathrm{Fix}(\Sigma)$. The branch is supercritical if $a + b < 0$ and subcritical if $a + b > 0$.

Remark 3.4 From the point of view of dynamics, this example is not particularly interesting. It can be shown that generically the local asymptotic dynamics consists of the relative equilibria corresponding to the three maximal isotropy subgroups. Moreover, the rotating waves in $\mathrm{Fix}(\Sigma)$ are generically unstable so that almost every trajectory that remains local in forward time is asymptotic to an ordinary equilibrium.

There are modifications of this example which do produce interesting dynamics. Asymptotically stable rotating waves exist when Γ is taken to be a suitable semi-direct product of $\mathbb{D}_4$ and T^4 acting absolutely irreducibly on $\mathbb{C}^4$. See also Remark 3.10(a). It seems highly likely that further modifications will lead to primary bifurcations to rather exotic dynamics. This will be the subject of future work.

A maximal isotropy subgroup that is not symmetry breaking

This is a slight variation on the previous example. Start with the semi-direct product of S_3 and T^3 acting on $\mathbb{C}^3$, but replace T^3 by the finite subgroup $(\mathbb{Z}_p)^3$ where the action of $\mathbb{Z}_p$ on $\mathbb{C}$ is generated by $z \to e^{2\pi i/p} z$. Provided

$p \geq 3$, $\mathbb{Z}_p$ acts irreducibly on $\mathbb{C}$ and the semi-direct product Γ' of S_3 with $(\mathbb{Z}_p)^3$ acts absolutely irreducibly on $\mathbb{C}^3$. Again there is a maximal isotropy subgroup Σ' with $\mathrm{Fix}(\Sigma') = \{(0, z, \overline{z})\}$.

Since Γ' is finite, maximal isotropy subgroups are real. In particular, Σ' is real. In addition, the only relative equilibria possible are ordinary equilibria. We show that provided $p \geq 5$, there are no branches of equilibria with isotropy Σ' for Γ'-equivariant bifurcation problems. Again we argue formally, but see Remark 3.5 below. The main point is that terms that are Γ'-equivariant but not Γ-equivariant enter only at order $p-1$ and higher. Hence for $p \geq 5$, the bifurcation equations truncated at cubic order are unchanged. Moreover, the dynamics in $\mathrm{Fix}(\Sigma)$ are completely determined at cubic order. In particular, the branch of normally hyperbolic rotating waves with isotropy Σ persists as a branch of normally hyperbolic invariant circles with isotropy subgroup Σ. Finally, the flow on the invariant circles remains nontrivial so that the branch consists of periodic solutions and there are no equilibria as required.

Remark 3.5 This formal argument can be made rigorous by a simple scaling argument. This is possible since the dynamics in $\mathrm{Fix}(\Sigma')$ is determined by a weighted homogeneous vector field (where z has weight 1 and λ has weight 2).

A quaternionic maximal isotropy subgroup

We take Γ to be a semi-direct product of a finite group G and eight copies $(\mathbf{SU}(2))^8$ of the unit quaternions acting on $\mathbb{H}^8 \cong \mathbb{R}^{32}$. The group G is itself a semi-direct product of $\mathbb{Z}_2$ and $\mathbb{D}_2^2$.

First identify $\mathbb{D}_2$ as the subgroup of the symmetric group S_4 generated by the permutations $\kappa = (12)(34)$ and $\tau = (13)(24)$. Denote points in $\mathbb{H}^8$ by $w = (u, v)$ where $u, v \in \mathbb{H}^4$ and define an action of $\mathbb{D}_2$ on $\mathbb{H}^8$ by

$$(u, v) \mapsto i(u, \kappa v)i, \tag{3.1}$$
$$(u, v) \mapsto j(u, \tau v)j. \tag{3.2}$$

(This notation means perform the required permutation of the v-coordinates and then multiply all coordinates on the left and the right by the given quaternion.)

Next let a second copy of $\mathbb{D}_2$ act on $\mathbb{H}^8$,

$$(u, v) \mapsto i(\kappa u, v)i, \quad (u, v) \mapsto j(\tau u, v)j.$$

Combined with the transformations (3.1,3.2), this defines an action of $\mathbb{D}_2^2$.

To complete the definition of the finite group G, add the copy of $\mathbb{Z}_2$ generated by

$$(u, v) \mapsto (v, u). \tag{3.3}$$

Since this transformation is in the normalizer of $\mathbb{D}_2^2$ it follows that G is a semi-direct product of $\mathbb{Z}_2$ with the normal subgroup $\mathbb{D}_2^2$.

Finally, the standard action of $\mathbf{SU}(2)$ by left multiplication on $\mathbb{H}$ induces an action of $(\mathbf{SU}(2))^8$ on $\mathbb{H}^8$ where each copy of $\mathbf{SU}(2)$ acts in the standard way on one copy of $\mathbb{H}$ and trivially on the remaining seven copies. Let Γ be the semi-direct product of G and $(\mathbf{SU}(2))^8$.

Remark 3.6 The finite group G is generated by the transformations (3.1,3.2,3.3). We can think of Γ as generated by $(\mathbf{SU}(2))^8$ together with these three transformations. Note that inside of Γ, left multiplication by i and j in the generators (3.1,3.2) is redundant. However, to realize Γ as a semi-direct product it is convenient to work with the generators as given.

