Operator renewal theory and mixing rates for dynamical systems with infinite measure

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Abstract

We develop a theory of operator renewal sequences in the context of infinite ergodic theory. For large classes of dynamical systems preserving an infinite measure, we determine the asymptotic behaviour of iterates L^n of the transfer operator. This was previously an intractable problem.

Examples of systems covered by our results include (i) parabolic rational maps of the complex plane and (ii) (not necessarily Markovian) nonuniformly expanding interval maps with indifferent fixed points.

In addition, we give a particularly simple proof of pointwise dual ergodicity (asymptotic behaviour of $\sum_{j=1}^{n} L^j$) for the class of systems under consideration.

In certain situations, including Pomeau-Manneville intermittency maps, we obtain higher order expansions for L^n and rates of mixing. Also, we obtain error estimates in the associated Dynkin-Lamperti arcsine laws.

1 Introduction

In finite ergodic theory, much recent interest has focussed on the statistical properties of smooth dynamical systems with strong hyperbolicity (expansion/contraction) properties. Landmark results include the proof of exponential decay of correlations for certain classes of uniformly hyperbolic flows [8, 12, 30] and planar dispersing billiards [42]. The latter is part of a general scheme [42, 43] for estimating decay of correlations, or mixing rates, for discrete time dynamical systems.

For systems with subexponential decay of correlations, most approaches yielded only upper bounds for mixing rates. Sarig [37] introduced a powerful new technique, *operator renewal theory*, to obtain precise asymptotics and hence sharp mixing rates. This is an extension of scalar renewal theory from probability theory to the operator situation. The technique was substantially extended and refined by Gouëzel [18, 19].

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Garsia & Lamperti [17] developed a theory of renewal sequences with infinite mean in the probabilistic setting. The techniques are very different from the finite mean case, and draw heavily on the theory of regular variation [7, 16, 27]. A natural question is to develop renewal operator theory in the infinite mean case.

Renewal sequences In probability theory, renewal sequences relate return probabilities to a specified "nice" set with first return probabilities. The analogue in ergodic theory arises in the study of first return maps.

Let (X,μ) be a measure space (finite or infinite), and $f : X \to X$ a conservative measure preserving map. Fix $Y \subset X$ with $\mu(Y) > 0$. Let $\varphi : Y \to \mathbb{Z}^+$ be the first return time $\varphi(y) = \inf\{n \geq 1 : f^n y \in Y\}$ (finite almost everywhere by conservativity). Let $L : L^1(X) \to L^1(X)$ denote the transfer operator (Perron-Frobenius operator) for f and define

$$T_n v = 1_Y L^n(1_Y v), \;\; n \geq 0, \qquad R_n v = 1_Y L^n(1_{\{\varphi=n\}} v), \;\; n \geq 1.$$

Note that T_n and R_n can be viewed as operators on $L^1(Y)$ with $T_0 = I$. Thus T_n corresponds to returns to Y and R_n corresponds to first returns to Y. The relationship $T_n = \sum_{j=1}^n T_{n-j} R_j$ generalises the classical notion of renewal sequences in probability theory.

In the infinite mean setting of Garsia & Lamperti [17], a crucial requirement is that the first return probabilities have regularly varying tails. In our setting it is natural to assume that the return time probabilities have regularly varying tails. Indeed, many of the results in the infinite ergodic theory literature rely crucially on such an assumption [1, 2, 3, 39, 41, 45]. Under this assumption, and certain functional-analytic hypotheses on the operators R_n, we obtain detailed results (Theorems 2.1, 2.2, 2.3) on the asymptotic behaviour of the operators T_n as $n \to \infty$. This has strong ramifications for the asymptotics of the iterates L^n, and hence for the underlying dynamical system.

Maps with indifferent fixed points An important class of examples is provided by interval maps with indifferent fixed points, in particular the Pomeau-Manneville intermittency maps [33] which are uniformly expanding away from an indifferent fixed point at 0. To fix notation, we focus on the version studied by Liverani *et al.* [31]:

$$fx = \begin{cases} x(1+2^\alpha x^\alpha), & 0 < x < \frac{1}{2} \\ 2x-1 & \frac{1}{2} < x < 1 \end{cases}. \tag{1.1}$$

When $\alpha = 0$, this is the doubling map, and Lebesgue measure is invariant, ergodic and exponentially mixing. For $\alpha \in (0,1)$, there is still a unique ergodic invariant probability measure μ absolutely continuous with respect to Lebesgue measure, but the rate of mixing is polynomial: $\int_X v\, w \circ f^n \, d\mu - \int_X v \, d\mu \int_X w \, d\mu \;\leq\; C_{v,w} n^{-(\beta-1)}$ where $\beta = \frac{1}{\alpha}$ for all $w \in L^\infty(X)$ and all v sufficiently regular (for example, Hölder continuous). Hu [25] proved that this decay rate is optimal; a special case of the theory of [18, 37].

For $\alpha \geq 1$, we are in the situation of infinite ergodic theory. There no longer exists an absolutely continuous invariant probability measure, but there is a unique (up to scaling) σ-finite, absolutely continuous invariant measure μ. Previous studies established *pointwise dual ergodicity*: $a_n^{-1}\sum_{j=1}^n L^j v \to \text{const} \int_X v\,d\mu$ almost everywhere for all $v \in L^1(X)$, where $a_n = n^\beta$ for $\beta \in (0,1)$ and $a_n = n/\log n$ when $\beta = 1$.

An important refinement is the limit theorem of Thaler [38] where the convergence of $a_n^{-1}\sum_{j=1}^n L^j v$ is shown to be uniform on compact subsets of $(0,1]$ for all observables of the form $v = u/h$ where u is Riemann integrable and h is the density.

The results of [38] are formulated for Markov maps of the interval with indifferent fixed points. Zweimüller [44, 45] relaxed the Markov condition and systematically studied a large class of non-Markovian nonuniformly expanding interval maps, called *AFN maps*. (See Section 11.3 for a precise definition of AFN map.) In particular, [44] obtained a spectral decomposition into basic (conservative and ergodic) sets and proved that for each basic set there is a σ-finite absolutely continuous invariant measure, unique up to scaling. The results in [38] on uniform pointwise dual ergodicity were extended in [45] to the class of AFN maps.

Understanding the asymptotics of L^n, rather than $\sum_{j=1}^n L^j$, turns out to be a much more difficult problem, even for (1.1). Previously, the sole success in this direction was obtained by Thaler [40]. However, the class of systems covered by [40] is quite restrictive and includes the family (1.1) only for $\beta = 1$.

It is this situation that we have sought to redress in this paper. It is convenient to describe our main results in the setting of AFN maps $f : X \to X$, though our general theory goes much further, as described later on. Let $X' \subset X$ denote the complement of the indifferent fixed points. For any compact subset $A \subset X'$, the construction in [44] yields a suitable first return set Y containing A. Fix such a set Y with first return time function $\varphi : Y \to \mathbb{Z}^+$. Then we assume that the tail probabilities are regularly varying: $\mu(\varphi > n) = \ell(n)n^{-\beta}$ where $\beta \in (0,1]$ and $\ell(x)$ is slowly varying ($\ell : (0,\infty) \to (0,\infty)$ is measurable and $\lim_{x\to\infty} \ell(\lambda x)/\ell(x) = 1$ for all $\lambda > 0$). (For (1.1), ℓ is asymptotically constant and $\beta = \frac{1}{\alpha}$.)

Now we can state our results for AFN maps. Set $d_\beta = \frac{1}{\pi}\sin\beta\pi$ for $\beta \in (0,1)$ and $d_1 = 1$. Define $m(x) = \ell(x)$ for $\beta \in (0,1)$ and $m(x) = \sum_{j=1}^{[x]} \ell(j)j^{-1}$ for $\beta = 1$.

Theorem 1.1 *Suppose that $f : X \to X$ is a topologically mixing AFN map with σ-finite absolutely continuous invariant measure μ and regularly varying tail probabilities. Consider observables of the form $v = \xi u$ where ξ is μ-integrable and of bounded variation on X, and u is Riemann integrable.*

(a) If $\beta \in (\frac{1}{2}, 1]$, then $\lim_{n\to\infty} m(n)n^{1-\beta}L^n v = d_\beta \int_X v\,d\mu$ uniformly on compact subsets of X'.

(b) If $\beta \in (0, \frac{1}{2}]$, then there is a subset $E \subset \mathbb{N}$ of zero density such that $\lim_{n\to\infty,\, n\notin E} \ell(n)n^{1-\beta}L^n v = d_\beta \int_X v\,d\mu$ pointwise on X'.

Moreover, if $v \geq 0$, then $\liminf_{n\to\infty} \ell(n)n^{1-\beta}L^n v = d_\beta \int_X v\,d\mu$ pointwise on X'.

(c) If $\beta \in (0, \frac{1}{2})$, then $L^n v = O(\ell(n) n^{-\beta})$ uniformly on compact subsets of X'.

Remark 1.2 (i) It is known that the asymptotic behaviour of L^n might be complicated. Chung [9, Section I.10] gives an example of a null recurrent Markov chain for which the ratio of n-step transition probabilities $p_{ij}^n / p_{k\ell}^{(n)}$ has no limit as $n \to \infty$ (the regular variation assumption on the return time probabilities is violated). Hajian & Kakutani [21] (see also [3, Proposition 1.4.7]) prove that there always exist *weakly wandering* sets W of positive measure. For such sets, $\int_W L^n 1_W \, d\mu = 0$ for infinitely many n. (Such indicator functions 1_W do not lie in our class of observables $v = \xi u$).

(ii) In the special case of the family (1.1), Theorem 1.1(a) recovers the result of Thaler [40] for $\beta = 1$ and the cases $\beta < 1$ are new. Parts (b,c) are probably not optimal for (1.1) but are the best one can expect in the general setting, see Remark 2.4.

