Decay in norm of transfer operators for semiflows

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Abstract

We establish decay of transfer operators in a Hölder norm for observables of uniformly and nonuniformly expanding semiflows with exponential decay of correlations. Our results fall short of establishing a spectral gap; indeed we prove that such a result is impossible for Hölder exponent greater than $\frac{1}{2}$.

1 Introduction

Decay of correlations (rates of mixing) and strong statistical properties are well-understood for Axiom A diffeomorphisms since the work of [9, 19, 20]. Mixing rates are computed with respect to any equilibrium measure with Hölder potential. Up to a finite cycle, such diffeomorphisms have exponential decay of correlations for Hölder observables. In the one-sided (uniformly expanding) setting, this is typically proved by establishing a spectral gap for the associated transfer operator L. Such a spectral gap yields a decay rate $\|L^n v - \int v\| \le C_v e^{-an}$ for v Hölder, where $\| \ \|$ is a suitable Hölder norm and a, C_v are positive constants. Decay of correlations for Hölder observables is an immediate consequence of such decay for L^n. This philosophy has been extended to large classes of nonuniformly expanding dynamical systems with exponential [23] and subexponential decay of correlations [24].

Many statistical properties such as the central limit theorem do not require strong control on L^n in norm, but such control is often useful for finer statistical properties. For example, rates of convergence in the central limit theorem [15] and the associated functional central limit theorem [4] rely heavily on control of operator norms.

In this paper, we consider analogous questions for uniformly and nonuniformly expanding semiflows. The usual techniques [13, 16, 18] bypass spectral gaps; the only exception that we know of being Tsujii [21]. However, the result in [21] is for suspension semiflows over the doubling map with a C^3 roof function, where the smoothness

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of the roof function is crucial and very restrictive. We note also a corresponding result for contact Anosov flows [22]; unfortunately it seems nontrivial to extend this to nonuniformly hyperbolic contact flows (or uniformly hyperbolic contact flows with unbounded distortion), see [7] which proves exponential decay of correlations for billiard flows with a contact structure but does not establish a spectral gap. Indeed, apart from [21, 22], there are previously no results on norm decay of transfer operators for semiflows and flows.

The results of Tsujii [21, 22] provide a spectral gap in an anisotropic Banach space. Our first result yields a restriction on the Banach spaces that can yield a spectral gap. We work in the following very general setting:

Let (Λ, d) be a bounded metric space Borel probability measure μ, and let $F_t : \Lambda \to \Lambda$ be a measure-preserving semiflow. We suppose that $t \to F_t$ is Lipschitz a.e. on Λ. Let $L_t : L^1(\Lambda) \to L^1(\Lambda)$ denote the transfer operator corresponding to F_t (so $\int_\Lambda L_t v \, w \, d\mu = \int_\Lambda v \, w \circ F_t \, d\mu$ for all $v \in L^1(\Lambda)$, $w \in L^\infty(\Lambda)$, $t > 0$). Let $v \in L^\infty(\Lambda)$ and define $v_t = \int_0^t v \circ F_r \, dr$ for $t \geq 0$.

Theorem 1.1 *Let $\eta \in (\frac{1}{2}, 1)$. Suppose that $L_t v \in C^\eta(\Lambda)$ for all $t > 0$ and that $\int_0^\infty \|L_t v\|_\eta \, dt < \infty$. Then v_t is a coboundary:*

$$v_t = \chi \circ F_t - \chi \quad \text{for all } t \geq 0, \text{ a.e. on } \Lambda$$

where $\chi = \int_0^\infty L_t v \, dt \in C^\eta(\Lambda)$. In particular, $\sup_{t \geq 0} |v_t|_\infty < \infty$.

Here, $|g|_\infty = \operatorname{ess\,sup}_\Lambda |g|$ and $\|g\|_\eta = |g|_\infty + \sup_{x \neq y} |g(x) - g(y)|/d(x,y)^\eta$. Theorem 1.1 implies that any Banach space admitting a spectral gap and embedded in $C^\eta(\Lambda)$ for some $\eta > \frac{1}{2}$ is cohomologically trivial. However, for nonuniformly expanding semiflows of the type considered in this paper and in the aforementioned references, coboundaries are known to be exceedingly rare, see for example [11, Section 2.3.3]. Hence, Theorem 1.1 can be viewed as an "anti-spectral gap" result for such continuous time dynamical systems.

The proof of Theorem 1.1 uses a Gordin-type argument [14] to obtain a martingale-coboundary decomposition. The martingale part has C^η-sample paths with $\eta > \frac{1}{2}$ and hence is constant, so the result follows.

Nevertheless, our aim of controlling L_t in Hölder norm for a large class of semiflows remains viable, and we obtain positive results in this direction. We consider uniformly and nonuniformly expanding semiflows satisfying a Dolgopyat-type estimate [13]. Such an estimate plays a key role in proving exponential decay of correlations for the semiflow. Our second main result, Theorem 3.2, shows how to use the Dolgopyat estimate to prove exponential decay of $L_t v$ in a Hölder norm for mean zero Hölder observables satisfying a good support condition. Apart from the Dolgopyat estimate, the main ingredient is an operator renewal equation for semiflows [17].

The remainder of the paper is organised as follows. In Section 2, we prove Theorem 1.1. In Section 3, we recall the set up for nonuniformly expanding semiflows with exponential decay of correlations and state Theorem 3.2 on decay in norm. In Section 4, we prove Theorem 3.2.

Notation We use "big O" and $\ll$ notation interchangeably, writing $a_n = O(b_n)$ or $a_n \ll b_n$ if there are constants $C > 0$, $n_0 \geq 1$ such that $a_n \leq Cb_n$ for all $n \geq n_0$.

2 Proof of Theorem 1.1

We begin by recalling the setup of Theorem 1.1. Let (Λ, d) be a bounded metric space with Borel probability measure μ, and let $F_t : \Lambda \to \Lambda$ be a measure-preserving semiflow. We suppose that $t \to F_t$ is Lipschitz a.e. on Λ. Let $L_t : L^1(\Lambda) \to L^1(\Lambda)$ denote the transfer operator corresponding to F_t.

Let $v \in L^\infty(\Lambda)$, with $L_t v \in C^\eta(\Lambda)$ for all $t > 0$ and $\int_0^\infty \|L_t v\|_\eta \, dt < \infty$ where $\eta \in (\frac{1}{2}, 1)$. Define $\chi = \int_0^\infty L_t v \, dt \in C^\eta(\Lambda)$, and

$$v_t = \int_0^t v \circ F_r \, dr, \qquad m_t = v_t - \chi \circ F_t + \chi,$$

for $t \geq 0$. Let $\mathcal{B}$ denote the Borel σ-algebra on Λ.

Proposition 2.1 *(i) $t \to m_t$ is C^η a.e. on Λ.*
(ii) $\mathbb{E}(m_t | F_t^{-1}\mathcal{B}) = 0$ for all $t \geq 0$.

