Testing for Chaos in Deterministic Systems with Noise

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Abstract

Recently, we introduced a new test for distinguishing regular from chaotic dynamics in deterministic dynamical systems and argued that the test had certain advantages over the traditional test for chaos using the maximal Lyapunov exponent.

In this paper, we investigate the capability of the test to cope with moderate amounts of noisy data. Comparisons are made between an improved version of our test and both the "tangent space" and "direct method" for computing the maximal Lyapunov exponent. The evidence of numerical experiments, ranging from the logistic map to an eight-dimensional Lorenz system of differential equations (the Lorenz 96 system), suggests that our method is superior to tangent space methods and that it compares very favourably with direct methods.

1 Introduction

Given time series data from a deterministic dynamical system, it is often of interest to determine whether the underlying dynamics are regular or chaotic. For example, heart rate data measured in electrocardiograms are believed to be chaotic for the case of healthy patients but show regularity in the case of congestive heart failure (see Wagner & Persson [19] for an overview).

The usual test of whether a deterministic dynamical system is chaotic or non-chaotic is the calculation of the maximal Lyapunov exponent λ. Standard references include [1, 4, 7, 8, 11, 17]. A positive maximal Lyapunov exponent indicates chaos: if $\lambda > 0$, then nearby trajectories separate exponentially and if $\lambda \leq 0$, then nearby trajectories stay close to each other. This approach has been widely used for dynamical systems whose equations are known. If the equations are not known or one wishes to examine experimental data, then λ may be estimated using the phase space reconstruction method of Takens [18] or by approximating the linearisation of the evolution operator.

In a previous paper [5], we introduced a binary test for distinguishing between regular and chaotic dynamics in deterministic dynamical systems. The test has two main advantages over computing the maximal Lyapunov exponent:

(i) The test applies directly to time series data, so that phase space reconstruction is not required. Moreover, the form and nature of the underlying dynamical system is irrelevant; the test applies equally well to continuous time systems and discrete time systems, to experimental data and maps, to ordinary differential equation and partial differential equations.

(ii) It is a binary test (in principle, the test yields 0 or 1), so a numerically computed value of 0.01 say yields a definite conclusion, whereas such a value for the maximal Lyapunov exponent would not.

The theoretical basis for the validity of any numerical test for chaos relies on the availability of unlimited noiseless data. In practice, it is necessary to work with limited amounts of contaminated data. The only way to determine the utility of the test in such situations is to try it out on different examples and see how it fares in comparison with existing methods. Such a comparison is carried out in this paper.

In Section 2, we describe an improved version of the test introduced in [5]. In Section 3, we briefly review the tangent space and direct methods for computing maximal Lyapunov exponents. In Section 4, we apply the test to the logistic map contaminated with measurement noise. Here, our test compares extremely favourably to the tangent space method for computing the Lyapunov exponent. In Section 5, the test is applied to contaminated data from an eight-dimensional ODE and is seen to compare favourably even to direct methods.

2 The (modified) test for chaos

Our test for chaos in this paper is a slightly modified version of the test that we introduced previously [5]. In this section, we describe the modified test and compare it with the one in [5]. We focus throughout on discrete data sets $\phi(n)$ sampled at times $n = 1, 2, 3, \ldots$ (The continuous time case is similar, cf. [5].) Here, $\phi(n)$ is a one-dimensional observable obtained from the underlying dynamics.

Choose a constant $c \in \mathbb{R}$ at random. In practice, we choose a number of different values of c as detailed below. For each value of c, define

$$p(n) = \sum_{j=1}^{n} \phi(j) \cos(jc), \quad n = 1, 2, 3, \ldots \tag{2.1}$$

Next, define the mean square displacement

$$M(n) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} [p(j+n) - p(j)]^2, \quad n = 1, 2, 3, \ldots$$

If the dynamics is regular (periodic or quasiperiodic), then with probability one $M(n)$ is a bounded function of n. However, if the dynamics is chaotic (in a fairly mild sense), then with probability one $M(n) = Vn + \mathcal{O}(1)$ for some $V > 0$. (For justifications of these statements, see [5] and the references therein.) Define the asymptotic growth rate of the mean square displacement

$$K = \lim_{n\to\infty} \frac{\log M(n)}{\log n}.$$

Then $K = 0$ signifies regular dynamics whereas $K = 1$ signifies chaotic dynamics.

Next suppose that we have a finite amount of data $\phi(n)$, $1 \le n \le N$. Define

$$M(n) = \frac{1}{N-n} \sum_{j=1}^{N-n} [p(j+n) - p(j)]^2.$$

Provided $n \ll N$ it is reasonable to expect that $M(n)$ scales with n in a similar way to before. To avoid logarithms of negative numbers, we plot $\log(M(n)+1)$ against $\log n$ for $1 \le n \le N_1$ for some choice of N_1, $1 \ll N_1 \ll N$. In practice, we have found it convenient to choose $N_1 = N/10$. This choice is made throughout the paper without further comment. (A disadvantage of our test is that we cannot use the full available data set for our statistics in the computation of the mean square displacement.) We define K to be the slope of the line of best fit on the resulting data set, using least square regression. The test is now that K close to zero signifies regular dynamics and K close to 1 implies chaotic dynamics.