(iii) In addition to yielding convergence results for L^n (rather than $\sum_{j=1}^n L^j$) our methods also cover much wider classes of observables than was previously possible. An indicative example is the family (1.1) where $\beta \in (\frac{1}{2}, 1]$. There is a constant $C > 1$ such that $C^{-1} x^{-\frac{1}{\beta}} \le h(x) \le C x^{-\frac{1}{\beta}}$. Consider observables of the form $v(x) = x^q$. Whereas the results of [38, 45] yield uniform convergence of $\sum_{j=1}^n L^j v$ (and of $L^n v$ when $\beta = 1$) on compact subsets of $(0, 1]$ if and only if $q\beta \ge 1$, our results apply for $(1 + q)\beta > 1$.

(iv) An immediate consequence of Theorem (1.1)(a) is that if $\beta \in (\frac{1}{2}, 1]$, v is of the required form $v = \xi u$, and $w \in L^1(X)$ is supported on a compact subset of X', then $\lim_{n\to\infty} m(n) \int_X v \, w \circ f^n \, d\mu = d_\beta \int_X v \, d\mu \int_X w \, d\mu$.

(v) The situation changes considerably if $\int_X v \, d\mu = 0$. We have the following result which has no counterpart in standard renewal theory, though Gouëzel [18] proves an analogous (and equally unexpected) result in the case of finite ergodic theory:

Theorem 1.3 *Suppose that $f : X \to X$ is a topologically mixing AFN map with regularly varying tail probabilities, $\beta \in (0, 1)$. Suppose that v is of bounded variation and is supported on a compact subset of X'. If $\int_X v \, d\mu = 0$, then $L^n v = O(\ell(n) n^{-\beta})$ uniformly on compact subsets of X'.*

Second order asymptotics and rates of mixing In certain situations, including the family (1.1), the tail probabilities satisfy $\mu(\varphi > n) = cn^{-\beta} + O(n^{-q})$ for some $q > 1$, $c > 0$. It is then possible to obtain higher order expansions of $L^n v$ on compact sets for bounded variation observables v supported on a compact subset of $(0, 1]$. For example, in the specific case of (1.1), $\beta \in (\frac{1}{2}, 1)$, we prove that $|n^{1-\beta} \int_0^1 v \, w \circ f^n \, d\mu - \int_0^1 v \, d\mu \int_0^1 w \, d\mu| \le C n^{-\gamma}$ where $\gamma = \min\{1 - \beta, \beta - \frac{1}{2}\}$. The rate of mixing is sharp for $\beta \ge \frac{3}{4}$, and we obtain precise second order asymptotics provided $\beta > \frac{3}{4}$.

These and related results such as error rates in the Dynkin-Lamperti arcsine laws [14, 29] are discussed in Section 9.

The remainder of the paper is organised as follows. In Section 2, we describe the general framework for our results on the renewal operators T_n. Sections 3, 4 and 5 contain the proofs for $\frac{1}{2} < \beta < 1$. In Section 6, we cover the case $\beta = 1$. In Section 7, we give a particularly elementary, self-contained, proof of pointwise dual ergodicity for all β. In Section 8, we prove our results for $0 < \beta \le \frac{1}{2}$. In Section 9, we formulate and prove results on higher order asymptotics. In Section 10, we show how to pass from $T_n v = 1_Y L^n(1_Y v)$ to L^n. Finally in Section 11, we show that our theory applies to large classes of examples including AFN maps (in particular, we prove Theorem 1.1) and systems for which the first return map is Gibbs-Markov. The latter includes parabolic rational maps of the complex plane [6].

Notation We use "big O" and $\ll$ notation interchangeably, writing $A_n = O(a_n)$ or $A_n \ll a_n$ as $n \to \infty$ if there is a constant $C > 0$ such that $\|A_n\| \le C a_n$ for all $n \ge 1$ (for A_n operators and $a_n \ge 0$ scalars). We write $A_n \sim c_n A$ as $n \to \infty$ if $\lim_{n\to\infty} \|A_n/c_n - A\| = 0$ (for A_n, A operators and $c_n > 0$ scalars).

2 General framework

Let (X, μ) be an infinite measure space, and $f : X \to X$ a conservative measure preserving map. Fix $Y \subset X$ with $\mu(Y) \in (0, \infty)$ and rescale μ so that $\mu(Y) = 1$. Let $\varphi : Y \to \mathbb{Z}^+$ be the first return time $\varphi(y) = \inf\{n \ge 1 : f^n y \in Y\}$ and define the first return map $F = f^\varphi : Y \to Y$.

The return time function $\varphi : Y \to \mathbb{Z}^+$ satisfies $\int_Y \varphi \, d\mu = \infty$. Our crucial assumption is that the tail probabilities are regularly varying:

$$\mu(y \in Y : \varphi(y) > n) = \ell(n) n^{-\beta} \text{ where } \ell \text{ is slowly varying and } \beta \in (0, 1].$$

Recall that the transfer operator $R : L^1(Y) \to L^1(Y)$ for the first return map $F : Y \to Y$ is defined via the formula $\int_Y Rv \, w \, d\mu = \int_Y v \, w \circ F \, d\mu$, $w \in L^\infty(Y)$. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\bar{\mathbb{D}} = \{z \in \mathbb{C} : |z| \le 1\}$. Given $z \in \bar{\mathbb{D}}$, we define $R(z) : L^1(Y) \to L^1(Y)$ to be the operator $R(z)v = R(z^\varphi v)$. Also, for each $n \ge 1$, we define $R_n : L^1(Y) \to L^1(Y)$, $R_n v = R(1_{\{\varphi = n\}} v)$. It is easily verified that $R(z) = \sum_{n=1}^\infty R_n z^n$.

Our assumptions on the first return map $F : Y \to Y$ are functional-analytic in nature. We assume that there is a function space $\mathcal{B} \subset L^\infty(Y)$ containing constant functions, with norm $\| \ \|$ satisfying $|v|_\infty \le \|v\|$ for $v \in \mathcal{B}$, such that for some constant $C > 0$:

(H1) For all $n \ge 1$, $R_n : \mathcal{B} \to \mathcal{B}$ is a bounded linear operator with $\|R_n\| \le C\mu(\varphi = n)$.

In particular, $z \mapsto R(z)$ is a continuous family of bounded linear operators on $\mathcal{B}$ for $z \in \bar{\mathbb{D}}$. Since $R(1) = R$ and $\mathcal{B}$ contains constant functions, 1 is an eigenvalue of $R(1)$. We require:

(H2) (i) The eigenvalue 1 is simple and isolated in the spectrum of $R(1)$.

(ii) For $z \in \bar{\mathbb{D}} \setminus \{1\}$, the spectrum of $R(z)$ does not contain 1.

Denote the spectral projection corresponding to the simple eigenvalue 1 for $R(1)$ by P. Then $(Pv)(y) \equiv \int_Y v\, d\mu$.

2.1 Asymptotics of T_n

We state our main results for the operators T_n. Since $T_n v = 1_Y L^n(1_Y v)$, we obtain precise results for the convergence of $L^n v$ on Y for observables v supported on Y. The restriction to Y is lifted in Section 10. Throughout, we assume regularly varying tails $\mu(\varphi > n) = \ell(n)n^{-\beta}$ and hypotheses (H1) and (H2). Set $d_\beta = \frac{1}{\pi}\sin\beta\pi$ for $\beta \in (0,1)$ and $d_1 = 1$. Define $m(n) = \ell(n)$ for $\beta \in (0,1)$ and $m(n) = \sum_{j=1}^n \ell(j)j^{-1}$ for $\beta = 1$.

In some of our statements, the observable v is not mentioned. Here, we are speaking of convergence of operators on the Banach space $\mathcal{B}$. So for example, Theorem 2.1 states that $\sup_{v\in\mathcal{B},\, \|v\|=1} \|m(n)n^{1-\beta}T_n v - d_\beta \int_Y v\, d\mu\| \to 0$ as $n \to \infty$. Since $\mathcal{B}$ is embedded in $L^\infty(Y)$, this immediately implies almost sure convergence (at a uniform rate) on Y. Redefining sequences on a set of measure zero, we obtain uniform convergence on Y. For brevity, we will speak of uniform convergence throughout this paper without further comment.

Theorem 2.1 *If $\beta \in (\frac{1}{2}, 1]$, then $\lim_{n\to\infty} m(n)n^{1-\beta}T_n = d_\beta P$.*

The next result gives upper bounds on the decay rate of T_n for $\beta \le \frac{1}{2}$, and an improved upper bound for $\beta \ge \frac{1}{2}$ when the observable is of mean zero.

Theorem 2.2 *(a) If $\beta = \frac{1}{2}$, then $T_n \ll \ell(n)n^{-\frac{1}{2}} \int_{1/n}^{\pi} \ell(1/\theta)^{-2}\theta^{-1}\, d\theta$.*

(b) If $\beta \in (0, \frac{1}{2})$, then $T_n \ll \ell(n)n^{-\beta}$.

(c) If $\beta \in (0,1)$, and $v \in \mathcal{B}$ satisfies $\int_Y v\, d\mu = 0$, then $T_n v \ll \ell(n)n^{-\beta}$.

As indicated in [17], the estimate in Theorem 2.2(b) is essentially optimal. However, we recover certain aspects of Theorem 2.1 even for $\beta \le \frac{1}{2}$. Recall that $E \subset \mathbb{N}$ has *density zero* if $\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^n 1_E(j) = 0$.

Theorem 2.3 *Let $\beta \in (0, \frac{1}{2}]$ and $v \in \mathcal{B}$.*

(a) For all $y \in Y$, there exists a set E of zero density such that $\lim_{n\to\infty,\, n\notin E} \ell(n)n^{1-\beta}(T_n v)(y) = d_\beta \int_Y v\, d\mu$.