Proof (i) For $0 \leq s \leq t \leq 1$ and $x \in \Lambda$,

$$\begin{aligned} |m_s(x) - m_t(x)| &\leq |v_s(x) - v_t(x)| + |\chi(F_s x) - \chi(F_t x)| \\ &\leq |s - t| |v|_\infty + |\chi|_\eta \, d(F_s x, F_t x)^\eta. \end{aligned}$$

Since $t \mapsto F_t$ is a.e. Lipschitz, it follows that $t \mapsto m_t$ is a.e. C^η.
(ii) Let $U_t v = v \circ F_t$, and recall that $L_t U_t = I$ and $\mathbb{E}(\cdot | F_t^{-1}\mathcal{B}) = U_t L_t$. Then

$$\begin{aligned} L_t m_t = L_t(v_t - U_t\chi + \chi) &= \textstyle\int_0^t L_t U_r v \, dr - \chi + \int_0^\infty L_t L_r v \, dr \\ &= \textstyle\int_0^t L_{t-r} v \, dr - \chi + \int_0^\infty L_{t+r} v \, dr = \int_0^t L_r v \, dr - \chi + \int_t^\infty L_r v \, dr = 0. \end{aligned}$$

Hence $\mathbb{E}(m_t | F_t^{-1}\mathcal{B}) = U_t L_t m_t = 0$. ∎

Proof of Theorem 1.1 Fix $T > 0$, and define

$$M_T(t) = m_T - m_{T-t} = m_t \circ F_{T-t}, \quad t \in [0, T].$$

Also, define the filtration $\mathcal{G}_{T,t} = F_{T-t}^{-1}\mathcal{B}$. It is immediate that $M_T(t) = m_t \circ F_{T-t}$ is $\mathcal{G}_{T,t}$-measurable. Also, for $s < t$ we have $M_T(t) - M_T(s) = m_{T-s} - m_{T-t} = m_{t-s} \circ F_{T-t}$, so

$$\mathbb{E}(M_T(t) - M_T(s) | \mathcal{G}_{T,s}) = \mathbb{E}(m_{t-s} \circ F_{T-t} | F_{T-s}^{-1}\mathcal{B}) = \mathbb{E}(m_{t-s} | F_{t-s}^{-1}\mathcal{B}) \circ F_{T-t} = 0$$

by Proposition 2.1(ii). Hence M_T is a martingale for each $T > 0$. Next,

$$|M_T(t)|_\infty = |m_t \circ F_{T-t}|_\infty \le |m_t|_\infty \le |v_t|_\infty + 2|\chi|_\infty \le T|v|_\infty + 2|\chi|_\infty.$$

Hence $M_T(t)$, $t \in [0,T]$, is a bounded martingale.

By Proposition 2.1(i), M_T has C^η sample paths. Since $\eta > \frac{1}{2}$, it follows from general martingale theory that $M_T \equiv 0$ a.e. Taking $t = T$, we obtain $m_T = 0$ a.e. Hence $v_T = \chi \circ F_T - \chi$ a.e. for all $T > 0$ as required.

For completeness, we include the argument that $M_T \equiv 0$ a.e. We require two standard properties of the quadratic variation process $t \mapsto [M_T](t)$; a reference for these is [12, Theorem 4.1]. First, $[M_T](t)$ is the limit in probability as $n \to \infty$ of

$$S_n(t) = \sum_{j=1}^{n} (M_T(jt/n) - M_T((j-1)t/n))^2.$$

Second (noting that $M_T(0) = 0$),

$$[M_T](t) = M_T(t)^2 - 2\int_0^t M_T\, dM_T,$$

where the stochastic integral has expectation zero. In particular, $\mathbb{E}([M_T]) \equiv \mathbb{E}(M_T^2)$.

Since M_T has Hölder sample paths with exponent $\eta > \frac{1}{2}$, we have a.e. that

$$|S_n(t)| = O(t^\eta n^{-(2\eta-1)}) \to 0 \quad \text{as } n \to \infty.$$

Hence $[M_T] \equiv 0$ a.e. It follows that $\mathbb{E}(M_T^2) \equiv 0$ and so $M_T \equiv 0$ a.e. ∎

3 Setup and statement of Theorem 3.2

In this section, we state our result on Hölder norm decay of transfer operators for uniformly and nonuniformly expanding semiflows.

Let (Y, d) be a bounded metric space with Borel probability measure μ and an at most countable measurable partition $\{Y_j\}$. Let $F : Y \to Y$ be a measure-preserving transformation such that F restricts to a measure-theoretic bijection from Y_j onto Y for each j. Let $g = d\mu/(d\mu \circ F)$ be the inverse Jacobian of F.

Fix $\eta \in (0,1)$. Assume that there are constants $\lambda > 1$ and $C > 0$ such that $d(F^n y, F^n y') \ge C\lambda^n d(y, y')$ and $|\log g(y) - \log g(y')| \le Cd(Fy, Fy')^\eta$ for all $y, y' \in Y_j$, $j \ge 1$, $n \ge 1$. In particular, F is a Gibbs-Markov map as in [2] (see also [1, 3]) with ergodic (and mixing) invariant measure μ.

Let $\varphi : Y \to [2, \infty)$ be a piecewise C^η roof function. We assume that there is a constant $C > 0$ such that

$$|\varphi(y) - \varphi(y')| \le Cd(Fy, Fy')^\eta \qquad (3.1)$$

for all $y, y' \in Y_j$, $j \geq 1$. Also, we assume exponential tails, namely that there exists $\delta_0 > 0$ such that

$$\textstyle\sum_j \mu(Y_j) e^{\delta_0 |1_{Y_j}\varphi|_\infty} < \infty. \tag{3.2}$$

Define the suspension $Y^\varphi = \{(y,u) \in Y \times [0,\infty) : u \in [0,\varphi(y)]\}/\sim$ where $(y,\varphi(y)) \sim (Fy, 0)$. The suspension semiflow $F_t : Y^\varphi \to Y^\varphi$ is given by $F_t(y,u) = (y, u+t)$ computed modulo identifications. We define the ergodic F_t-invariant probability measure $\mu^\varphi = (\mu \times \text{Lebesgue})/\bar\varphi$ where $\bar\varphi = \int_Y \varphi \, d\mu$. [1]

Let $L_t : L^1(Y^\varphi) \to L^1(Y^\varphi)$ denote the transfer operator for F_t and let $R_0 : L^1(Y) \to L^1(Y)$ denote the transfer operator for F. Recall (see for example [2]) that $(R_0 v)(y) = \sum_j g(y_j) v(y_j)$ where y_j is the unique preimage of y under $F|Y_j$, and there is a constant $C > 0$ such that

$$|g(y)| \leq C\mu(Y_j), \qquad |g(y) - g(y')| \leq C\mu(Y_j) d(Fy, Fy')^\eta, \tag{3.3}$$

for all $y, y' \in Y_j$, $j \geq 1$.

Function space on Y^φ Let $Y_j^\varphi = \{(y,u) \in Y^\varphi : y \in Y_j\}$. Fix $\eta \in (0,1]$, $\delta > 0$. For $v : Y^\varphi \to \mathbb{R}$, define $|v|_{\delta,\infty} = \sup_{(y,u)\in Y^\varphi} e^{-\delta u}|v(y,u)|$ and

$$\|v\|_{\delta,\eta} = |v|_{\delta,\infty} + |v|_{\delta,\eta}, \qquad |v|_{\delta,\eta} = \sup_{j\geq 1} \sup_{(y,u),(y',u)\in Y_j^\varphi,\, y\neq y'} e^{-\delta u} \frac{|v(y,u) - v(y',u)|}{d(y,y')^\eta}.$$

Then $\mathcal{F}_{\delta,\eta}(Y^\varphi)$ consists of observables $v : Y^\varphi \to \mathbb{R}$ with $\|v\|_{\delta,\eta} < \infty$.

Next, define $\partial_u v$ to be the partial derivative of v with respect to u at points $(y,u) \in Y^\varphi$ with $u \in (0, \varphi(y))$ and to be the appropriate one-sided partial derivative when $u \in \{0, \varphi(y)\}$. For $m \geq 0$, define $\mathcal{F}_{\delta,\eta,m}(Y^\varphi)$ to consist of observables $v : Y^\varphi \to \mathbb{R}$ such that $\partial_u^j v \in \mathcal{F}_{\delta,\eta}(Y^\varphi)$ for $j = 0, 1, \ldots, m$, with norm $\|v\|_{\delta,\eta,m} = \max_{j=0,\ldots,m} \|\partial_u^j v\|_{\delta,\eta}$.

Definition 3.1 We say that a function $u : Y^\varphi \to \mathbb{R}$ has *good support* if there exists $r > 0$ such that $\operatorname{supp} v \subset \{(y,u) \in Y \times \mathbb{R} : u \in [r, \varphi(y) - r]\}$.

For functions with good support, $\partial_u v$ coincides with the derivative $\partial_t v = \lim_{h\to 0}(v \circ F_h - v)/h$ in the flow direction.