For short data sets (with or without noise) there is an inconvenient resonance phenomenon, where a given choice of c may resonate with frequencies in the underlying dynamics. For example, if the signal is 2π-periodic and we choose c to be an integer, then a simple argument using the Fourier series for $\phi(n)$ shows that typically $p(n)$ will grow linearly yielding $K = 2$. A near-integer choice of c results in $K = 0$ eventually but the convergence is very slow; for a small number of iterates the computed value of K is likely to be closer to 1 than to 0. The situation for quasiperiodic dynamics is even worse since "bad" choices of c are now dense. Nevertheless, there is zero probability of making a bad choice in theory. In practice, numerical experimentation shows that even for short data series, the bad choices of c are rare but do occur from time-to-time.

Our resolution of this problem is to choose several values of c at random, computing K for each choice of c. Then we take the median value of K. (We do not take the average of K, since when c does fail it can fail quite badly.)

To demonstrate the improvement in the modified test, we consider the logistic map

$$x_{n+1} = \mu x_n(1 - x_n),$$

varying the parameter μ in the range $3.5 \leq \mu \leq 4$ in increments of 0.001. Starting with initial condition 0.0001, we use $N = 1,000$ iterates, after a transient of $20,000$ iterates. We take $\phi(n) = x_n$.

In Figure 1, we show how the old method [5] compares with the modified test, where for the modified test we take the median of 100 different values for c. It is clear that there is a dramatic improvement in our test for chaos. (A comparison with traditional Lyapunov exponent methods is carried out in Section 4.) An obvious criticism is that K is always far from 1 for such short data sets. In later sections, we demonstrate that this issue is of little consequence in comparison with traditional methods. Our purpose in this section is solely to compare the modified test with our old test.

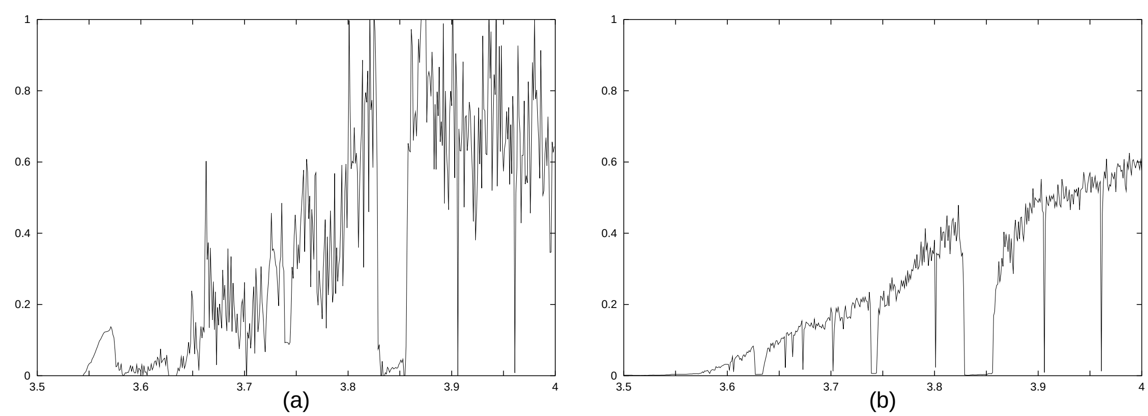

Figure 1: Plots of K versus μ for the logistic map $f(x) = \mu x(1-x)$, with $3.5 \leq \mu \leq 4$ using $1,000$ iterates and noise-free data. (a) The old test [5]. (b) The new test with 100 values of c.

In Figure 2, we compare the effects of taking 1, 10, 100 and 1000 different values of c. It is clear from these results that taking one choice of c is not effective and that taking 10 values of c is a dramatic improvement, though some periodic windows are still poorly defined. This is remedied by going to 100 values of c, and increasing to 1000 does not make a significant further improvement. In the remainder of the paper, we use 100 values of c.