(b) If $v \ge 0$, then $\liminf_{n\to\infty} \ell(n)n^{1-\beta}T_n v = d_\beta \int_Y v\, d\mu$ pointwise on Y.

Remark 2.4 In general, Theorem 2.1 fails for $\beta \le \frac{1}{2}$. However, there is the possibility of proving the result for all β under additional hypotheses. Indeed, Gouëzel [20] informs us that he is able to prove Theorem 2.1 for all $\beta \in (0, 1)$ under the additional assumption that $\mu(\varphi = n) = O(\ell(n)n^{-(\beta+1)})$. In particular, Gouëzel's result applies to the family (1.1).

It is worth recalling the situation from the scalar case (where the T_n are probabilities instead of operators). Under the additional assumption $\mu(\varphi = n) = O(\ell(n)n^{-(\beta+1)})$, Garsia & Lamperti [17] were able to extend Theorem 2.1 to the case $\beta \in (\frac{1}{4}, \frac{1}{2})$. This is the only part of [17] that we are unable to generalise to the operator setting. However, an argument of Doney [13] applies to all $\beta \in (0, 1)$ and according to Gouëzel [20] this argument can be extended to the operator case.

Remark 2.5 An immediate consequence of Theorem 2.1 is that Y is a Darling-Kac set whenever $\beta > \frac{1}{2}$. We refer to Aaronson [3, Chapter 3] and Aaronson, Denker & Urbanksi [6, Section 1] for definitions and numerous consequences. Other consequences include the Dynkin-Lamperti arcsine laws, see Thaler [39]. Indeed our main result significantly simplifies the derivation of the arcsine laws, see [40].

2.2 Preliminaries

For convenience, we state Karamata's Theorem on the integration of regularly varying sequences [7, 16].

Proposition 2.6 *Suppose that ℓ is slowly varying.*

(a) If $p > -1$, then $\sum_{j=1}^{n} \ell(j)j^p \sim \frac{1}{p+1}\ell(n)n^{p+1}$ as $n \to \infty$.

(b) The function $\tilde{\ell}(x) = \sum_{j=1}^{[x]} \ell(j)j^{-1}$ is slowly varying and $\lim_{n\to\infty} \ell(n)/\tilde{\ell}(n) = 0$. ∎

The following consequence of (H1) and regularly varying tails is standard.

Proposition 2.7 *There is a constant $C > 0$ such that $\|R(\rho e^{i(\theta+h)}) - R(\rho e^{i\theta})\| \le Cm(1/h)h^\beta$ and $\|R(\rho) - R(1)\| \le Cm(\frac{1}{1-\rho})(1-\rho)^\beta$ for all $\theta \in [0, 2\pi)$, $\rho \in (0, 1]$, $h > 0$.*

Proof We sketch the calculation for the first estimate. By Proposition 2.6,

$$R(\rho e^{i(\theta+h)}) - R(\rho e^{i\theta}) \ll h\sum_{j=1}^{N} j\mu(\varphi = j) + \sum_{j>N} \mu(\varphi = j) \ll hm(N)N^{1-\beta} + m(N)N^{-\beta},$$

so the result follows with $N = [h^{-1}]$. ∎

By (H2), there exists $\epsilon > 0$ such that $R(z)$ has a continuous family of simple eigenvalues $\lambda(z)$ for $z \in \bar{\mathbb{D}} \cap B_\epsilon(1)$ with $\lambda(1) = 1$. Let $P(z) : \mathcal{B} \to \mathcal{B}$ denote the corresponding family of spectral projections, with complementary projections $Q(z) = I - P(z)$ and $P(1) = P$. Also, let $v(z) \in \mathcal{B}$ denote the corresponding family of eigenfunctions normalised so that $\int_Y v(z)\, d\mu = 1$ for all z. In particular, $v(1) \equiv 1$.

Corollary 2.8 *The estimates for $R(z)$ in Proposition 2.7 are inherited by the families $P(z)$, $Q(z)$, $\lambda(z)$ and $v(z)$, where defined.*

Proof This is a standard consequence of perturbation theory for analytic families of operators [28]. ∎

We have defined the bounded linear operators $T_n, R_n : \mathcal{B} \to \mathcal{B}$ given by

$$T_n v = 1_Y L^n(1_Y v), \ \ n \geq 0, \qquad R_n v = 1_Y L^n(1_{\{\varphi=n\}} v) = R(1_{\{\varphi=n\}} v), \ \ n \geq 1.$$

The power series

$$T(z) = \sum_{n=0}^{\infty} T_n z^n, \ \ z \in \mathbb{D}, \qquad R(z) = \sum_{n=1}^{\infty} R_n z^n, \ \ z \in \bar{\mathbb{D}},$$

are analytic on the open unit disk $\mathbb{D}$, and $R(z)$ is continuous on $\bar{\mathbb{D}}$ by (H1). We have the usual relation $T_n = \sum_{j=1}^{n} T_{n-j} R_j$ for $n \geq 1$, and it follows that $T(z) = I + T(z)R(z)$ on $\mathbb{D}$. Hence $T(z) = (I - R(z))^{-1}$ on $\mathbb{D}$. By (H2)(ii), $T(z)$ extends continuously to $\bar{\mathbb{D}} - \{1\}$ via the formula $T(z) = (I - R(z))^{-1}$.

Proposition 2.9 *There exists $\epsilon, C > 0$ such that $\|T(z) - (1 - \lambda(z))^{-1} P(z)\| \leq C$ for $z \in \bar{\mathbb{D}} \cap B_\epsilon(1)$, $z \neq 1$, and $\|T(z)\| \leq C$ for $z \in \bar{\mathbb{D}} \setminus B_\epsilon(1)$.*

Proof Choose $\epsilon > 0$ so that the family of simple eigenvalues $\lambda(z)$ is defined on $\bar{\mathbb{D}} \cap B_\epsilon(1)$. For $z \in \bar{\mathbb{D}} \cap B_\epsilon(1)$, we can write $R(z) = \lambda(z)P(z) + R(z)Q(z)$. If in addition $z \neq 1$, then we have (in an obvious notation)

$$T(z) = (1 - \lambda(z))^{-1} P(z) + (I - R(z))^{-1} Q(z).$$

By (H2), the second term is uniformly bounded in the operator norm, and $T(z)$ is uniformly bounded for $z \in \bar{\mathbb{D}} \setminus B_\epsilon(1)$. ∎

2.3 Strategy of the proof of Theorem 2.1

Our aim is to compute the operators T_n defined above. Most of the analysis is carried out on the unit circle S^1, so it is convenient to abuse notation, writing $T(\theta)$ instead of $T(e^{i\theta})$ and so on. For $\beta \in (\frac{1}{2}, 1)$, our treatment follows Garsia & Lamperti [17] but there are some significant differences in two of the three steps.

The first step, Section 3, is to study the singularity for $T(\theta)$ at $\theta = 0$. The argument in [17] is scalar and similar results can be found in [15, 22]. In our situation, the key is to use the fact that the return time φ lies in the domain of a stable law, and a Nagaev-type argument due to Aaronson & Denker [4] shows that $(1-\lambda(\theta))^{-1} \sim \mathrm{const}\,\ell(1/\theta)^{-1}\theta^{-\beta}$ when $\beta \in (0,1)$. By Corollary 2.8 and Proposition 2.9, $T(\theta) \sim \mathrm{const}\,\ell(1/\theta)^{-1}\theta^{-\beta}P$.

In particular $T(\theta) \in L^1$ with Fourier coefficients $\widehat{T}_n$. The second step is to relate T_n to $\widehat{T}_n$. In the scalar case, [17] invokes ideas of Herglotz [24] on analytic functions with positive real part. A different approach is required here since we are working with operators. In Section 4, we verify that $T_n = \widehat{T}_n$.

In the final step, Section 5, we investigate the behaviour of $\widehat{T}_n$ as $n \to \infty$, directly following [17].

2.4 Tower extensions

The following tower construction will be required in Sections 7 and 10.

Starting from the first return map $F = f^\varphi : Y \to Y$, there is a standard way of constructing an extension $f_\Delta : \Delta \to \Delta$ of the underlying map $f : X \to X$. Define the *tower* $\Delta = \{(y,j) \in Y \times \mathbb{Z} : 0 \le j < \varphi(y)\}$ and the *tower map* $f_\Delta : \Delta \to \Delta$ given by $f_\Delta(y,j) = (y, j+1)$ for $j \le \varphi(y) - 2$ and $f_\Delta(y,j) = (Fy, 0)$ for $j = \varphi(y) - 1$.

The base of the tower $\{(y,0) : y \in Y\}$ is naturally identified with Y and so we may regard Y as a subset of both X and Δ. Let μ_Δ be the unique f_Δ-invariant measure on Δ that agrees with the underlying measure μ on the common subset Y.

Define the projection $\pi : \Delta \to X$, $\pi(y,j) = f^j y$. Then $\pi f_\Delta = f\pi$ and $\pi_*\mu_\Delta = \mu$. Thus f_Δ is an extension of f with the same first return map $F : Y \to Y$ and return time function $\varphi : Y \to \mathbb{Z}^+$ as the original map.

3 Asymptotics of $T(\theta)$

In this section, we obtain an asymptotic expression for $T(\theta)$ as $\theta \to 0^+$. Throughout, $\beta \in (0,1)$. The main part of the analysis is to understand the asymptotics of the leading eigenvalue $\lambda(\theta)$. In certain situations, we obtain a higher order expansion. Let $c_\beta = -i \int_0^\infty e^{i\sigma}\sigma^{-\beta}\, d\sigma$.