Let

$$\mathcal{F}^0_{\delta,\eta,m}(Y^\varphi) = \{v \in \mathcal{F}_{\delta,\eta,m}(Y^\varphi) : \textstyle\int_{Y^\varphi} v \, d\mu^\varphi = 0\}.$$

We write $\mathcal{F}_{\delta,\eta}(Y^\varphi)$ and $\mathcal{F}^0_{\delta,\eta}(Y^\varphi)$ when $m = 0$.

Function space on Y For $v : Y \to \mathbb{R}$, define

$$\|v\|_\eta = |v|_\infty + |v|_\eta, \qquad |v|_\eta = \sup_{j\geq 1} \sup_{y,y'\in Y_j,\, y\neq y'} |v(y) - v(y')|/d(y,y')^\eta.$$

Let $\mathcal{F}_\eta(Y)$ consist of observables $v : Y \to \mathbb{R}$ with $\|v\|_\eta < \infty$.

[1] We call such semiflows "nonuniformly expanding" since they are the continuous time analogue of maps that are nonuniformly expanding in the sense of Young [23]. "Uniformly expanding" semiflows are those with φ bounded; they have bounded distortion as well as uniform expansion.

Dolgopyat estimate Define the twisted transfer operators

$$\widehat{R}_0(s) : L^1(Y) \to L^1(Y), \qquad \widehat{R}_0(s)v = R_0(e^{-s\varphi}v).$$

We assume that there exists $\gamma \in (0,1)$, $\epsilon > 0$, $m_0 \ge 0$, $A, D > 0$ such that

$$\|\widehat{R}_0(s)^n\|_{\mathcal{F}_\eta(Y)\mapsto\mathcal{F}_\eta(Y)} \le |b|^{m_0}\gamma^n \tag{3.4}$$

for all $s = a + ib \in \mathbb{C}$ with $|a| < \epsilon$, $|b| \ge D$ and all $n \ge A\log|b|$. Such an assumption holds in the settings of [5, 6, 8, 13].

Now we can state our main result on norm decay for L_t.

Theorem 3.2 *Under these assumptions, there exists $\epsilon > 0$, $m \ge 1$, $C > 0$ such that*

$$\|L_t v\|_{\delta,\eta,1} \le Ce^{-\epsilon t}\|v\|_{\delta,\eta,m} \quad \textit{for all } t > 0$$

for all $v \in \mathcal{F}^0_{\delta,\eta,m}(Y^\varphi)$ with good support.

Remark 3.3 (a) There is no contradiction to Theorem 1.1 since the norm on $\mathcal{F}_{\delta,\eta,1}$ gives no Hölder control in the flow direction when passing through points of the form $(y, \varphi(y))$.
(b) The lack of control in (a) is also the barrier for mollification arguments of the type usually used to pass from $\mathcal{F}_{\delta,\eta,m}$ to C^η observables on an ambient space. In fact, such arguments are doomed to fail at the operator level by Theorem 1.1 when $\eta > \frac{1}{2}$ and hence seem unlikely for any η.

Remark 3.4 Usually, we can take $m_0 \in (0,1)$ in (3.4) in which case $m = 3$ suffices in Theorem 3.2.

There are numerous simplifications when $\{Y_j\}$ is a finite partition. In particular, conditions (3.1) and (3.2) are redundant and we can take $\delta = 0$.

4 Proof of Theorem 3.2

Our proof of norm decay is broken into three parts. In Subsection 4.1, we recall a continuous-time operator renewal equation [17] which enables estimates of Laplace transforms of transfer operators at the level of Y. In Subsection 4.2, we show how to pass to estimates of Laplace transforms of L_t. In Subsection 4.3, we invert the Laplace transform to obtain norm decay of L_t.

4.1 Operator renewal equation

Let $\widetilde{Y} = Y \times [0,1]$ and define

$$\widetilde{F} : \widetilde{Y} \to \widetilde{Y}, \qquad \widetilde{F}(y,u) = (Fy, u),$$

with transfer operator $\widetilde{R} : L^1(\widetilde{Y}) \to L^1(\widetilde{Y})$. Also, define

$$\widetilde{\varphi} : \widetilde{Y} \to [2, \infty), \quad \widetilde{\varphi}(y, u) = \varphi(y).$$

Define the twisted transfer operators

$$\widehat{R}(s) : L^1(\widetilde{Y}) \to L^1(\widetilde{Y}), \qquad \widehat{R}(s)v = \widetilde{R}(e^{-s\widetilde{\varphi}}v).$$

Let $\widetilde{Y}_j = Y_j \times [0,1]$. For $v : \widetilde{Y} \to \mathbb{R}$, define

$$\|v\|_\eta = |v|_\infty + |v|_\eta, \qquad |v|_\eta = \sup_{j\ge 1} \sup_{(y,u),(y',u)\in\widetilde{Y}_j,\, y\neq y'} |v(y,u) - v(y',u)|/d(y,y')^\eta.$$

Let $\mathcal{F}_\eta(\widetilde{Y})$ consist of observables $v : \widetilde{Y} \to \mathbb{R}$ with $\|v\|_\eta < \infty$. Let

$$\mathcal{F}^0_\eta(\widetilde{Y}) = \{v \in \mathcal{F}_\eta(\widetilde{Y}) : \textstyle\int_{\widetilde{Y}} v\, d\tilde{\mu} = 0\}$$

where $\tilde{\mu} = \mu \times \mathrm{Leb}_{[0,1]}$.

Lemma 4.1 *Write $s = a + ib \in \mathbb{C}$. There exists $\epsilon > 0$, $m_1 \ge 0$, $C > 0$ such that*

(a) $s \mapsto (I - \widehat{R}(s))^{-1} : \mathcal{F}^0_\eta(\widetilde{Y}) \to \mathcal{F}_\eta(\widetilde{Y})$ is analytic on $\{|a| < \epsilon\}$;

(b) $s \mapsto (I - \widehat{R}(s))^{-1} : \mathcal{F}_\eta(\widetilde{Y}) \to \mathcal{F}_\eta(\widetilde{Y})$ is analytic on $\{|a| < \epsilon\}$ except for a simple pole at $s = 0$;

(c) $\|(I - \widehat{R}(s))^{-1}\|_{\mathcal{F}_\eta(\widetilde{Y})\mapsto\mathcal{F}_\eta(\widetilde{Y})} \le C|b|^{m_1}$ for $|a| \le \epsilon$, $|b| \ge 1$.

Proof It suffices to verify these properties for $Z(s) = (I - \widehat{R}_0(s))^{-1}$ on Y. They immediately transfer to $(I - \widehat{R}(s))^{-1}$ on $\widetilde{Y}$ since $(\widehat{R}v)(y,u) = (\widehat{R}_0 v^u)(y)$ where $v^u(y) = v(y,u)$.

The arguments for passing from (3.4) to the desired properties for $Z(s)$ are standard. For completeness, we sketch these details now recalling arguments from [5]. Define $\mathcal{F}_\eta(Y)$ with norm $\|\ \|_\eta$ by restricting to $u = 0$ (this coincides with the usual Hölder space on Y). Let A, D, ϵ and m_0 be as in (3.4). Increase A and D so that $D > 1$ and $|b|^{m_0}\gamma^{[A\log|b|]} \le \frac{1}{2}$ for $|b| \ge D$. Suppose that $|a| \le \epsilon$, $|b| \ge D$. Then $\|\widehat{R}_0(s)^{[A\log|b|]}\|_\eta \le |b|^{m_0}\gamma^{[A\log|b|]} \le \frac{1}{2}$ and $\|(I - \widehat{R}_0(s)^{[A\log|b|]})^{-1}\|_\eta \le 2$.