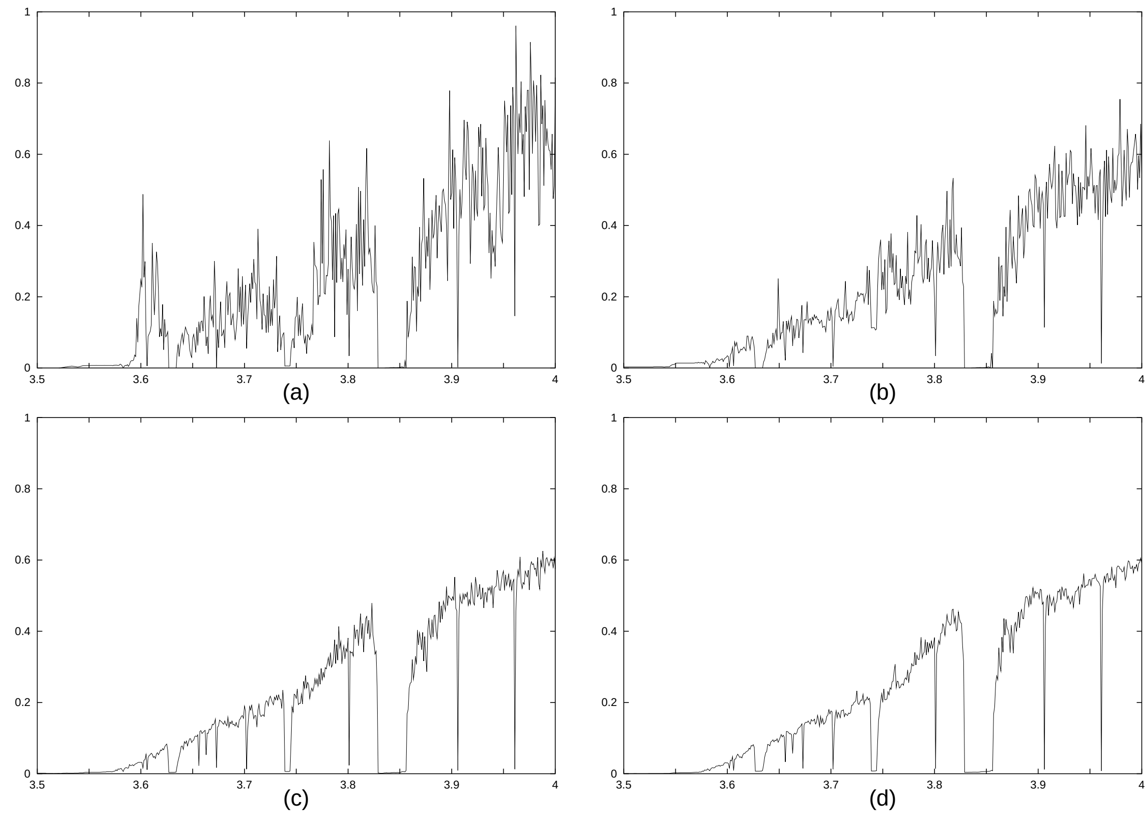

Figure 2: Plots of K versus μ for the logistic map $f(x) = \mu x(1-x)$, with $3.5 \leq \mu \leq 4$ using $1,000$ iterates and noise-free data. The median value of K is taken from m values of c where (a) $m = 1$, (b) $m = 10$, (c) $m = 100$, (d) $m = 1000$.

Remark 2.1 There are two differences between the modified test presented here and the test introduced in [5]. The first is that we now use several values of c. The second is that in [5] we defined $p(n)$ by the more complicated prescription:

$$\left.\begin{array}{l} \theta(n+1) = \theta(n) + c + \phi(n) \\ p(n+1) = p(n) + \phi(n)\cos(\theta(n)) \end{array}\right\}. \tag{2.2}$$

We note that the modified $p(n)$ in equation (2.1) can be recovered by removing the $\phi(n)$ term in the formula for $\theta(n+1)$ in equations (2.2).

Numerical experimentation (specifically with the forced van der Pol oscillator considered in [5]) shows that the modified definition of p in (2.1) is inferior for long data sets with no noise when only one choice of c is used. This was our reason for including the extra $\phi(n)$ term in (2.2). However, this term is no longer advantageous when 100 choices of c are sampled. Moreover, (2.1) now has the advantage that $p(n)$ scales linearly with the data and so is far less susceptible to measurement noise than the test in [5].

3 Traditional methods

Suppose that $\phi(n)$ is the data set originating from an unknown deterministic dynamical system. (As usual ϕ is some one-dimensional observable of the system under investigation.) To compute the maximal Lyapunov exponent, it is first necessary to reconstruct the dynamics [18]. Define the m-dimensional *delay vector*

$$\xi_n = \{\phi(n), \phi(n+\tau), \phi(n+2\tau), \cdots, \phi(n+(m-1)\tau)\}.$$

Here, the delay time $\tau > 0$ is chosen to be an integer. If the underlying dynamics $\mathbf{x}_n$ lies inside a d-dimensional phase space, then the delay reconstruction map $\mathbf{x}_n \mapsto \xi_n$ is an embedding provided $m > 2d+1$ (corresponding to Whitney's embedding theorem [20]). Sauer *et al.* [14, 15] extended Takens' work by showing that it suffices to take $m > 2d_0(A)$ where $d_0(A)$ is the fractal box-counting dimension of the attractor. Under these conditions, the reconstructed dynamics $\xi_1, \xi_2, \ldots$ in $\mathbb{R}^m$ faithfully represents the underlying dynamics.

The maximal Lyapunov exponent for a map $\xi_{n+1} = f(\xi_n)$ is defined to be

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \log(\|Df^n_{\xi_1}\|). \tag{3.1}$$

There are (at least) two main numerical approaches for computing the maximal Lyapunov exponent: the *tangent space method* and the *direct method*.