Lemma 3.1 *Let $\beta \in (0,1)$. As $\theta \to 0^+$,*

(a) $\lambda(\theta) = 1 - c_\beta \ell(1/\theta)\theta^\beta(1 + o(1))$.

(b) $T(\theta) - (1-\lambda(\theta))^{-1}P = O(1)$.

(c) $T(\theta) = c_\beta^{-1}\ell(1/\theta)^{-1}\theta^{-\beta}(1 + o(1))P + O(1)$.

Proof (a) This is part of [4, Theorem 5.1]. (The main ideas of the proof are reproduced later in the proof of Lemma 3.2 and Lemma 4.1.)

(b) By Proposition 2.9,

$$T(\theta) = (1-\lambda(\theta))^{-1}P(\theta) + O(1) = (1-\lambda(\theta))^{-1}P + (1-\lambda(\theta))^{-1}(P(\theta)-P) + O(1).$$

By Corollary 2.8, $P(\theta) - P \ll \ell(1/\theta)\theta^\beta$, so the result follows from (a).
(c) is immediate by (a) and (b). ∎

The following expansion for $\lambda(\theta)$ will be used in proving results on second order asymptotics (Section 9).

Lemma 3.2 *Suppose that $\mu(\varphi > n) = c(n^{-\beta} + H(n))$, where $c > 0$ and $H(n) = O(n^{-q})$, $q > 1$. Let $H_1(x) = [x]^{-\beta} - x^{-\beta} + H([x])$ and set $c_H = \int_0^\infty H_1(x)\,dx$. Then*

$$\lambda(\theta) = 1 - cc_\beta\theta^\beta + icc_H\theta + O(\theta^{2\beta}, \theta^q), \quad \textit{as } \theta \to 0^+.$$

Proof We follow [4, Theorem 5.1]. Recall that $v(\theta)$ is the eigenfunction corresponding to $\lambda(\theta)$ normalised so that $\int v(\theta)\,d\mu = 1$. Since $v(0) \equiv 1$, it follows from Corollary 2.8 that $|v(\theta) - 1|_\infty \ll \theta^\beta$. Let $\mathcal{F}_0$ denote the σ-algebra generated by φ. Define the step function $\hat{v}(\theta) : [0,\infty) \to \mathbb{C}$ given by $\hat{v}(\theta) \circ \varphi = E(v(\theta)|\mathcal{F}_0)$ and the distribution function $G(x) = \mu(\varphi \le x)$. Then

$$\begin{aligned} \lambda(\theta) &= \textstyle\int_Y R(\theta)v(\theta)\,d\mu = \int_Y e^{i\theta\varphi}v(\theta)\,d\mu = 1 + \int_Y (e^{i\theta\varphi} - 1)v(\theta)\,d\mu \\ &= 1 + \textstyle\int_0^\infty (e^{i\theta x} - 1)\hat{v}_\theta(x)\,dG(x), \end{aligned}$$

where $\sup_{x\ge 0}|\hat{v}_\theta(x) - 1| \ll \theta^\beta$ and $1 - G(x) = c(x^{-\beta} + H_1(x))$. Here $H_1(x) = O(x^{-q'})$ as $x \to \infty$, where $q' = \min\{q, \beta+1\} > 1$. In particular, H_1 is integrable.

Write $\hat{v}_\theta = 1 + v_\theta^1 - v_\theta^2 + iv_\theta^3 - iv_\theta^4$, where $v_\theta^s \ge 0$ and $\sup_x |v_\theta^s(x)| \ll \theta^\beta$ for $s = 1,2,3,4$. Define the positive measure $dG_\theta^s = v_\theta^s\,dG$. Then

$$\begin{aligned} \lambda(\theta) &= 1 + \int_0^\infty (e^{i\theta x} - 1)\,dG(x) + \sum_{s=1}^4 q_s \int_0^\infty (e^{i\theta x} - 1)\,dG_\theta^s(x) \\ &= 1 + i\theta\int_0^\infty e^{i\theta x}(1 - G(x))\,dx + \sum_{s=1}^4 q_s i\theta \int_0^\infty e^{i\theta x} g_\theta^s(x)(1 - G(x))\,dx, \end{aligned}$$

where $q_1 = 1$, $q_2 = -1$, $q_3 = i$, $q_4 = -i$, $g_\theta^s(x) = \int_x^\infty v_\theta^s(u)\,dG(u)/\int_x^\infty dG(u) \ll \theta^\beta$.

Now,

$$\begin{aligned} i\theta\int_0^\infty e^{i\theta x}(1 - G(x))\,dx &= ic\theta\int_0^\infty e^{i\theta x}x^{-\beta}\,dx + ic\theta\int_0^\infty e^{i\theta x}H_1(x)\,dx \\ &= ic\theta^\beta\int_0^\infty e^{i\sigma}\sigma^{-\beta}\,d\sigma + ic\theta\int_0^\infty H_1(x)\,dx + ic\theta\int_0^\infty (e^{i\theta x} - 1)H_1(x)\,dx \\ &= -cc_\beta\theta^\beta + icc_H\theta + ic\theta A \end{aligned}$$

where

$$A = \int_0^\infty (e^{i\theta x} - 1) H_1(x)\, dx = \int_0^{1/\theta} (e^{i\theta x} - 1) H_1(x)\, dx + \int_{1/\theta}^\infty (e^{i\theta x} - 1) H_1(x)\, dx$$
$$\ll \theta \int_0^{1/\theta} x H_1(x)\, dx + \int_{1/\theta}^\infty H_1(x)\, dx \ll \theta \int_0^{1/\theta} x^{1-q'}\, dx + \int_{1/\theta}^\infty x^{-q'}\, dx \ll \theta^{q'-1}.$$

It remains to estimate the terms $\int_0^\infty e^{i\theta x} g_\theta^s(x)(1 - G(x))\, dx$. We give the details for $\int_0^\infty \sin\theta x\, g_\theta^s(x)(1 - G(x))\, dx$; the case with sin replaced by cos is identical. Since $x \mapsto g_\theta^s(x)(1 - G(x))$ is decreasing for each fixed θ, we can write

$$\int_0^\infty \sin\theta x\, g_\theta^s(x)(1 - G(x))\, dx \le \int_0^{\pi/\theta} \sin\theta x\, g_\theta^s(x)(1 - G(x))\, dx$$
$$\ll \theta^\beta \int_0^{\pi/\theta} x^{-\beta}\, dx \ll \theta^{2\beta-1},$$

giving the required upper bound, and the lower bound is obtained in the same way. ∎

4 Identification of the Fourier coefficients

Let $\widehat{R}_n$ and $\widehat{T}_n$ denote the Fourier coefficients of $R(\theta)$ and $T(\theta)$. By (H1), R is uniformly absolutely summable on S^1. Therefore $\widehat{R}_n = R_n$. In this section, we verify that $\widehat{T}_n = T_n$ for all $n \ge 0$. Throughout, $\beta \in (0, 1)$.

Lemma 4.1 *There exists $\epsilon, C > 0$ such that $|1 - \lambda(z)| \ge C\ell(1/\theta)\theta^\beta$ for all $z = \rho e^{i\theta} \in \bar{\mathbb{D}} \cap B_\epsilon(1)$.*

Proof We start off by mimicking the proof of Lemma 3.2. Consider functions $v_z : [0, \infty) \to [0, \infty)$ satisfying either (i) $v_z \equiv 1$ or (ii) $|v_z|_\infty = o(1)$ as $z \to 1$. Write $z = e^{-u+i\theta}$, $0 \le u, \theta < \epsilon$. Then the expansion $\lambda(z) - 1$ leads to a linear combination of five integrals of the form

$$I = \int_0^\infty (e^{(-u+i\theta)x} - 1) v_z(x)\, dG(x) = (-u + i\theta) \int_0^\infty e^{(-u+i\theta)x} g_z(x)(1 - G(x)) dx, \tag{4.1}$$

where $g_z \ge 0$ and either (i) $g_z \equiv 1$ or (ii) $|g_z|_\infty = o(1)$ as $z \to 1$. Moreover there is one integral of type (i) and we show that this satisfies the desired lower bound, whilst the four integrals of type (ii) are negligible.

Recall that $1 - G(x) = \mu(\varphi > x) = \ell([x])[x]^{-\beta} = x^{-\beta} h(x)$ where $h(x) = \ell(x)(1 + o(1))$. Substituting $\sigma = \theta x$,

$$I = (-u + i\theta)\ell(1/\theta)\theta^{\beta-1} \int_0^\infty e^{-\sigma y} e^{i\sigma} g_z(\sigma/\theta)\sigma^{-\beta} h(\sigma/\theta)\ell(1/\theta)^{-1}\, d\sigma,$$

where $y = u/\theta$. We estimate the oscillatory integrals in the same way that alternating series are estimated in Leibnitz's theorem, making extensive use of the fact that $\sigma \mapsto e^{-\sigma y} g_z(\sigma/\theta)\sigma^{-\beta} h(\sigma/\theta)$ is decreasing for each fixed θ, y.