As in [5, Proposition 2.5], we can shrink ϵ so that $s \to \widehat{R}_0(s)$ is continuous on $\mathcal{F}_\eta(Y)$ for $|a| \le \epsilon$. The simple eigenvalue 1 for $\widehat{R}_0(0) = R_0$ extends to a continuous family of simple eigenvalues $\lambda(s)$ for $|s| \le \epsilon$. Hence we can choose ϵ so that $\frac{1}{2} < \lambda(a) < 2$ for $|a| \le \epsilon$. By [5, Corollary 2.8], $\|\widehat{R}_0(s)^n\|_\eta \ll |b|\lambda(a)^n \le |b|2^n$ for all $n \ge 1$, $|a| \le \epsilon$, $|b| \ge D$. Hence

$$\begin{aligned}\|Z(s)\|_\eta &\le \big(1 + \|\widehat{R}_0(s)\|_\eta + \cdots + \|\widehat{R}_0(s)^{[A\log|b|]-1}\|_\eta\big)\|(I - \widehat{R}_0(s)^{[A\log|b|]})^{-1}\|_\eta \\ &\ll (\log|b|)\,|b|\,2^{A\log|b|} \le |b|^{m_1},\end{aligned}$$

with $m_1 = 1 + A\log 2$. This proves analyticity on the region $\{|a| < \epsilon, |b| > D\}$ with the desired estimates for property (c) on this region.

For $|a| \le \epsilon$, $|b| \le D$, we recall arguments from the proof of [5, Lemma 2.22] (where $\widehat{R}_0(s)$ is denoted Q_s). For ϵ sufficiently small, the part of spectrum of $\widehat{R}_0(s)$ that is close to 1 consists only of isolated eigenvalues. Also, the spectral radius of $\widehat{R}_0(s)$ is at most $\lambda(a)$ and $\lambda(a) < 1$ for $a \in [0, \epsilon]$, so $s \mapsto Z(s)$ is analytic on $\{0 < a < \epsilon\}$.

Suppose that $\widehat{R}_0(ib)v = v$ for some $v \in \mathcal{F}_\eta(Y)$, $b \ne 0$. Choose $q \ge 1$ such that $q|b| > D$. Since $\widehat{R}_0(s)$ is the L^2 adjoint of $v \mapsto e^{s\varphi} v \circ F$, we have $e^{ib\varphi} v \circ F = v$. Hence $e^{iqb\varphi} v^q \circ F = v^q$ and so $\widehat{R}_0(iqb)v^q = v^q$. But $\|Z(iqb)v^q\|_\eta < \infty$, so $v = 0$. Hence $1 \notin \operatorname{spec} \widehat{R}_0(ib)$ for all $b \ne 0$. It follows that for all $b \ne 0$ there exists an open set $U_b \subset \mathbb{C}$ containing ib such that $1 \notin \operatorname{spec} \widehat{R}_0(s)$ for all $s \in U_b$, and so $s \mapsto Z(s)$ is analytic on U_b.

Next, we recall that for s near to zero, $\lambda(s) = 1 + cs + O(s^2)$ where $c < 0$. Hence $s \mapsto Z(s)$ has a simple pole at zero. It follows that there exists $\epsilon > 0$ such that $s \mapsto Z(s)$ is analytic on $\{|a| < \epsilon, |b| < 2D\}$ except for a simple pole at $s = 0$. Combining this with the estimates on $\{|a| < \epsilon, |b| \ge D\}$ we have proved properties (b) and (c) for $Z(s)$.

Finally, the spectral projection π corresponding to the eigenvalue $\lambda(0) = 1$ for $\widehat{R}_0(0) = R$ is given by $\pi v = \int_Y v\, d\mu$. Hence the pole disappears on restriction to observables of mean zero, proving property (a) for $Z(s)$. ∎

Next define
$$T_t v = 1_{\widetilde{Y}} L_t(1_{\widetilde{Y}} v), \qquad U_t v = 1_{\widetilde{Y}} L_t(1_{\{\widetilde{\varphi} > t\}} v)$$
and
$$\widehat{T}(s) = \int_0^\infty e^{-st} T_t\, dt, \qquad \widehat{U}(s) = \int_0^\infty e^{-st} U_t\, dt,$$

By [17, Theorem 3.3], we have the operator renewal equation
$$\widehat{T} = \widehat{U}(I - \widehat{R})^{-1}.$$

Proposition 4.2 *There exists $\epsilon > 0$, $C > 0$ such that $s \mapsto \widehat{U}(s) : \mathcal{F}_\eta(\widetilde{Y}) \to \mathcal{F}_\eta(\widetilde{Y})$ is analytic on $\{|a| < \epsilon\}$ and $\|\widehat{U}(s)\|_{\mathcal{F}_\eta(\widetilde{Y}) \mapsto \mathcal{F}_\eta(\widetilde{Y})} \le C|s|$ for $|a| \le \epsilon$.*

Proof By [17, Proposition 3.4],
$$(U_t v)(y, u) = \begin{cases} v(y, u-t) 1_{[t,1]}(u) & 0 \le t \le 1 \\ (\widetilde{R} v_t)(y, u) & t > 1 \end{cases}$$
where $v_t(y, u) = 1_{\{t < \varphi(y) < t+1-u\}} v(y, u - t + \varphi(y))$. Hence $\widehat{U}(s) = \widehat{U}_1(s) + \widehat{U}_2(s)$ where
$$(\widehat{U}_1(s)v)(y, u) = \int_0^u e^{-st} v(y, u-t)\, dt, \qquad \widehat{U}_2(s)v = \int_1^\infty e^{-st} \widetilde{R} v_t\, dt.$$

It is clear that $\|\widehat{U}_1(s)v\|_\eta \le e^\epsilon \|v\|_\eta$. We focus attention on the second term

$$(\widehat{U}_2(s)v)(y,u) = \textstyle\sum_j g(y_j) \displaystyle\int_1^\infty e^{-st} v_t(y_j,u)\,dt = \textstyle\sum_j g(y_j) \widehat{V}(s)(y_j,u),$$

where $\widehat{V}(s)(y,u) = \int_u^1 e^{s(t-u-\varphi)} v(y,t)\,dt$. Clearly, $|1_{Y_j}\widehat{V}(s)|_\infty \le e^{\epsilon|1_{Y_j}\varphi|_\infty}|v|_\infty$. Also,

$$\widehat{V}(s)(y,u) - \widehat{V}(s)(y',u) = I + J,$$

where

$$\begin{aligned} I &= \int_u^1 (e^{s(t-u-\varphi(y))} - e^{s(t-u-\varphi(y'))}) v(y,t)\,dt, \\ J &= \int_u^1 e^{s(t-u-\varphi(y'))} (v(y,t) - v(y',t))\,dt. \end{aligned}$$

For $y, y' \in Y_j$,

$$|I| \le |v|_\infty \int_u^1 e^{\epsilon(|1_{Y_j}\varphi|_\infty + u - t)} |s||\varphi(y) - \varphi(y')|\,dt \ll |s||v|_\infty\, e^{\epsilon|1_{Y_j}\varphi|_\infty} d(Fy, Fy')^\eta$$

by (3.1), and

$$|J| \le \int_u^1 e^{\epsilon(|1_{Y_j}\varphi|_\infty + u - t)} |v(y,t) - v(y',t)|\,dt \le e^{\epsilon|1_{Y_j}\varphi|_\infty} |v|_\eta\, d(y,y')^\eta.$$

Hence $|\widehat{V}(s)(y,u) - \widehat{V}(s)(y',u)|_\eta \ll |s| e^{\epsilon|1_{Y_j}\varphi|_\infty} \|v\|_\eta\, d(Fy, Fy')^\eta$.