In tangent space methods, a model is fitted to the data to approximate the Jacobian. By (3.1), the maximal Lyapunov exponent is defined as the product of the Jacobian of the the linearised flow along the given trajectory. If the underlying equations are not known the Jacobian can be approximated. Such tangent space methods were first introduced by Eckmann & Ruelle [3], and then further developed by Sano & Sawada [13] and Eckmann *et al.* [4].

Direct methods use the fact that nearby trajectories separate on average asymptotically at rate $e^{\lambda n}$. The original algorithm by Wolf *et al.* [21] allows the computation of the whole spectrum of Lyapunov exponents. Rosenstein *et al.* [12] and Kantz [6] concentrated attention on the maximal Lyapunov exponent, and developed a more robust method. In the method by Rosenstein *et al.* [12] for each m-dimensional vector ξ in the reconstructed phase space, its nearest neighbour $\xi^\star$ is determined. Rosenstein *et al.* calculate $C(k) =< \log|f^k(\xi) - f^k(\xi^\star)| >$ where the angles denote averaging over all $\xi = \xi_1, \xi_2, \ldots$ The function $C(k)$ shows roughly three different regimes. An initial regime of flat increase, a subsequent interval with exponential behaviour, and finally a plateau (because the separation cannot go beyond the extension of the attractor). The maximal Lyapunov exponent is determined by the slope of $C(k)$ in the usually quite short range of exponential behaviour.

Remark 3.1 There are a number of inherent problems of both the tangent space method and the direct method, linked to phase space reconstruction. These difficulties, which include the choice of embedding dimension m and delay parameter τ, are well-documented and we refer to Casdagli *et al.* [2] and Schreiber & Kantz [16] for detailed discussions.

4 Comparison with tangent space methods

In this section, we compare our test with the tangent space method. Note that the definition of the maximal Lyapunov exponent (3.1) is itself a tangent space method. We use the method developed by Sano & Sawada [13]. In particular we study the influence of measurement noise.

As discussed in Section 3, tangent space methods seek to approximate the Jacobian $Df^n(x_0)$. In the case of measurement (additive) noise, the noise is amplified by higher order nonlinearities. Similarly, if the dimensionality of the underlying dynamical system is large, the evaluation of the diagonalised Jacobian requires extensive multiplication of the individual matrix elements of the Jacobian, and noise is again amplified by this process.

Neither of these problems arises in our test. Measurement noise only enters our diagnostic variable $p(n)$ linearly via the observation $\phi(n)$ irrespective of the underlying dynamical system.

We consider an illustrative example. Our contaminated data has the form $\widetilde{\phi}(n) = \phi(n) + (\mathcal{N}/100)\eta_n$ where x_n is the clean data, $\mathcal{N}$ is the noise-level in percent, and $\eta_1, \eta_2, \dots$ are i.i.d. random variables drawn from a uniform distribution on $[-1, 1]$. We note that the results are similar in the case of normally distributed noise.

As in Section 2, we study the logistic map

$$x_{n+1} = \mu x_n(1 - x_n), \tag{4.1}$$

varying the parameter μ in the range $3.5 \le \mu \le 4$ in increments of 0.001. We again use data sets consisting of $N = 1,000$ iterates, after a transient of $20,000$ iterates starting from the initial condition 0.0001. Throughout, we take $\phi(n) = x_n$ as the observable.

Our results for noise-free data are shown in the first column of Figure 3. As a benchmark, we compute the "exact" Lyapunov exponent making use of the explicitly given form of the map in (4.1), see Figure 3(a), Next, we use the tangent space method proposed by Sano & Sawada [13] for approximating the dynamics on the tangent space using phase-space reconstruction, see Figure 3(b). The results from our modified test (Section 2) with 100 values of c are shown in Figure 3(c).

Now we add measurement noise with noise-level $\mathcal{N} = 1\%$. As can be seen from Figure 3(e), the phase space reconstruction method by Sano & Sawada [13] performs rather poorly. Moreover, the numerical values and the overall qualitative behaviour of the maximal Lyapunov exponent with respect to varying μ depends very sensitively on the particular choice of the embedding dimension m and the value chosen to define nearest neighbours. In contrast, our test performs as well as it did in the noise-free case, see Figure 3(f).

It turns out that the tangent space method copes poorly with measurement noise even when phase space reconstruction is not required. To see that this is the case, we fed contaminated data into the exact expression for the Jacobian computed analytically from (4.1). In Figure 4, we show the results for the "exact" tangent space method compared with our test, both with 10% measurement noise. This seems to be conclusive evidence that our test is better than the tangent space method. We have also verified that our test deals comfortably with 20% noise.

There is another advantage of our method, even in the noise-free case. Suppose that a whole scan through the bifurcation parameter is not available but instead data is only available for one particular value of the bifurcation parameter. A value of 0.2 for the maximal Lyapunov exponent is inconclusive (see for example the periodic windows in Figure 3), whereas a value of $K = 0.01$ indicates regular dynamics. In this sense, our test is a better absolute test than the Lyapunov exponent.