Write $|I| = (u^2 + \theta^2)^{1/2} \ell(1/\theta)\theta^{\beta-1} A$, where in cases (i) and (ii) respectively,

$$A_{(i)} \geq \int_0^{3\pi/2} \cos\sigma\, e^{-\sigma y} \sigma^{-\beta} h(\sigma/\theta)\ell(1/\theta)^{-1}\, d\sigma,$$

$$A_{(ii)} \leq \Big(\int_0^{\pi/2} \cos\sigma + \int_0^{\pi} \sin\sigma\Big) e^{-\sigma y}\sigma^{-\beta} g_z(\sigma/\theta) h(\sigma/\theta)\ell(1/\theta)^{-1}\, d\sigma.$$

We divide the region $y > 0$ into the regions $y \in (0, 1/\delta]$ and $y \geq 1/\delta$ where δ is chosen sufficiently small. We have the Potter's bounds [34], [7, Theorem 1.5.6]: $C^{-1}\sigma^{\beta} \leq h(\sigma/\theta)\ell(1/\theta)^{-1} \leq C\sigma^{-1}$ uniformly in $\theta > 0$, $\sigma \in (0, 2\pi]$, where C is a constant. Hence

$$\begin{aligned} A_{(i)} &\geq \frac{\sqrt{2}}{2} \int_0^{\pi/4} e^{-\sigma y}\sigma^{-\beta} h(\sigma/\theta)\ell(1/\theta)^{-1}\, d\sigma + O\Big(\int_{\pi/4}^{3\pi/2} e^{-\sigma y}\, d\sigma\Big) \\ &= \frac{\sqrt{2}}{2} \int_0^{\pi/4} e^{-\sigma y}\sigma^{-\beta} h(\sigma/\theta)\ell(1/\theta)^{-1}\, d\sigma + O(y^{-1} e^{-(\pi/4)y}), \end{aligned}$$

$$\begin{aligned} A_{(ii)} &\leq 2|g_z|_\infty \int_0^{\pi/4} e^{-\sigma y}\sigma^{-\beta} h(\sigma/\theta)\ell(1/\theta)^{-1}\, d\sigma + O\Big(\int_{\pi/4}^{\pi} e^{-\sigma y}\, d\sigma\Big) \\ &\ll |g_z|_\infty A_{(i)} + O(y^{-1} e^{-(\pi/4)y}). \end{aligned}$$

Since $|g_z|_\infty = o(1)$, we can choose ϵ sufficiently small that

$$|1 - \lambda(z)| \gg u\ell(1/\theta)\theta^{\beta-1} \Big\{ \int_0^{\pi/4} e^{-\sigma y}\sigma^{-\beta} h(\sigma/\theta)\ell(1/\theta)^{-1}\, d\sigma + O(y^{-1} e^{-(\pi/4)y}) \Big\}.$$

Furthermore,

$$\int_0^{\pi/4} e^{-\sigma y}\sigma^{-\beta} h(\sigma/\theta)\ell(1/\theta)^{-1}\, d\sigma \gg \int_0^{\pi/4} e^{-\sigma y}\, d\sigma = y^{-1} + O(y^{-1} e^{-(\pi/4)y}).$$

For $y \geq 1/\delta$ with δ sufficiently small, the terms $O(y^{-1}e^{-(\pi/4)y})$ are negligible so that $1 - \lambda(z) \gg u\ell(1/\theta)\theta^{\beta-1} y^{-1} = \ell(1/\theta)\theta^{\beta}$.

It remains to consider the complementary region $y \in (0, 1/\delta]$. Note that $\int_0^{3\pi/2} e^{-\sigma y} \cos\sigma\, \sigma^{-\beta}\, d\sigma$ depends continuously on y and is positive for all $y \geq 0$. It follows by compactness that $\int_0^{3\pi/2} e^{-\sigma y} \cos\sigma\, \sigma^{-\beta}\, d\sigma$ is bounded away from zero for $y \in [0, 1/\delta]$. Moreover there exists $b \in (0, 3\pi/2)$ such that $\int_b^{3\pi/2} e^{-\sigma y} \cos\sigma\, \sigma^{-\beta}\, d\sigma$ is bounded away from zero for $y \in [0, 1/\delta]$. By uniform convergence of slowly varying functions [7, Theorem 1.2.1], we can shrink ϵ if necessary so that $|h(\sigma/\theta)/\ell(1/\theta) - 1|$

is as small as desired, uniformly in $\sigma \in [b, 3\pi/2]$ and $\theta \in (0, \epsilon]$. Hence $A_{(i)} \geq \int_0^{3\pi/2} \cos\sigma\, e^{-\sigma y}\sigma^{-\beta} h(\sigma/\theta)\ell(1/\theta)^{-1}\, d\sigma$ which is bounded away from zero for $y \in (0, 1/\delta]$.

Also, by Potter's bounds for any $\beta' \in (\beta, 1)$,

$$A_{(ii)} \leq 2|g_z|_\infty \int_0^\pi e^{-\sigma y}\sigma^{-\beta} h(\sigma/\theta)\ell(1/\theta)^{-1}\, d\sigma \ll |g_z|_\infty \int_0^\pi \sigma^{-\beta'}\, d\sigma \ll |g_z|_\infty = o(1).$$

Hence $A_{(ii)}$ is negligible relative to $A_{(i)}$ and we obtain $1 - \lambda(z) \gg \theta\ell(1/\theta)\theta^{\beta-1} = \ell(1/\theta)\theta^\beta$ completing the proof. ∎

It is convenient in the next result (and crucial in Section 6) to discuss the real part of an operator. We recall that the operators T_n are defined on the real Banach space $\mathcal{B}$. Passing to the complexification, there is a natural conjugation $u+iv \mapsto u-iv$ on $\mathcal{B}$. Given an operator $A : \mathcal{B} \to \mathcal{B}$, define the conjugate $\bar{A} : \mathcal{B} \to \mathcal{B}$ by setting $\bar{A}v = \overline{A\bar{v}}$, and the real part $\operatorname{Re} A = \frac{1}{2}(A + \bar{A})$. In the case of the operator $T(z) = \sum_{n=0}^\infty T_n z^n$, this coincides with the definitions $\overline{T(z)} = T(\bar{z})$ and $\operatorname{Re} T(z) = \sum_{n=0}^\infty T_n \operatorname{Re}(z^n)$.

Corollary 4.2 *$T_n = \widehat{T}_n = \frac{1}{\pi} \operatorname{Re} \int_0^\pi T(e^{i\theta})e^{-in\theta}\, d\theta$ for all $n \geq 0$.*

Proof Since $T(z) = \sum_{j=0}^\infty T_j z^j$ is analytic on the open unit disk $\mathbb{D}$, $T_n = \frac{1}{2\pi}\rho^{-n} \int_0^{2\pi} T(\rho e^{i\theta})e^{-in\theta}\, d\theta$, for all $\rho \in (0, 1)$.

By Proposition 2.9, $T(z) = O(1)$ on $\bar{\mathbb{D}} \setminus B_\epsilon(1)$. Further, on $\bar{\mathbb{D}} \cap B_\epsilon(1)$, $T(z) = (1 - \lambda(z))^{-1}P(z) + O(1) \ll (1 - \lambda(z))^{-1} + O(1)$. By Lemma 4.1, $\|T(\rho e^{i\theta})\| \ll \ell(1/\theta)^{-1}\theta^{-\beta}$ for $z = \rho e^{i\theta} \in \bar{\mathbb{D}}$ uniformly in ρ. Since $\ell(1/\theta)^{-1}\theta^{-\beta}$ is integrable, it follows from the dominated convergence theorem (as $\rho \to 1$) that $T_n = \frac{1}{2\pi}\int_0^{2\pi} T(e^{i\theta})e^{-in\theta}\, d\theta = \widehat{T}_n$. Since $T(\bar{z}) = \overline{T(z)}$, we obtain the expression $\frac{1}{\pi} \operatorname{Re} \int_0^\pi T(e^{i\theta})e^{-in\theta}\, d\theta$. ∎

5 Convergence for $\beta \in (\frac{1}{2}, 1)$

In this section, we prove Theorem 2.1 for $\beta \in (\frac{1}{2}, 1)$.

Lemma 5.1 *Let $\beta \in (\frac{1}{2}, 1)$. Let $n \geq 1$, $a \in [1, n]$. Then for any $\beta' \in (0, \beta)$,*

$$\ell(n)n^{1-\beta} \int_{a/n}^\pi T(\theta)e^{-in\theta}\, d\theta \ll a^{-(2\beta'-1)}.$$

If ℓ is asymptotically constant, then the result holds with $\beta' = \beta$.

Proof By Lemma 3.1(c), we have the estimate $\|T(\theta)\| \ll \ell(1/\theta)^{-1}\theta^{-\beta}$. The proof uses this fact together with Proposition 2.7, and follows Garsia & Lamperti [17, p. 231]. We give the details partly for completeness and partly because we want to make explicit certain estimates that will be used in Section 9.

First, write

$$I = \int_{a/n}^{\pi} T(\theta)e^{-in\theta}\,d\theta = -\int_{(a+\pi)/n}^{\pi+\pi/n} T(\theta-\pi/n)e^{-in\theta}\,d\theta,$$

so

$$2I = \int_{a/n}^{\pi} T(\theta)e^{-in\theta}\,d\theta - \int_{(a+\pi)/n}^{\pi+\pi/n} T(\theta-\pi/n)e^{-in\theta}\,d\theta = I_1 + I_2 + I_3,$$

where

$$I_1 = \int_{\pi}^{\pi+\pi/n} T(\theta-\pi/n)e^{-in\theta}\,d\theta, \qquad I_2 = \int_{a/n}^{(a+\pi)/n} T(\theta-\pi/n)e^{-in\theta}\,d\theta,$$

$$I_3 = \int_{(a+\pi)/n}^{\pi} \{T(\theta) - T(\theta-\pi/n)\}e^{-in\theta}\,d\theta.$$

Clearly, $I_1 \ll 1/n$, while

$$\begin{aligned} I_2 &\ll \int_{a/n}^{(a+\pi)/n} \ell(1/\theta)^{-1}\theta^{-\beta}\,d\theta \ll \ell(n)^{-1}n^{-(1-\beta)}\int_a^{a+\pi}[\ell(n)/\ell(n/\sigma)]\sigma^{-\beta}\,d\sigma \\ &\ll \ell(n)^{-1}n^{-(1-\beta)}\int_a^{a+\pi}\sigma^{-\beta'}\,d\sigma = \ell(n)^{-1}n^{-(1-\beta)}a^{1-\beta'}\{(1+\pi/a)^{1-\beta'}-1\} \\ &\ll \ell(n)^{-1}n^{-(1-\beta)}a^{-\beta'}. \end{aligned}$$