Proposition 3.7 *The group Γ acts absolutely irreducibly on $\mathbb{H}^8$.*

Proof The standard action of $\mathbf{SU}(2)$ on $\mathbb{H}$ (by left multiplication) is irreducible and since G permutes the eight copies of $\mathbb{H}$ transitively, it follows that the action of Γ on $\mathbb{H}^8$ is irreducible. We shall recover this information in the following calculation which shows that the action is in fact absolutely irreducible.

The commuting linear maps for the action of $\mathbf{SU}(2)$ on $\mathbb{H}$ are given by $q \mapsto qa$ where $a \in \mathbb{H}$ (right multiplication). It follows that the commuting linear maps for the action of $(\mathbf{SU}(2))^8$ have the form

$$Lw = (u_1 a_1, u_2 a_2, u_3 a_3, u_4 a_4, v_1 b_1, v_2 b_2, v_3 b_3, v_4 b_4),$$

where $a_r, b_r \in \mathbb{H}$.

Next we take into account the additional constraints coming from the action of G. As noted previously (and as is easily checked), it follows from commutativity with G that L is determined by a_1 and that if a_1 is real then $L = a_1 I$ so that the action is absolutely irreducible. Hence it suffices to show that a_1 commutes with the quaternions i and j. Commutativity with (3.1,3.2) yields the equations for the first component of Lw:

$$iu_1a_1i = iu_1ia_1, \quad ju_1a_1j = ju_1ja_1,$$

and taking $u_1 \neq 0$ we have $a_1 i = i a_1$, $a_1 j = j a_1$. ■

Let $(\mathbf{SU}(2))^6$ denote the subgroup of $(\mathbf{SU}(2))^8$ obtained by deleting the copies of $\mathbf{SU}(2)$ that act on the first components of u and v. Define Σ to be the semi-direct product of the subgroup $\mathbb{Z}_2$ (generated by (3.3)) and $(\mathbf{SU}(2))^6$.

Proposition 3.8 *The subgroup* Σ *is a quaternionic maximal isotropy subgroup of* Γ *and* $\dim \mathrm{Fix}(\Sigma) = 4$.

Proof As was the case for Proposition 3.2 it is easily checked that Σ is a maximal isotropy subgroup with fixed-point subspace $\mathrm{Fix}(\Sigma) = (u_1, 0, 0, 0, u_1, 0, 0, 0) \cong \mathbb{H}$. The diagonal subgroup of $(\mathbf{SU}(2))^8$ acts fixed-point freely on $\mathrm{Fix}(\Sigma)$:

$$(u_1, 0, 0, 0, u_1, 0, 0, 0) \mapsto (qu_1, 0, 0, 0, qu_1, 0, 0, 0),\ q \in \mathbf{SU}(2),$$

and hence Σ is quaternionic. ■

Corollary 3.9 *Generically there is a branch of Hopf fibrations with isotropy* Σ *in* Γ*-equivariant bifurcation problems.*

Remark 3.10 (a) A computation shows that the branch of Hopf fibrations is asymptotically stable for an open set of cubic order coefficients in the Taylor expansion of the Γ-bifurcation problem f. Indeed, our example gives rise to maximal isotropy subgroups of all three types and to branches of asymptotically stable equilibria, rotating waves and Hopf fibrations corresponding to these subgroups.
(b) The space of commuting linear maps for a real irreducible representations is a real division ring and hence is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. As pointed out in [17], the classification of maximal isotropy subgroups into those of real, complex and quaternionic type is reminiscent of, but different from, this trichotomy. Our examples confirm the viewpoint that these trichotomies are unrelated.

Indeed, for any pair of real division rings D_1 and D_2 it is possible to find a compact Lie group Γ acting irreducibly such that the commuting linear maps are isomorphic to D_1 and so that there is a maximal isotropy subgroup Σ of type D_2. (We are making the obvious correspondence between the possibilities $D_\Sigma^0 \cong 1$, S^1, $\mathbf{SU}(2)$ and $D_2 = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$.) This is easy if $D_1 = D_2$

and to obtain $D_2 \subset D_1$ we discretize the group Γ. For example if $\Gamma = S^1$ acting in the standard way on $\mathbb{C}$ with $\Sigma = 1$ then $D_1 = D_2 = \mathbb{C}$. Now replace S^1 by $\mathbb{Z}_p$ where $p \geq 3$ and we have $D_1 = \mathbb{C}$, $D_2 = \mathbb{R}$.

The most difficult cases occur when D_1 is a proper subset of D_2. We have given examples when $D_1 = \mathbb{R}$ and $D_2 = \mathbb{C}$ or $D_2 = \mathbb{H}$. To obtain $D_1 = \mathbb{C}$, $D_2 = \mathbb{H}$ we can change the example in this subsection by removing the left and right multiplication by j in the generator (3.2). However we note that much simpler examples can be found, in particular with Γ acting on $\mathbb{H}^3$.

(c) Several people have pointed out that there is a 'twisted' wreath product structure in the examples in this section. Each group Γ is a semi-direct product of a finite group G and n copies of a group H acting on a vector space V. The full group Γ acts on V^n and G is isomorphic to a subgroup of S_n. Moreover, G acts on the n copies of V as a permutation corresponding to the permutation in S_n combined with a 'twist'. In our examples, the twist takes the form of complex conjugation or quaternionic multiplication. Without this twist, Γ is a standard *wreath product*. See [18] for an overview of the occurrence of wreath products in equivariant dynamical systems.

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