It follows from the estimates for $1_{Y_j}\widehat{V}(s)$ together with (3.3) that $\|\widehat{U}_2(s)v\|_\eta \ll \sum_j |s|\mu(Y_j) e^{\epsilon|1_{Y_j}\varphi|_\infty} \|v\|_\eta$. By (3.2), $\|\widehat{U}_2(s)v\|_\eta \ll |s|\|v\|_\eta$ for ϵ sufficiently small. We conclude that $\|\widehat{U}(s)v\|_\eta \ll |s|\|v\|_\eta$. ∎

4.2 From $\widehat{T}$ on $\widetilde{Y}$ to $\widehat{L}$ on Y^φ

Lemma 4.1 and Proposition 4.2 yield analyticity and estimates for $\widehat{T} = \widehat{U}(I - \widehat{R})^{-1}$ on $\widetilde{Y}$. In this subsection, we show how these properties are inherited by $\widehat{L}(s) = \int_0^\infty e^{-st} L_t\,dt$ on Y^φ.

Remark 4.3 The approach in this subsection is similar to that in [10, Section 5] but there are some important differences. The rationale behind the two step decomposition in Propositions 4.4 and 4.5 below is that the discreteness of the decomposition in Proposition 4.4 simplifies many formulas significantly. In particular, the previously problematic term E_t in [10] becomes elementary (and vanishes for large t when φ is bounded). The decomposition in Proposition 4.5 remains continuous to simplify the estimates in Proposition 4.8.

Since the setting in [10] is different (infinite ergodic theory, reinducing) we keep the exposition here self-contained even where the estimates coincide with those in [10].

Define

$$\begin{aligned} A_n : L^1(\widetilde{Y}) &\to L^1(Y^\varphi), & (A_n v)(y,u) &= 1_{\{n\le u<n+1\}}(L_n v)(y,u),\ n\ge 0,\\ E_t : L^1(Y^\varphi) &\to L^1(Y^\varphi), & (E_t v)(y,u) &= 1_{\{[t]+1\le u\le \varphi(y)\}}(L_t v)(y,u),\ t>0. \end{aligned}$$

Proposition 4.4 $L_t = \sum_{j=0}^{[t]} A_j 1_{\widetilde{Y}} L_{t-j} + E_t$ *for* $t>0$.

Proof For $y\in Y$, $u\in(0,\varphi(y))$,

$$\begin{aligned} (L_t v)(y,u) &= \sum_{j=0}^{[t]} 1_{\{j\le u<j+1\}}(L_t v)(y,u) + 1_{\{[t]+1\le u\le\varphi(y)\}}(L_t v)(y,u)\\ &= \sum_{j=0}^{[t]} (A_j L_{t-j} v)(y,u) + (E_t v)(y,u). \end{aligned}$$

Now use that $A_n = A_n 1_{\widetilde{Y}}$. ∎

Next, define

$$\begin{aligned} B_t : L^1(Y^\varphi) &\to L^1(\widetilde{Y}), & B_t v &= 1_{\widetilde{Y}} L_t(1_{\Delta_t} v),\\ G_t : L^1(Y^\varphi) &\to L^1(\widetilde{Y}), & G_t v &= B_t(\omega(t)v),\\ H_t : L^1(Y^\varphi) &\to L^1(\widetilde{Y}), & H_t v &= 1_{\widetilde{Y}} L_t(1_{\Delta_t'} v), \end{aligned}$$

for $t>0$, where

$$\begin{aligned} &\Delta_t = \{(y,u)\in Y^\varphi : \varphi(y)-t\le u<\varphi(y)-t+1\}\\ &\Delta_t' = \{(y,u)\in Y^\varphi : u<\varphi(y)-t\}, \qquad \omega(t)(y,u) = \varphi(y)-u-t+1. \end{aligned}$$

Proposition 4.5 $1_{\widetilde{Y}} L_t = \int_0^t T_{t-\tau} B_\tau\, d\tau + G_t + H_t$ *for* $t>0$.

Proof Let $y\in Y$, $u\in[0,\varphi(y)]$. Then

$$\begin{aligned} \int_0^t 1_{\Delta_\tau}(y,u)\, d\tau &= \int_0^t 1_{\{\varphi(y)-u\le\tau\le\varphi(y)-u+1\}}\, d\tau\\ &= 1_{\{t\ge\varphi(y)-u+1\}} + 1_{\{\varphi(y)-u\le t<\varphi(y)-u+1\}}(t-\varphi(y)+u)\\ &= 1 - 1_{\{t<\varphi(y)-u+1\}} + 1_{\{\varphi(y)-u\le t<\varphi(y)-u+1\}}(t-\varphi(y)+u)\\ &= 1 - 1_{\Delta_t'}(y,u) + 1_{\Delta_t}(y,u)(t-\varphi(y)+u-1). \end{aligned}$$

Hence $\int_0^t 1_{\Delta_\tau}\,d\tau = 1 - 1_{\Delta_t}\omega(t) - 1_{\Delta'_t}$. It follows that

$$\begin{aligned}\int_0^t T_{t-\tau}B_\tau v\,d\tau &= 1_{\widetilde Y}\int_0^t L_{t-\tau}1_{\widetilde Y}B_\tau v\,d\tau = 1_{\widetilde Y}\int_0^t L_{t-\tau}B_\tau v\,d\tau \\ &= 1_{\widetilde Y}\int_0^t L_{t-\tau}L_\tau(1_{\Delta_\tau}v)\,d\tau = 1_{\widetilde Y}L_t\Big(\int_0^t 1_{\Delta_\tau}v\,d\tau\Big) = 1_{\widetilde Y}L_t v - G_t v - H_t v\end{aligned}$$

as required. ∎

We have already defined the Laplace transforms $\widehat L(s)$ and $\widehat T(s)$ for $s = a+ib$ with $a>0$. Similarly, define

$$\widehat B(s) = \int_0^\infty e^{-st}B_t\,dt, \qquad \widehat E(s) = \int_0^\infty e^{-st}E_t\,dt,$$
$$\widehat G(s) = \int_0^\infty e^{-st}G_t\,dt, \qquad \widehat H(s) = \int_0^\infty e^{-st}H_t\,dt.$$

Also, we define the discrete transform $\widehat A(s) = \sum_{n=0}^\infty e^{-sn}A_n$.

Corollary 4.6 *$\widehat L(s) = \widehat A(s)\widehat T(s)\widehat B(s) + \widehat A(s)\widehat G(s) + \widehat A(s)\widehat H(s) + \widehat E(s)$ for $a>0$.*

Proof By Proposition 4.4,

$$\begin{aligned}\widehat L(s) - \widehat E(s) &= \int_0^\infty e^{-st}\sum_{j=0}^{[t]} A_j 1_{\widetilde Y}L_{t-j}\,dt = \sum_{j=0}^\infty e^{-sj}A_j 1_{\widetilde Y}\int_j^\infty e^{-s(t-j)}L_{t-j}\,dt \\ &= \widehat A(s)1_{\widetilde Y}\int_0^\infty e^{-st}L_t\,dt = \widehat A(s)1_{\widetilde Y}\widehat L(s).\end{aligned}$$

Hence $\widehat L = \widehat A 1_{\widetilde Y}\widehat L + \widehat E$. In addition, by Proposition 4.5, $1_{\widetilde Y}\widehat L = \widehat T\widehat B + \widehat G + \widehat H$. ∎

Proposition 4.7 *Let $\delta > \epsilon > 0$. Then there is a constant $C>0$ such that*

(a) $\|\widehat A(s)\|_{\mathcal F_\eta(\widetilde Y)\to\mathcal F_{\delta,\eta}(Y^\varphi)} \le 1$,

(b) $\|\widehat E(s)\|_{\mathcal F_{\delta,\eta}(Y^\varphi)\to\mathcal F_{\delta,\eta}(Y^\varphi)} \le C$,

(c) $\|\widehat H(s)\|_{\mathcal F_{\delta,\eta}(Y^\varphi)\to\mathcal F_\eta(\widetilde Y)} \le e^\delta$,

for $|a|\le\epsilon$.