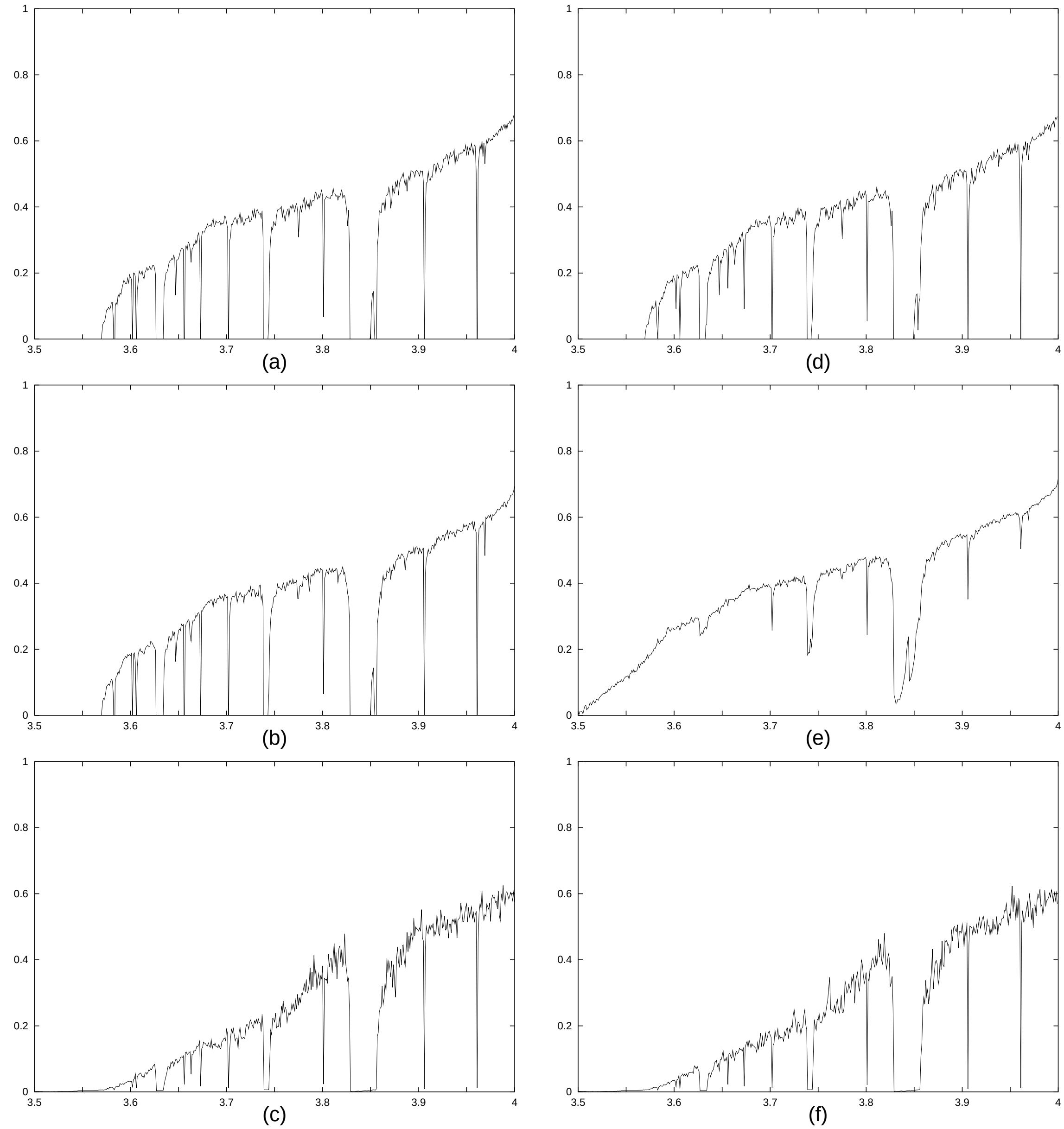

Figure 3: Plots of the Lyapunov exponent and K versus μ for the logistic map (4.1) with $3.5 \leq \mu \leq 4$ using $1,000$ iterates. First row: "Exact" Lyapunov exponent. Second row: Method by Sano & Sawada with $m = 2$ and 0.000001 as an allowed distance to define nearest neighbours. Third row: Our test with 100 different values of c. The first column uses noise-free data. The second column incorporates 1% measurement noise.