By the resolvent identity and Proposition 2.7 (with $m(x) = \ell(x)$),

$$\begin{aligned} I_3 &\ll \int_{(a+\pi)/n}^{\pi} \|T(\theta)\|\|T(\theta-\pi/n)\|\|R(\theta)-R(\theta-\pi/n)\|\,d\theta \\ &\ll \ell(n/\pi)n^{-\beta}\int_{(a+\pi)/n}^{\pi}\ell(1/\theta)^{-1}\ell(1/(\theta-\pi/n))^{-1}\theta^{-\beta}(\theta-\pi/n)^{-\beta}\,d\theta \\ &= \ell(n/\pi)n^{-\beta}\int_{a/n}^{\pi-\pi/n}\ell(1/(\theta+\pi/n))^{-1}\ell(1/\theta)^{-1}(\theta+\pi/n)^{-\beta}\theta^{-\beta}\,d\theta. \end{aligned}$$

By Potter's bounds, $\ell(1/(\theta+\pi/n))^{-1} \ll \ell(1/\theta)^{-1}$ for $n\theta \ge 1$. Hence,

$$\begin{aligned} I_3 &\ll \ell(n)n^{-\beta}\int_{a/n}^{\pi}\ell(1/\theta)^{-2}\theta^{-2\beta}\,d\theta = \ell(n)^{-1}n^{-(1-\beta)}\int_a^{n\pi}[\ell(n)/\ell(n/\sigma)]^2\sigma^{-2\beta}\,d\sigma \\ &\ll \ell(n)^{-1}n^{-(1-\beta)}\int_a^{n\pi}\sigma^{-2\beta'}\,d\sigma \ll \ell(n)^{-1}n^{-(1-\beta)}a^{-(2\beta'-1)}. \end{aligned}$$

Altogether, we obtain $\ell(n)n^{1-\beta}I \ll n^{-\beta} + a^{-\beta'} + a^{-(2\beta'-1)} \ll a^{-(2\beta'-1)}$ as required. ∎

Lemma 5.2 *Let $\beta \in (0,1)$. Let $n \geq 1$, $a \in (0, \epsilon n)$. Then*

$$\lim_{a\to\infty}\lim_{n\to\infty} \ell(n)n^{1-\beta} \int_0^{a/n} (1-\lambda(\theta))^{-1} e^{-in\theta}\, d\theta = d'_\beta,$$

where $d'_\beta = i\int_0^\infty e^{-i\sigma}\sigma^{-\beta}\, d\sigma / \int_0^\infty e^{i\sigma}\sigma^{-\beta}\, d\sigma$.

Proof This is identical to [17, Lemma 3.4.1] and we give the proof only for completeness. By Lemma 3.1, we can write $(1-\lambda(\theta))^{-1} = c_\beta^{-1}\ell(1/\theta)^{-1}\theta^{-\beta}h(\theta)$ where $c_\beta = -i\int_0^\infty e^{i\sigma}\sigma^{-\beta}\, d\sigma$ and $\lim_{\theta\to 0} h(\theta) = 1$. Hence

$$\begin{aligned}\int_0^{a/n} (1-\lambda(\theta))^{-1} e^{-in\theta}\, d\theta &= n^{-1}\int_0^a (1-\lambda(\sigma/n))^{-1} e^{-i\sigma}\, d\sigma \\ &= c_\beta^{-1} n^{-(1-\beta)} \int_0^a e^{-i\sigma}\sigma^{-\beta}\ell(n/\sigma)^{-1} h(\sigma/n)\, d\sigma,\end{aligned}$$

so that

$$\ell(n)n^{1-\beta}\int_0^{a/n} (1-\lambda(\theta))^{-1} e^{-in\theta}\, d\theta = c_\beta^{-1}\int_0^a e^{-i\sigma}\sigma^{-\beta}[\ell(n/\ell(n/\sigma)]h(\sigma/n)\, d\sigma.$$

For fixed a, it follows from the dominated convergence theorem that

$$\lim_{n\to\infty} \ell(n)n^{1-\beta}\int_0^{a/n} (1-\lambda(\theta))^{-1} e^{-in\theta}\, d\theta = c_\beta^{-1}\int_0^a e^{-i\sigma}\sigma^{-\beta}\, d\sigma,$$

and the result follows. ∎

Proof of Theorem 2.1, $\beta \in (\frac{1}{2}, 1)$. By Section 4, $T_n = \frac{1}{\pi}\operatorname{Re}\int_0^\pi T(\theta)e^{-in\theta}\, d\theta$. Let

$$\begin{aligned}D(a,n) &= \int_0^\pi T(\theta)e^{-in\theta}\, d\theta - \int_0^{a/n} (1-\lambda(\theta))^{-1} P e^{-in\theta}\, d\theta \\ &= \int_0^{a/n} \big\{T(\theta) - (1-\lambda(\theta))^{-1}P\big\} e^{-in\theta}\, d\theta + \int_{a/n}^\pi T(\theta)Pe^{-in\theta}\, d\theta,\end{aligned}$$

so $D(a,n) \ll a/n + \ell(n)^{-1}n^{-(1-\beta)}a^{-(2\beta'-1)}$ by Lemma 3.1(b) and Lemma 5.1. Hence $\lim_{a\to\infty}\lim_{n\to\infty}\ell(n)n^{1-\beta}D(a,n) = 0$. By Lemma 5.2, $\lim_{a\to\infty}\lim_{n\to\infty}\ell(n)n^{1-\beta}T_n = \frac{1}{\pi}\operatorname{Re} d'_\beta = d_\beta$. The result follows since T_n is independent of a. ∎

6 Convergence for $\beta = 1$

In this section, we prove Theorem 2.1 in the case $\beta = 1$. There are several differences from the case $\beta \in (\frac{1}{2}, 1)$. First, $T(e^{i\theta}) \notin L^1$; instead it is shown below that $\operatorname{Re} T(e^{i\theta})$ is integrable.

Estimating $\mathrm{Re}\{(1-\lambda(z))^{-1}\}$ is slightly easier than in Section 4 but estimating $\mathrm{Re}\, T(z)$ is harder since $\mathrm{Re}\{(1-\lambda(z))^{-1}(P(z)-P)\}$ is not dominated by $\mathrm{Re}\{(1-\lambda(z))^{-1}\}$. As a consequence, $\mathrm{Re}\, T(z) = \mathrm{Re}\, T(\rho e^{i\theta})$ is not dominated by a single integrable function of θ, see Lemma 6.4 below.

We have $\mu(\varphi > n) = \ell(n)n^{-1}$, where ℓ is slowly varying and $\sum \ell(n)n^{-1} = \infty$. Let $\tilde{\ell}(x) = m(x) = \sum_{j=1}^{[x]} \ell(j)j^{-1}$. Then $\tilde{\ell}$ is monotone increasing and $\lim_{n\to\infty} \tilde{\ell}(n) = \infty$. By Proposition 2.6(b), $\tilde{\ell}$ is slowly varying and $\ell(n)/\tilde{\ell}(n) \to 0$ as $n \to \infty$. Up to asymptotic equivalence, we have the alternative definitions $\tilde{\ell}(x) = \int_1^x \ell(y)y^{-1}\, dy$ and $\tilde{\ell}(x) = \int_0^x (1-G(y))\, dy$ where $G(x) = \mu(\varphi \le x)$.

Proposition 6.1 $\displaystyle\int_0^{1/y} \frac{\ell(1/\theta)}{\theta(\tilde{\ell}(1/\theta))^2}\, d\theta = \int_y^\infty \frac{\ell(x)}{x(\tilde{\ell}(x))^2}\, dx \sim \frac{1}{\tilde{\ell}(y)}.$

Proof Note that $-(\tilde{\ell}(x))^{-1}$ is an antiderivative of $\ell(x)x^{-1}(\tilde{\ell}(x))^{-2}$. ∎

6.1 Identification of Fourier coefficients

Write $z = e^{-u+i\theta}$, $u \in [0,1]$, $\theta \in [0,\pi]$. Given a function $g_\theta(x) \ge 0$ with $|g_\theta|_\infty \le C\theta^{1-\epsilon}$ for constants $C > 0$, $\epsilon \in (0,1)$, such that $x \to g_\theta(x)(1-G(x))$ is decreasing for each fixed θ, define

$$J_C = \int_0^\infty e^{-ux} \cos\theta x\, g_\theta(x)(1-G(x))\, dx, \quad J_S = \int_0^\infty e^{-ux} \sin\theta x\, g_\theta(x)(1-G(x))\, dx.$$

Let I_C and I_S be the corresponding integrals in the case $g_\theta \equiv 1$.