Proof (a) Let $v \in \mathcal{F}_\eta(\widetilde{Y})$. Let $(y,u), (y',u) \in Y_j^\varphi$, $j \ge 1$. Since $(A_n v)(y,u) = 1_{\{n\le u<n+1\}} v(y,u-n)$,

$$(\widehat{A}(s)v)(y,u) = \sum_{n=0}^\infty e^{-sn} 1_{\{n\le u<n+1\}} v(y,u-n) = e^{-s[u]} v(y,u-[u]).$$

Hence

$$|(\widehat{A}(s)v)(y,u)| \le e^{\epsilon u}|v|_\infty, \quad |(\widehat{A}(s)v)(y,u) - (\widehat{A}(s)v)(y',u)| \le e^{\epsilon u}|v|_\eta \, d(y,y')^\eta.$$

That is, $|\widehat{A}(s)v|_{\epsilon,\infty} \le |v|_\infty$, $|\widehat{A}(s)v|_{\epsilon,\eta} \le |v|_\eta$. Hence $\|\widehat{A}(s)v\|_{\delta,\eta} \le \|\widehat{A}(s)v\|_{\epsilon,\eta} \le \|v\|_\eta$.

(b) We take $C = 1/(\delta-\epsilon)$. Let $v \in \mathcal{F}_{\delta,\eta}(Y^\varphi)$. Let $(y,u), (y',u) \in Y_j^\varphi$, $j \ge 1$. Note that $(E_t v)(y,u) = 1_{\{[t]+1\le u\}} v(y,u-t)$, so

$$(\widehat{E}(s)v)(y,u) = \int_0^\infty e^{-st} 1_{\{[t]+1\le u\}} v(y,u-t)\, dt.$$

Hence

$$|(\widehat{E}(s)v)(y,u)| \le \int_0^\infty e^{\epsilon t}|v|_{\delta,\infty}\, e^{\delta(u-t)}\, dt = C|v|_{\delta,\infty}\, e^{\delta u},$$

and

$$|(\widehat{E}(s)v)(y,u) - (\widehat{E}(s)v)(y',u)| \le \int_0^\infty e^{\epsilon t}|v|_{\delta,\eta}\, d(y,y')^\eta e^{\delta(u-t)}\, dt = Ce^{\delta u}|v|_{\delta,\eta}\, d(y,y')^\eta.$$

That is, $|\widehat{E}(s)v|_{\delta,\infty} \le |v|_{\delta,\infty}$ and $|\widehat{E}(s)v|_{\delta,\eta} \le |v|_{\delta,\eta}$.

(c) Let $v \in \mathcal{F}_{\epsilon,\eta}(Y^\varphi)$. Let $(y,u), (y',u) \in \widetilde{Y}_j$, $j \ge 1$. Then $(H_t v)(y,u) = 1_{\{t<u\}} v(y,u-t)$ and $(\widehat{H}(s)v)(y,u) = \int_0^u e^{-st} v(y,u-t)\, dt$. Hence,

$$|\widehat{H}(s)v|_\infty \le e^\delta |v|_{\delta,\infty} \quad \text{and} \quad |(\widehat{H}(s)v)(y,u) - (\widehat{H}(s)v)(y',u)| \le e^\delta |v|_{\delta,\eta}\, d(y,y')^\eta.$$

The result follows. ∎

Proposition 4.8 *There exists $\delta > \epsilon > 0$, $C > 0$ such that*

$$\|\widehat{B}(s)\|_{\mathcal{F}_{\delta,\eta}(Y^\varphi)\to\mathcal{F}_\eta(\widetilde{Y})} \le C|s| \quad \textit{and} \quad \|\widehat{G}(s)\|_{\mathcal{F}_{\delta,\eta}(Y^\varphi)\to\mathcal{F}_\eta(\widetilde{Y})} \le C|s| \quad \textit{for } |a| \le \epsilon.$$

Proof Let $v \in L^1(Y^\varphi)$, $w \in L^\infty(\widetilde{Y})$. Using that $F_t(y,u) = (Fy, u+t-\varphi(y))$ for $(y,u) \in \Delta_t$,

$$\begin{aligned}
\int_{\widetilde{Y}} B_t v\, w\, d\tilde{\mu} &= \bar{\varphi}\int_{Y^\varphi} L_t(1_{\Delta_t} v)\, w\, d\mu^\varphi = \bar{\varphi}\int_{Y^\varphi} 1_{\Delta_t} v\, w \circ F_t\, d\mu^\varphi \\
&= \int_Y \int_0^{\varphi(y)} 1_{\{0\le u+t-\varphi(y)<1\}} v(y,u) w(Fy, u+t-\varphi)\, du\, d\mu \\
&= \int_Y \int_{t-\varphi(y)}^t 1_{\{0\le u<1\}} v(y, u+\varphi(y)-t) w(Fy,u)\, du\, d\mu \\
&= \int_{\widetilde{Y}} v_t\, w \circ \widetilde{F}\, d\tilde{\mu} = \int_{\widetilde{Y}} \widetilde{R} v_t\, w\, d\tilde{\mu}
\end{aligned}$$

where $v_t(y,u)=1_{\{0<u+\varphi(y)-t<\varphi(y)\}}v(y,u+\varphi(y)-t)$.

Hence $B_tv=\widetilde{R}v_t$ and it follows immediately that $G_tv=\widetilde{R}(\omega(t)v)_t$. But

$$(\omega(t)v)_t(y,u)=1_{\{0<u+\varphi(y)-t<\varphi(y)\}}(\omega(t)v)(y,u+\varphi(y)-t)=(1-u)v_t(y,u),$$

so $(G_tv)(y,u)=(1-u)(B_tv)(y,u)$.

Next, $\widehat{B}(s)v=\widetilde{R}\widehat{V}(s)$ where

$$\begin{aligned}\widehat{V}(s)(y,u)=\int_0^\infty e^{-st}v_t(y,u)\,dt&=\int_u^{u+\varphi(y)}e^{-st}v(y,u+\varphi(y)-t)\,dt\\&=\int_0^{\varphi(y)}e^{-s(\varphi(y)+u-t)}v(y,t)\,dt.\end{aligned}$$

It is immediate that

$$(\widehat{G}(s)v)(y,u)=(1-u)(\widehat{B}(s)v)(y,u). \tag{4.1}$$

Suppose that $\delta>\epsilon>0$ are fixed. Let $v\in\mathcal{F}_{\delta,\eta}(Y^\varphi)$. Let (y,u), $(y',u)\in\widetilde{Y}_j$, $j\ge1$. Then

$$|\widehat{V}(s)(y,u)|\le\int_0^{\varphi(y)}e^{-a\,(\varphi(y)+u-t)}|v|_{\delta,\infty}\,e^{\delta t}\,dt\ll e^{\delta\varphi(y)}|v|_{\delta,\infty}$$

and so $|1_{Y_j}\widehat{V}(s)|_\infty\ll e^{\delta|1_{Y_j}\varphi|_\infty}|v|_{\delta,\infty}$.

Next, suppose without loss that $\varphi(y')\le\varphi(y)$. Then

$$\widehat{V}(s)(y,u)-\widehat{V}(s)(y',u)=J_1+J_2+J_3$$

where

$$\begin{aligned}J_1&=\int_0^{\varphi(y)}(e^{-s(\varphi(y)+u-t)}-e^{-s(\varphi(y')+u-t)})v(y,t)\,dt,\\J_2&=\int_0^{\varphi(y)}e^{-s(\varphi(y')+u-t)}(v(y,t)-v(y',t))\,dt,\\J_3&=\int_{\varphi(y')}^{\varphi(y)}e^{-s(\varphi(y')+u-t)}v(y',t)\,dt.\end{aligned}$$