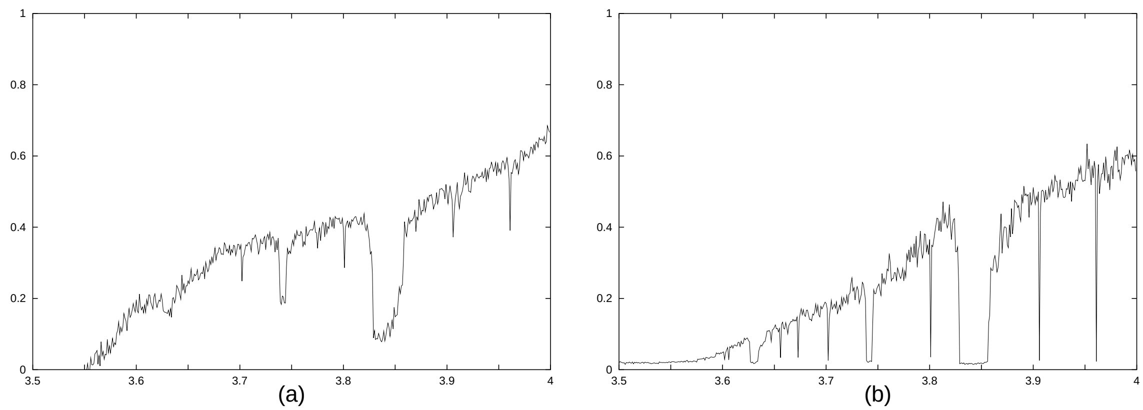

Figure 4: Plots of the Lyapunov exponent and K versus μ for the logistic map (4.1) with $3.5 \le \mu \le 4$ using $1,000$ iterates with 10% noise. (a): "Exact" Lyapunov exponent. (b): Our test with 100 different values of c.

5 Comparison with direct methods

In this section, we compare our test for chaos against the direct method for computing the maximal Lyapunov exponent. In particular, we use the direct method proposed by Rosenstein *et al.* [12].

The standard Lorenz equation is a 3-dimensional truncation of a certain system of partial differential equations (PDEs). This is too simple a system to present a challenge for testing for chaos. Hence, we study the n-dimensional Lorenz system, known as the Lorenz 96 system:

$$\frac{dx_i}{dt} = x_{i-1}(x_{i+1} - x_{i-2}) - x_i + r \quad \text{with} \quad i = 1, \cdots, n \; . \tag{5.1}$$

This system of ODEs was first introduced by Lorenz as an idealised model for the atmosphere [9], and has been studied by Orrell & Smith [10]. In the following we choose $n = 8$. For $3.8 \leq r \leq 3.9$ there are periodic windows, whereas for $5.4 \leq r \leq 6.8$ most regular windows are due to quasiperiodic dynamics [10]. Throughout, we integrate the system using a time step of 0.05 and calculate until $250,000$ units of time after a transient of $75,000$ units of time. However, we use only the data recorded after each 2.5 units of time, yielding a time series of $N = 10,000$ data points. In this way, we obtain a discrete time series mimicking the situation for experimental data. As an observable we take $\phi = x_2 + x_3 + x_4$, so $\phi(n) = x_2(t) + x_3(t) + x_4(t)$ with $t = 2.5n$.

We focus first on the range $3.8 \leq r \leq 3.94$ with increments of 0.00025. The "exact" maximal Lyapunov exponent calculated by solving the analytic linearised flow for a noise free trajectory is shown in Figure 5(a) and serves again as a benchmark to compare maximal Lyapunov exponents with our test.

In Figure 5(b,c) we show the results for noise-free data for the direct method and for our method. We used $m = 4$ as the embedding dimension, having first verified that $m = 4$ yields the the best results when compared to the "exact" maximal Lyapunov exponents. The actual numerical values of the maximal Lyapunov exponents are very sensitive on the choice of the embedding dimension m in the direct method; for larger embedding dimensions there are convergence problems resulting in maximal Lyapunov exponents that are too small. Both tests identify all periodic windows correctly. Our test indicates regular behaviour at $r = 3.8175$ whereas the "exact" maximal Lyapunov exponent is positive at that value of r indicating chaotic behaviour. As a matter of fact there is a periodic window close by at $r = 3.817525$ which we checked by using the "exact" maximal Lyapunov exponent.

The results for i.i.d. measurement noise using a noise-level of 10% are shown in Figure 6. The methods seem to perform roughly on an equal level. However, the direct method fails as an absolute test. For example, within the periodic window in

$3.82675 \leq r \leq 3.828$ the value of the maximal Lyapunov exponents are 95% of its neighbouring chaotic values. In contrast, our test shows clear distinctions and works as an absolute test.