Proposition 6.2 *As $u, \theta \to 0^+$,*

$$\begin{aligned}
&|I_S| \ll \theta u^{-1}\ell(1/u), && I_C = \tilde{\ell}(1/u)(1+o(1)) + O(\theta u^{-1}\ell(1/u)),\\
&|I_S| \ll \ell(1/\theta), && I_C = \tilde{\ell}(1/\theta)(1+o(1)) + O(u\theta^{-1}\ell(1/\theta)),\\
&|J_S| \ll \theta^{2-\epsilon}u^{-1}\ell(1/u), && |J_C| \ll \theta^{2-\epsilon}u^{-1}\ell(1/u) + \theta^{1-\epsilon}\tilde{\ell}(1/u),\\
&|J_S| \ll \theta^{1-2\epsilon}, && |J_C| \ll \theta^{1-2\epsilon} + u\theta^{-2\epsilon}.
\end{aligned}$$

Proof First,

$$\begin{aligned}
|J_S| &\ll \theta|g_\theta|_\infty \int_0^\infty e^{-ux}\Big|\frac{\sin\theta x}{\theta x}\Big|\ell(x)\, dx \ll \theta|g_\theta|_\infty \int_0^\infty e^{-ux}\ell(x)\, dx\\
&= \theta|g_\theta|_\infty u^{-1}\ell(1/u)\int_0^\infty e^{-\sigma}\frac{\ell(\sigma/u)}{\ell(1/u)}\, d\sigma \ll \theta|g_\theta|_\infty u^{-1}\ell(1/u).
\end{aligned}$$

This gives the first estimate for J_S and taking $g_\theta = 1$ we obtain the first estimate for I_S. Alternatively, we make the substitution $\sigma = \theta x$. Using the oscillation of $\sin\sigma$ and the fact that $\sigma \mapsto e^{-u\sigma/\theta} g_\theta(\sigma/\theta)(1 - G(\sigma/\theta))$ is decreasing,

$$\begin{aligned} 0 \le J_S = \theta^{-1} \int_0^\infty e^{-u\sigma/\theta} \sin\sigma\, g_\theta(\sigma/\theta)(1 - G(\sigma/\theta))\, d\sigma \\ \le \theta^{-1} \int_0^\pi e^{-u\sigma/\theta} \sin\sigma\, g_\theta(\sigma/\theta)(1 - G(\sigma/\theta))\, d\sigma, \end{aligned}$$

and so $|J_S| \le |g_\theta|_\infty \int_0^\pi \sin\sigma\, \ell(\sigma/\theta)\sigma^{-1}\, d\sigma \ll |g_\theta|_\infty \ell(1/\theta)$, yielding the remaining estimates for I_S and J_S.

In the estimates for J_C and I_C, we use the fact that $\int_0^x (1 - G(x))\, dx = \tilde{\ell}(x)(1 + o(1))$. Note that

$$\Big| \int_{1/u}^\infty e^{-ux} \cos\theta x\, g_\theta(x)(1 - G(x))\, dx \Big| \le |g_\theta|_\infty \int_1^\infty e^{-\sigma} \ell(\sigma/u)\sigma^{-1}\, d\sigma \ll |g_\theta|_\infty \ell(1/u).$$

For the integral over $[0, 1/u]$, write $e^{-ux} \cos\theta x = \{e^{-ux}(\cos\theta x - 1)\} + \{e^{-ux} - 1\} + 1$. This yields three integrals, the first of which is estimated by $|g_\theta|_\infty \theta u^{-1} \ell(1/u)$ (like J_S) and the second by $|g_\theta|_\infty \ell(1/u)$. This leaves $\int_0^{1/u} g_\theta(x)(1 - G(x))\, dx \ll |g_\theta|_\infty \int_0^{1/u} (1 - G(x))\, dx = |g_\theta|_\infty \tilde{\ell}(1/u)(1 + o(1))$ completing the first estimate for J_C. Setting $g_\theta = 1$ yields the first asymptotic expression for I_C. The remaining estimate for J_C is obtained by splitting the range of integration into $[0, 1/\theta]$ and $[1/\theta, \infty)$ and combining the above arguments for J_C (first estimate) and J_S (second estimate). Again the final expression for I_C follows by setting $g_\theta = 1$. ∎

Corollary 6.3 *Let $z = e^{-u+i\theta} \in B_\epsilon(1)$, ϵ sufficiently small, $u > 0$, $\theta \ge 0$. Then*

$$\begin{aligned} |1 - \lambda(e^{-u+i\theta})|^{-1} &\ll \frac{1}{u\tilde{\ell}(1/u)}, \quad \text{for } \theta \in [0, u], \\ |1 - \lambda(e^{-u+i\theta})|^{-1} &\ll \frac{1}{(u + \theta)\tilde{\ell}(1/\theta)}, \quad \text{for } \theta \ge u. \\ |\operatorname{Re}\{(1 - \lambda(e^{-u+i\theta}))^{-1}\}| &\ll \frac{u}{(u^2 + \theta^2)\tilde{\ell}(1/\theta)} + \frac{\theta\ell(1/\theta)}{(u^2 + \theta^2)\tilde{\ell}(1/\theta)^2}, \quad \text{for } \theta \ge u. \end{aligned}$$

Proof Recall from the proof of Lemma 4.1 (in particular (4.1)) that $1 - \lambda(z)$ is the sum of five integrals of the form

$$K = (u - i\theta) \int_0^\infty e^{(-u+i\theta)x} g_z(x)(1 - G(x))dx,$$

where $g_z \ge 0$ and either (i) $g_z \equiv 1$ or by Corollary 2.8 (ii) $|g_z|_\infty = O(m(1/\theta)\theta)$ as $\theta \to 0$. In case (i), $K = uI_C + \theta I_S + i(uI_S - \theta I_C)$. In case (ii), K consists of terms of the form uJ_C, uJ_S, θJ_C, θJ_S. Hence

$$\operatorname{Re}(1 - \lambda(e^{-u+i\theta})) = uI_C + \theta I_S + E_1, \qquad \operatorname{Im} \lambda(e^{-u+i\theta}) = uI_S - \theta I_C + E_2,$$

where $E_j = O((|u|+|\theta|)(|J_C|+|J_S|))$, $j=1,2$.

For $\theta \in [0,u]$, we use the first estimates in Proposition 6.2 to obtain $\operatorname{Re}(1-\lambda(e^{-u+i\theta})) \sim u\tilde{\ell}(1/u)$. Hence $|1-\lambda(e^{-u+i\theta})| \ge |\operatorname{Re}(1-\lambda(e^{-u+i\theta}))| \gg u\tilde{\ell}(1/u)$.

For $\theta \ge u$ we use the second estimates in Proposition 6.2 to obtain

$$\operatorname{Re}(1-\lambda(e^{-u+i\theta})) \sim u\tilde{\ell}(1/\theta) + O(\theta\ell(1/\theta)), \quad \operatorname{Im}\lambda(e^{-u+i\theta}) \sim -\theta\tilde{\ell}(1/\theta).$$

Hence $|1-\lambda(e^{-u+i\theta})| \gg (u+\theta)\tilde{\ell}(1/\theta)$. Finally,

$$\operatorname{Re}\{(1-\lambda(e^{-u+i\theta}))^{-1}\} = \frac{\operatorname{Re}\{1-\lambda(e^{-u+i\theta})\}}{|1-\lambda(e^{-u+i\theta})|^2} \ll \frac{u\tilde{\ell}(1/\theta)+\theta\ell(1/\theta)}{(u^2+\theta^2)\tilde{\ell}(1/\theta)^2},$$

completing the proof. ∎

Lemma 6.4 *For $u \in [0,1]$, $\theta \in [0,\pi]$, we have $\operatorname{Re} T(e^{-u+i\theta}) \ll h_u(\theta)+g(\theta)$ where*

$$h_u(\theta) = \frac{u\tilde{\ell}(1/u)}{u+\theta} + \frac{1}{u\tilde{\ell}(1/u)}1_{[0,u]}(\theta) + \frac{1}{\tilde{\ell}(1/\theta)}\frac{u}{u^2+\theta^2}, \quad g(\theta) = \frac{\ell(1/\theta)}{\tilde{\ell}(1/\theta)^2}\frac{1}{\theta}.$$

Proof By Proposition 2.9, for $z = e^{-u+i\theta} \in B_\epsilon(1)$, ϵ sufficiently small,

$$T(z) = (1-\lambda(z))^{-1}P + (1-\lambda(z))^{-1}(P(z)-P) + O(1).$$

By Corollary 6.3,

$$\begin{aligned}\operatorname{Re}\{(1-\lambda(z))^{-1}\} &\ll \frac{1}{u\tilde{\ell}(1/u)}1_{[0,u]} + \Big\{\frac{u}{(u^2+\theta^2)\tilde{\ell}(1/\theta)} + \frac{\theta\ell(1/\theta)}{(u^2+\theta^2)\tilde{\ell}(1/\theta)^2}\Big\}1_{[u,\epsilon]} \\ &\le \frac{1}{u\tilde{\ell}(1/u)}1_{[0,u]} + \frac{u}{(u^2+\theta^2)\tilde{\ell}(1/\theta)} + \frac{\ell(1/\theta)}{\theta\tilde{\ell}(1/\theta)^2}.\end{aligned}$$

By Corollary 2.8, $P(e^{-u+i\theta}) - P(e^{-u}) \ll \tilde{\ell}(1/\theta)\theta$ uniformly in u, and $P(e^{-u}) - P(1) \ll \tilde{\ell}(1/u)u$. Combining this with the estimates for $(1-\lambda(e^{-u+i\theta}))^{-1}$,

$$\begin{aligned}(1-\lambda(z))^{-1}(P(z)-P(1)) &\ll \Big(\frac{1}{u\tilde{\ell}(1/u)}1_{[0,u]} + \frac{1}{(u+\theta)\tilde{\ell}(1/\theta)}1_{[u,\epsilon]}\Big)(\theta\tilde{\ell}(1/\theta)+u\tilde{\ell}(1/u)) \\ &\ll 1 + \frac{1}{u\tilde{\ell}(1/u)}1_{[0,u]} + \frac{u\tilde{\ell}(1/u)}{u+\theta}.\end{aligned}$$

This proves the result. ∎

Remark 6.5 By similar but much simpler calculations, we obtain the estimates $|\operatorname{Re}\{(1-\lambda(e^{i\theta}))^{-1}\}| \ll g(\theta)$ for $\theta \in (0,\epsilon)$ and $\operatorname{Re} T(e^{i\theta}) \ll g(\theta)$ for $\theta \in (0,\pi]$.