For notational convenience we suppose that $a\in(-\epsilon,0)$ since the range $a\ge0$ is simpler. Using (3.1),

$$\begin{aligned}|J_1|&\le\int_0^{\varphi(y)}e^{\epsilon(|1_{Y_j}\varphi|_\infty+1-t)}|s||\varphi(y)-\varphi(y')||v|_{\delta,\infty}\,e^{\delta t}\,dt\\&\ll|s|\varphi(y)e^{\delta|1_{Y_j}\varphi|_\infty}\,d(Fy,Fy')^\eta|v|_{\delta,\infty}\ll|s|e^{2\delta|1_{Y_j}\varphi|_\infty}\,d(Fy,Fy')^\eta|v|_{\delta,\infty},\\|J_2|&\le\int_0^{\varphi(y)}e^{\epsilon(|1_{Y_j}\varphi|_\infty+1-t)}|v|_{\delta,\eta}\,e^{\delta t}d(y,y')^\eta\,dt\ll e^{\delta|1_{Y_j}\varphi|_\infty}\,d(y,y')^\eta|v|_{\delta,\eta},\\|J_3|&\le\int_{\varphi(y')}^{\varphi(y)}e^{\epsilon(|1_{Y_j}\varphi|_\infty+1-t)}|v|_{\delta,\infty}\,e^{\delta t}\,dt\ll e^{2\delta|1_{Y_j}\varphi|_\infty}|v|_{\delta,\infty}\,d(Fy,Fy')^\eta.\end{aligned}$$

Hence
$$|\widehat{V}(s)(y,u)-\widehat{V}(s)(y,u)|\ll|s|e^{2\delta|1_{Y_j}\varphi|_\infty}\|v\|_{\delta,\eta}\,d(Fy,Fy')^\eta.$$
Now, for $(y,u)\in\widetilde{Y}$,
$$(\widehat{B}(s)v)(y,u)=(\widetilde{R}\widehat{V}(s))(y,u)=\textstyle\sum_j g(y_j)\widehat{V}(s)(y_j,u),$$
where y_j is the unique preimage of y under $F|Y_j$. It follows from the estimates for $\widehat{V}(s)$ together with (3.3) that
$$\|\widehat{B}(s)v\|_\eta\ll|s|\textstyle\sum_j\mu(Y_j)e^{2\delta|1_{Y_j}\varphi|_\infty}\|v\|_{\delta,\eta}.$$
Shrinking δ, the desired estimate for $\widehat{B}$ follows from (3.2). Finally, the estimate for $\widehat{G}$ follows from (4.1). ∎

Proposition 4.9 *$\int_{\widetilde{Y}}\widehat{B}(0)v\,d\tilde{\mu}=\bar{\varphi}\int_{Y^\varphi}v\,d\mu^\varphi$ for $v\in L^1(Y^\varphi)$.*

Proof By the definition of $\widehat{B}$,
$$\begin{aligned}\int_{\widetilde{Y}}\widehat{B}(0)v\,d\tilde{\mu}&=\int_{\widetilde{Y}}\int_0^\infty L_t(1_{\Delta_t}v)\,dt\,d\tilde{\mu}=\bar{\varphi}\int_0^\infty\int_{Y^\varphi}L_t(1_{\Delta_t}v)\,d\mu^\varphi\,dt\\&=\bar{\varphi}\int_0^\infty\int_{Y^\varphi}1_{\Delta_t}v\,d\mu^\varphi\,dt=\bar{\varphi}\int_{Y^\varphi}\int_0^\infty 1_{\{\varphi-u<t<\varphi-u+1\}}v\,dt\,d\mu^\varphi=\bar{\varphi}\int_{Y^\varphi}v\,d\mu^\varphi,\end{aligned}$$
as required. ∎

Lemma 4.10 *Write $s=a+ib\in\mathbb{C}$. There exists $\epsilon>0$, $\delta>0$, $m_2\geq 0$, $C>0$ such that*

(a) $s\mapsto\widehat{L}(s):\mathcal{F}^0_{\delta,\eta}(Y^\varphi)\to\mathcal{F}_{\delta,\eta}(Y^\varphi)$ is analytic on $\{|a|<\epsilon\}$;

(b) $s\mapsto\widehat{L}(s):\mathcal{F}_{\delta,\eta}(Y^\varphi)\to\mathcal{F}_{\delta,\eta}(Y^\varphi)$ is analytic on $\{|a|<\epsilon\}$ except for a simple pole at $s=0$;

(c) $\|\widehat{L}(s)v\|_{\delta,\eta}\leq C|b|^{m_2}\|v\|_{\delta,\eta}$ for $|a|\leq\epsilon$, $|b|\geq 1$, $v\in\mathcal{F}_{\delta,\eta}(Y^\varphi)$.

Proof Recall that
$$\widehat{L}=\widehat{A}\widehat{T}\widehat{B}+\widehat{A}\widehat{G}+\widehat{A}\widehat{H}+\widehat{E},\qquad\widehat{T}=\widehat{U}(I-\widehat{R})^{-1}$$
where $\widehat{U}$, $\widehat{A}$, $\widehat{B}$, $\widehat{G}$, $\widehat{H}$ and $\widehat{E}$ are analytic by Propositions 4.2, 4.7 and 4.8. Hence part (b) follows immediately from Lemma 4.1(b). Also, part (c) follows using Lemma 4.1(c).

By Proposition 4.9, $\widehat{B}(0)(\mathcal{F}^0_{\delta,\eta}(Y^\varphi))\subset\mathcal{F}^0_\eta(\widetilde{Y})$. Hence the simple pole at $s=0$ for $(I-\widehat{R})^{-1}\widehat{B}$ disappears on restriction to $\mathcal{F}^0_{\delta,\eta}(Y^\varphi)$ by Lemma 4.1(a). This proves part (a). ∎

4.3 Moving the contour of integration

Proposition 4.11 *Let $m \ge 1$. Let $v \in \mathcal{F}_{\delta,\eta,m}(Y^\varphi)$ with good support. Then $\widehat{L}(s)v = \sum_{j=0}^{m-1}(-1)^j s^{-(j+1)}\partial_t^j v + (-1)^m s^{-m}\widehat{L}(s)\partial_t^m v$ for $a > 0$.*

Proof Recall that $\operatorname{supp} v \subset \{(y,u) \in Y^\varphi : u \in [r, \varphi(y) - r]\}$ for some $r > 0$. For $h \in [0,r]$, we can define $(\Psi_h v)(y,u) = v(y, u-h)$ and then $(\Psi_h v) \circ F_h = v$.

Let $w \in L^\infty(Y^\varphi)$ and write $\rho_{v,w}(t) = \int_{Y^\varphi} v\, w_t \, d\mu^\varphi$ where $w_t = w \circ F_t$. Then for $h \in [0,r]$,

$$\rho_{v,w}(t+h) = \int_{Y^\varphi} v\, w_t \circ F_h \, d\mu^\varphi = \int_{Y^\varphi} (\Psi_h v) \circ F_h \, w_t \circ F_h \, d\mu^\varphi = \int_{Y^\varphi} \Psi_h v \, w_t \, d\mu^\varphi.$$

Hence $h^{-1}(\rho_{v,w}(t+h) - \rho_{v,w}(t)) = \int_{Y^\varphi} h^{-1}(\Psi_h v - v)\, w_t \, d\mu^\varphi$ so

$$\rho'_{v,w}(t) = -\int_{Y^\varphi} \partial_t v \, w_t \, d\mu^\varphi = -\int_{Y^\varphi} \partial_t v \, w \circ F_t \, d\mu^\varphi = -\rho_{\partial_t v, w}(t).$$

Inductively, $\rho^{(j)}_{v,w}(t) = (-1)^j \rho_{\partial_t^j v, w}(t)$.