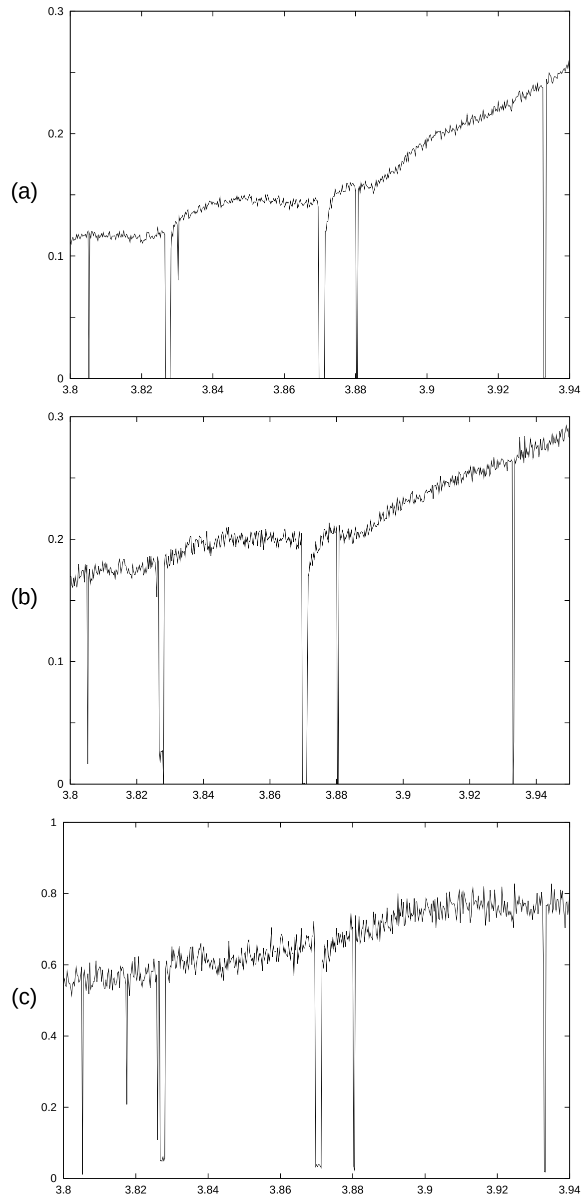

Figure 5: Plots of the maximal Lyapunov exponent and K versus r for the 8-dimensional Lorenz 96 system (5.1) in the regime $3.8 \leq r \leq 3.94$ using $N = 10,000$ noise-free data points. (a): "Exact" maximal Lyapunov exponent. (b): Maximal Lyapunov exponent, direct method [12]. (c): Our test.

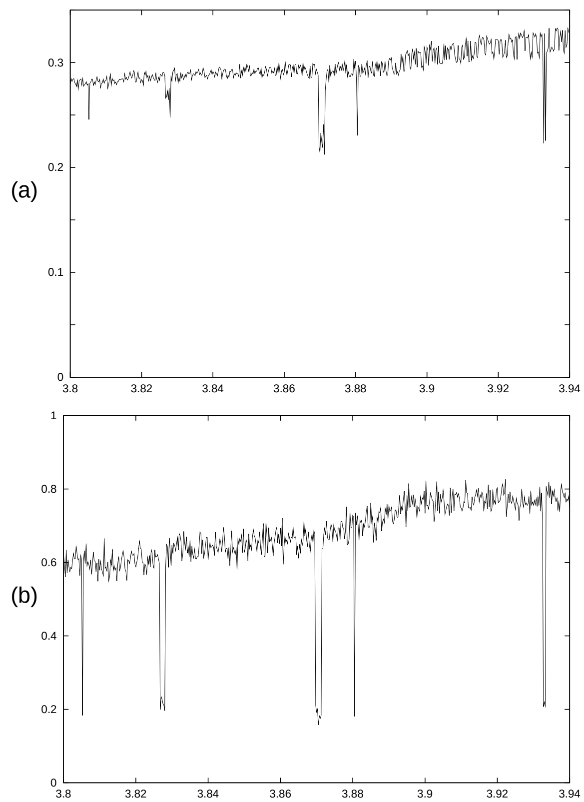

Figure 6: Plots of the maximal Lyapunov exponent and K versus r for the 8-dimensional Lorenz 96 system (5.1) in the regime $3.8 \leq r \leq 3.94$ using $N = 10,000$ data points with 10% measurement noise. (a): Maximal Lyapunov, direct method [12]. (b): Our test.

Next, we consider the quasiperiodic regime $5.25 \leq r \leq 5.5$ increasing in increments of 0.0005. The "exact" maximal Lyapunov exponent is shown in Figure 7(a). In Figure 7(b,c), we show the results for noise-free data for the direct method and for our method.

Whereas the K test shows a distinction of regular quasiperiodic motion and chaotic motion near $r = 5.25$, the direct method cannot properly resolve this. Also, the regular windows at $5.47 \leq r \leq 5.473$ and at $5.449 \leq r \leq 5.4575$ are not resolved, and the chaotic peak within the latter quasiperiodic window is indistinguishable from the values of the neighbouring regular region.

In Figure 8, we show the results for the quasiperiodic regime when 10% measurement noise is added. Here, the direct method fails both as a relative test in the interval $5.449 \leq r \leq 5.4575$ and as an absolute test overall. In contrast our method works well as a relative and absolute test.