Now $\int_{Y^\varphi} \widehat{L}(s) v\, w\, d\mu^\varphi = \int_0^\infty e^{-st} \int_{Y^\varphi} L_t v\, w \, d\mu^\varphi \, dt = \int_0^\infty e^{-st} \rho_{v,w}(t)\, dt$, so repeatedly integrating by parts,

$$\begin{aligned}
\int_{Y^\varphi} \widehat{L}(s) v\, w\, d\mu^\varphi &= \sum_{j=0}^{m-1} s^{-(j+1)} \rho^{(j)}_{v,w}(0) + s^{-m} \int_0^\infty e^{-st} \rho^{(m)}_{v,w}(t)\, dt \\
&= \sum_{j=0}^{m-1} (-1)^j s^{-(j+1)} \rho_{\partial_t^j v, w}(0) + (-1)^m s^{-m} \int_0^\infty e^{-st} \rho_{\partial_t^m v, w}(t)\, dt \\
&= \int_{Y^\varphi} \sum_{j=0}^{m-1} (-1)^j s^{-(j+1)} \partial_t^j v\, w\, d\mu^\varphi + (-1)^m s^{-m} \int_0^\infty e^{-st} \rho_{\partial_t^m v, w}(t)\, dt.
\end{aligned}$$

Finally, $\int_0^\infty e^{-st} \rho_{\partial_t^m v, w}(t)\, dt = \int_{Y^\varphi} \widehat{L}(s) \partial_t^m v\, w\, d\mu^\varphi$ and the result follows since $w \in L^\infty(Y^\varphi)$ is arbitrary. ∎

We can now estimate $\|L_t v\|_{\delta,\eta}$.

Corollary 4.12 *Under the assumptions of Theorem 3.2, there exists $\epsilon > 0$, $m_3 \ge 1$, $C > 0$ such that*

$$\|L_t v\|_{\delta,\eta} \le C e^{-\epsilon t} \|v\|_{\delta,\eta,m_3} \quad \textit{for all } t > 0$$

for all $v \in \mathcal{F}^0_{\delta,\eta,m_3}(Y^\varphi)$ with good support.

Proof Let $m_3 = m_2 + 2$. By Lemma 4.10(a), $\widehat{L}(s) : \mathcal{F}^0_{\delta,\eta,m_3}(Y^\varphi) \to \mathcal{F}_{\delta,\eta}(Y^\varphi)$ is analytic for $|a| \le \epsilon$. The alternative expression in Proposition 4.11 is also analytic on this region (the apparent singularity at $s = 0$ is removable by the equality with the analytic function $\widehat{L}$). Hence we can move the contour of integration to $s = -\epsilon + ib$ when computing the inverse Laplace transform, to obtain

$$\begin{aligned} L_t v &= \int_{-\infty}^{\infty} e^{st}\Big(\sum_{j=0}^{m_3-1}(-1)^j s^{-(j+1)}\partial_t^j v + (-1)^{m_3}s^{-m_3}\widehat{L}(s)\partial_t^{m_3}v\Big)\,db \\ &= e^{-\epsilon t}\sum_{j=0}^{m_3-1}(-1)^j\partial_t^j v\int_{-\infty}^{\infty} e^{ibt}s^{-(j+1)}\,db + (-1)^{m_3}e^{-\epsilon t}\int_{-\infty}^{\infty} e^{ibt}s^{-m_3}\widehat{L}(s)\partial_t^{m_3}v\,db. \end{aligned}$$

The final term is estimated using Lemma 4.10(b,c):

$$\Big\|\int_{-\infty}^{\infty} e^{ibt}s^{-m_3}\widehat{L}(s)\partial_t^{m_3}v\,db\Big\|_{\delta,\eta} \ll \int_{-\infty}^{\infty}(1+|b|)^{-(m_2+2)}(1+|b|)^{m_2}\|\partial_t^{m_3}v\|_{\delta,\eta}\,db \ll \|v\|_{\delta,\eta,m_3}.$$

Clearly, the integrals $\int_{-\infty}^{\infty} e^{ibt}s^{-(j+1)}\,db$ converge absolutely for $j \geq 1$, while the integral for $j = 0$ converges as an improper Riemann integral. Hence altogether we obtain that $\|L_t v\|_{\delta,\eta} \ll e^{-\epsilon t}\|v\|_{\delta,\eta,m_3}$. ∎

For the proof of Theorem 3.2, it remains to estimate $\|\partial_u L_t v\|_{\delta,\eta}$. Recall that the transfer operator R_0 for F has weight function g. We have the pointwise formula $(R_0^k v)(y) = \sum_{F^k y'=y} g_k(y')v(y')$ where $g_k = g \ldots g\circ F^{k-1}$. Let $\varphi_k = \sum_{j=0}^{k-1}\varphi\circ F^j$.

Proposition 4.13 *Let $v \in L^1(Y^\varphi)$. Then for all $t > 0$, $(y,u) \in Y^\varphi$,*

$$(L_t v)(y,u) = \sum_{k=0}^{[t/2]}\sum_{F^k y'=y} g_k(y')1_{\{0\le u-t+\varphi_k(y')<\varphi(y')\}}v(y', u-t+\varphi_k(y')).$$

Proof The lap number $N_t(y,u) \in [0,t/2]\cap\mathbb{N}$ is the unique integer $k \geq 0$ such that $u + t - \varphi_k(y) \in [0,\varphi(F^k y))$. In particular, $F_t(y,u) = (F^{N_t(y,u)}y, u + t - \varphi_{N_t(y,u)}(y))$. For $w \in L^\infty(Y^\varphi)$,

$$\begin{aligned} \int_{Y^\varphi} L_t(1_{\{N_t=k\}}v)\,w\,d\mu^\varphi &= \int_{Y^\varphi} 1_{\{N_t=k\}}v\,w\circ F_t\,d\mu^\varphi \\ &= \bar\varphi^{-1}\int_Y\int_0^{\varphi(y)} 1_{\{0\le u+t-\varphi_k(y)<\varphi(F^k y)\}}v(y,u)\,w(F^k y, u+t-\varphi_k(y))\,du\,d\mu \\ &= \bar\varphi^{-1}\int_Y\int_0^{\varphi(F^k y)} 1_{\{0\le u-t+\varphi_k(y)<\varphi(y)\}}v(y,u-t+\varphi_k(y))\,w(F^k y,u)\,du\,d\mu. \end{aligned}$$

Writing $v_{t,k}^u(y) = 1_{\{0\le u-t+\varphi_k(y)<\varphi(y)\}}v(y,u-t+\varphi_k(y))$ and $w^u(y) = w(y,u)$,

$$\begin{aligned} \int_{Y^\varphi} L_t(1_{\{N_t=k\}}v)\,w\,d\mu^\varphi &= \bar\varphi^{-1}\int_0^\infty\int_Y 1_{\{u<\varphi\circ F^k\}}v_{t,k}^u\,w^u\circ F^k\,d\mu\,du \\ &= \bar\varphi^{-1}\int_0^\infty\int_Y 1_{\{u<\varphi\}}R_0^k v_{t,k}^u\,w^u\,d\mu\,du = \int_{Y^\varphi}(R_0^k v_{t,k}^u)(y)\,w(y,u)\,d\mu^\varphi. \end{aligned}$$

Hence,

$$(L_t v)(y,u) = \sum_{k=0}^{[t/2]} (L_t(1_{\{N_t=k\}} v)(y,u) = \sum_{k=0}^{[t/2]} (R_0^k v_{t,k}^u)(y).$$

The result follows from the pointwise formula for R_0^k. ∎

Proof of Theorem 3.2 Let $m = m_3+1$. By Corollary 4.12, $\|L_t v\|_{\delta,\eta} \ll e^{-\epsilon t}\|v\|_{\delta,\eta,m}$.

Recall that ∂_u denotes the ordinary derivative with respect to u at $0 < u < \varphi(y)$ and denotes the appropriate one-sided derivative at $u = 0$ and $u = \varphi(y)$. Since v has good support, the indicator functions in the right-hand side of the formula in Proposition 4.13 are constant on the support of v. It follows that $\partial_u L_t v = L_t(\partial_u v)$. By Corollary 4.12,

$$\|\partial_u L_t v\|_{\delta,\eta} = \|L_t(\partial_u v)\|_{\delta,\eta} \ll e^{-\epsilon t}\|\partial_u v\|_{\delta,\eta,m_3} \le e^{-\epsilon t}\|v\|_{\delta,\eta,m}.$$

Hence, $\|L_t v\|_{\delta,\eta,1} \ll e^{-\epsilon t}\|v\|_{\delta,\eta,m}$ as required. ∎

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