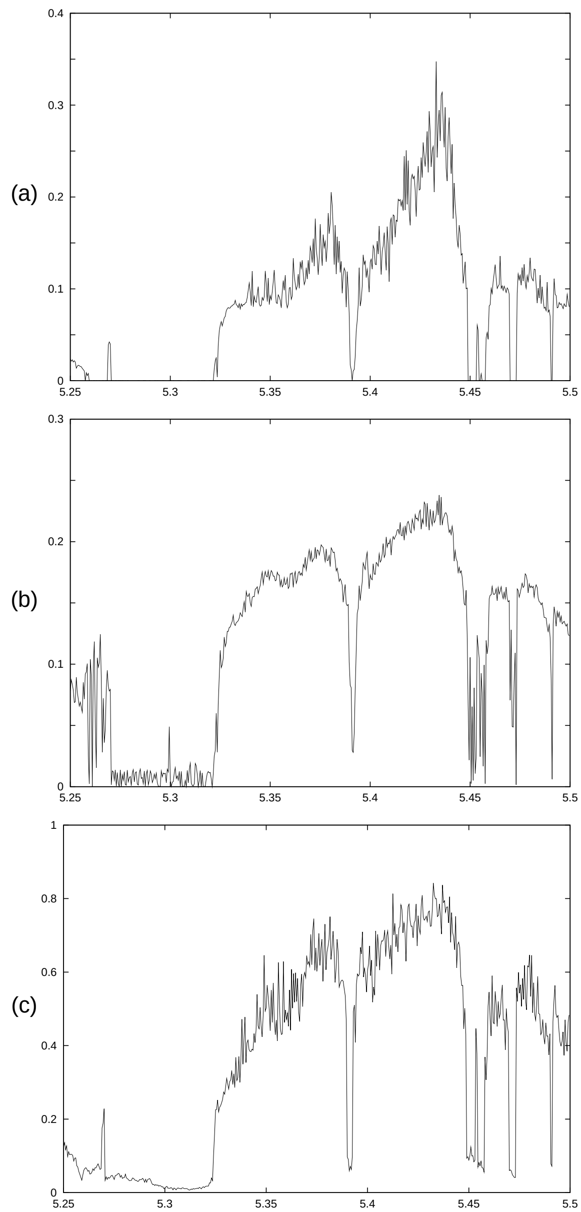

Figure 7: Plots of the maximal Lyapunov exponent and K versus r for the 8-dimensional Lorenz 96 system (5.1) in the quasiperiodic regime $5.25 \leq r \leq 5.5$ using $N = 10,000$ noise-free data points. (a): "Exact" maximal Lyapunov exponent. (b): Maximal Lyapunov exponent, direct method [12]. (c): Our test.

(a)

(b)

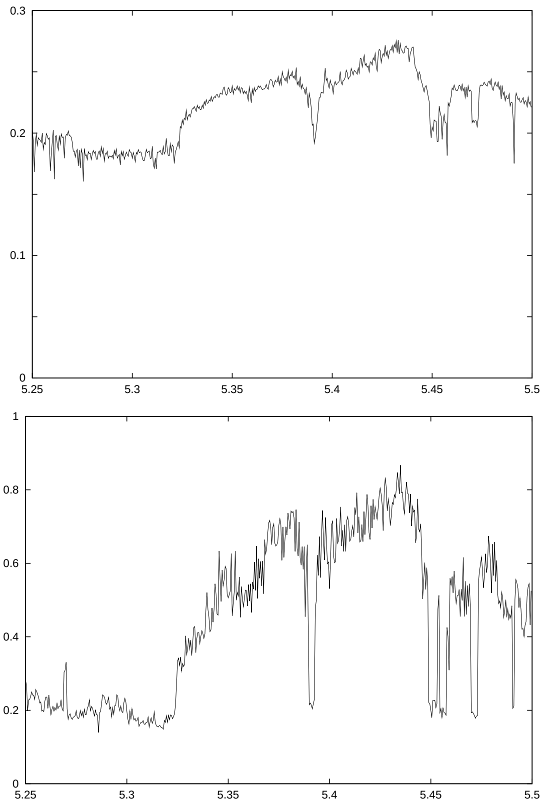

Figure 8: Plots of the maximal Lyapunov exponent and K versus r for the 8-dimensional Lorenz 96 system (5.1) in the quasiperiodic regime $5.25 \leq r \leq 5.5$ using $N = 10,000$ data points with 10% measurement noise. (a): Maximal Lyapunov exponent, direct method [12]. (b): Our test.

6 Summary

We have presented a modification of our test for chaos [5] which better handles moderate amount of contaminated data. In particular we have simplified the diagnostic equation, and we now compute the asymptotic growth rate K as the median value over realizations over many different values of c.

We have shown that our test is much better at coping with measurement noise than tangent space methods. Moreover, in the case of high dimensional ODE's we found that our method compares well also with the direct method proposed by Rosenstein *et al.* [12] in the noise-free case, and shows much better results in the case of underlying quasiperiodic dynamics in the presence of measurement noise. Our test works well as a relative test and also as an absolute test for moderately short time series contaminated by measurement noise.

Since our test applies directly to the time-series data, bypassing the need for phase space reconstruction, we anticipate that the advantages of our method will be magnified for data from partial differential equations and from real experiments. (However, the comparison is complicated by the increased difficulty in generating the data and the fact that often in these situations the "correct" answer is not known beforehand, which is why a reliable test is required in the first place.) This will be the subject of future work.

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