Corollary 6.6 *For $n \geq 1$, $\lim_{u\to 0} \int_0^\pi \cos n\theta \operatorname{Re} T(e^{-u+i\theta})\,d\theta = \int_0^\pi \cos n\theta \operatorname{Re} T(e^{i\theta})\,d\theta$.*

Proof The function $g(\theta) = \frac{\ell(1/\theta)}{\tilde{\ell}(1/\theta)^2}\frac{1}{\theta}$ lies in L^1 by Proposition 6.1. Note that $\operatorname{Re} T(e^{-u+i\theta}) \to \operatorname{Re} T(e^{i\theta})$ and $h_u(\theta) \to 0$ pointwise (for all $\theta \neq 0$). We claim that $\int_0^\pi h_u(\theta)\,d\theta \to 0$. The result then follows from the dominated convergence theorem (more precisely the extended version stated in [35, p. 92]).

The claim is easy to check for the first two terms in h_u. For the third term $k_u(\theta) = \frac{1}{\tilde{\ell}(1/\theta)}\frac{u}{u^2+\theta^2}$, we compute for $b \in (0,\pi)$ that

$$\int_0^\pi k_u(\theta)\,d\theta = \int_0^b k_u(\theta)\,d\theta + \int_b^\pi k_u(\theta)\,d\theta \ll \frac{1}{\tilde{\ell}(1/b)}u^{-1}b + u\int_b^\pi \theta^{-2}\,d\theta$$
$$\ll \frac{1}{\tilde{\ell}(1/b)}u^{-1}b + ub^{-1},$$

where the implied constant is independent of b and u. Define $b = b(u)$ such that $u = b(\tilde{\ell}(1/b))^{-1/2}$. In particular, $b \to 0$ as $u \to 0$ and so $\int_0^\pi k_u(\theta)\,d\theta \ll (\tilde{\ell}(1/b))^{-1/2} \to 0$ as required. ∎

Corollary 6.7 $\operatorname{Re} T \in L^1$ *and $T_n = \frac{2}{\pi}\int_0^\pi \cos n\theta \operatorname{Re} T(e^{i\theta})\,d\theta$ for all $n \geq 1$.*

Proof The function $\theta \mapsto T(\rho e^{i\theta})$ is integrable for each fixed $\rho < 1$. Moreover, the power series for $T(z)$ is uniformly convergent on compact subsets of $\mathbb{D}$, so we obtain

$$\int_0^\pi \cos n\theta \operatorname{Re} T(\rho e^{i\theta})\,d\theta = \sum_{j=0}^\infty T_j\rho^j \int_0^\pi \cos n\theta \cos j\theta\,d\theta = \tfrac{\pi}{2}T_n\rho^n.$$

By Corollary 6.6, $T_n = \frac{2}{\pi}\rho^{-n}\int_0^\pi \cos n\theta \operatorname{Re} T(\rho e^{i\theta})\,d\theta \to \frac{2}{\pi}\int_0^\pi \cos n\theta \operatorname{Re} T(e^{i\theta})\,d\theta$, as $\rho = e^{-u} \to 1$. ∎

6.2 Asymptotics of T_n

The calculations in this subsection are restricted to the unit circle, so we revert to writing $T(\theta)$ instead of $T(e^{i\theta})$ and so on. First we determine the asymptotics of $\operatorname{Re}\{(1-\lambda(\theta))^{-1}\}$ (see also [4]).

Lemma 6.8 $\operatorname{Re}\{(1-\lambda(\theta))^{-1}\} = \frac{\pi}{2}g(\theta)(1+o(1))$ *as $\theta \to 0^+$, where* $g(\theta) = \dfrac{\ell(1/\theta)}{\theta(\tilde{\ell}(1/\theta))^2}$.

Proof By Remark 6.5, $\operatorname{Re}\{(1-\lambda(\theta))^{-1}\} \ll g(\theta)$. We claim that $I_S \sim \frac{\pi}{2}\ell(1/\theta)$ from which the result follows easily.

Let $M \geq 3\pi$. Since $\sigma \mapsto \ell(\sigma)/\sigma$ is decreasing, we have the oscillatory integral estimate

$$\frac{1}{\ell(1/\theta)}I_S = \int_0^\infty \frac{\sin\sigma}{\sigma}\frac{\ell(\sigma/\theta)}{\ell(1/\theta)}\,d\sigma = \int_0^M \frac{\sin\sigma}{\sigma}\frac{\ell(\sigma/\theta)}{\ell(1/\theta)}\,d\sigma + F_M,$$

where

$$|F_M| \le 2 \sup_{\sigma\in[M-2\pi,M+2\pi]} \frac{\ell(\sigma/\theta)}{\sigma\ell(1/\theta)}.$$

By Potter's bounds, for any $\delta > 0$, $F_M = O(1/M^{1-\delta})$. Hence, $\lim_{\theta\to 0} \ell(1/\theta)^{-1} I_S = \int_0^M \frac{\sin\sigma}{\sigma}\, d\sigma + O(1/M^{1-\delta})$. Let $M \to \infty$ to verify the claim. ∎

Corollary 6.9 *Let $a > 0$. Then $\lim_{n\to\infty} \tilde{\ell}(n) \int_0^{a/n} \mathrm{Re}\{(1-\lambda(\theta))^{-1}\}\, d\theta = \frac{\pi}{2}$.*

Proof By Lemma 6.8, we can write $\mathrm{Re}\{(1-\lambda(\theta))^{-1}\} = \frac{\pi}{2} g(\theta)(1+h(\theta))$ where $h(\theta) = o(1)$ as $\theta \to 0^+$. Let $H(n) = \sup_{\theta\in[0,a/n]} |h(\theta)|$, so $H(n) = o(1)$ as $n \to \infty$. Then

$$\int_0^{a/n} \mathrm{Re}\{(1-\lambda(\theta))^{-1}\}\, d\theta = \frac{\pi}{2} \int_0^{a/n} g(\theta)\, d\theta + O\Big(H(n) \int_0^{a/n} g(\theta)\, d\theta\Big).$$

By Proposition 6.1, $\int_0^{a/n} g(\theta)\, d\theta = \tilde{\ell}(n/a)^{-1}$. Hence

$$\tilde{\ell}(n) \int_0^{a/n} \mathrm{Re}\{(1-\lambda(\theta))^{-1}\}\, d\theta = \frac{\pi}{2} \frac{\tilde{\ell}(n)}{\tilde{\ell}(n/a)}(1+o(1)) \to \frac{\pi}{2},$$

as $n \to \infty$. ∎

Proof of Theorem 2.1, $\beta = 1$. By Remark 6.5 and Proposition 2.6(b), $T(\theta) \ll (\theta\tilde{\ell}(1/\theta))^{-1}$. Let $\delta > 0$. By the argument in the proof of Lemma 5.1, we obtain

$$\tilde{\ell}(n) \int_{a/n}^{\pi} \cos n\theta\, T(\theta)\, d\theta \ll a^{-(1-\delta)}, \tag{6.1}$$

for $a \in [1, n]$, $n \ge 1$. Also, we have

$$\tilde{\ell}(n) \int_0^{a/n} \cos n\theta\, T(\theta)\, d\theta = \tilde{\ell}(n) \int_0^{a/n} \cos n\theta\, (1-\lambda(\theta))^{-1}\, d\theta\, P + O(an^{-(1-\delta)}). \tag{6.2}$$

By Lemma 6.8,

$$\begin{aligned}&\tilde{\ell}(n) \int_0^{a/n} (\cos n\theta - 1)\, \mathrm{Re}\{(1-\lambda(\theta))^{-1}\}\, d\theta \\ &\qquad \ll \tilde{\ell}(n) \int_0^{a} (\cos\sigma - 1) \ell(n/\sigma)(\tilde{\ell}(n/\sigma))^{-2} \sigma^{-1}\, d\sigma \ll \ell(n)(\tilde{\ell}(n))^{-1} a.\end{aligned} \tag{6.3}$$

Combining estimates (6.1), (6.2) and (6.3) with Corollary 6.9, we obtain

$$\lim_{a\to\infty} \lim_{n\to\infty} \tilde{\ell}(n) \int_0^{\pi} \cos n\theta\, \mathrm{Re}\, T(\theta)\, d\theta = \frac{\pi}{2} P,$$

and hence $\tilde{\ell}(n) T_n \to P$ by Corollary 6.7. ∎

7 Pointwise dual ergodicity

In this section, we give an elementary proof of pointwise dual ergodicity for the class of systems under consideration for all $\beta \in (0,1]$. We assume our general framework from Section 2, except that we do not require H2(ii).

Proposition 7.1 *$T(s) \sim \begin{cases} \tilde\ell(\frac{1}{1-s})^{-1}(1-s)^{-1}P, & \beta=1, \\ \Gamma(1-\beta)^{-1}\ell(\frac{1}{1-s})^{-1}(1-s)^{-\beta}P, & \beta\in(0,1), \end{cases}$ as $s \to 1^-$.*

Proof This is similar to the proof of Lemma 3.1, but much simpler since the integrals are absolutely convergent. By Proposition 2.9, for $s \in (1-\epsilon, 1]$,

$$T(s) = (1-\lambda(s))^{-1}P + (1-\lambda(s))^{-1}(P(s)-P) + O(1).$$

By Corollary 2.8, $P(s) - P \ll m(\frac{1}{1-s})(1-s)^\beta$, so it suffices to establish the desired asymptotic expression for $(1-\lambda(s))^{-1}$.

Setting $s = e^{-u}$ we have $\lambda(s) = 1 + \int_0^\infty (e^{-ux}-1)\hat v_u(x)dG(x)$, where $G(x) = \mu(\varphi \le x)$ and $|\hat v_u - 1|_\infty = o(1)$ as $u \to 0$. Writing $d\hat G_u = v_u\,dG$ and integrating by parts,

$$\lambda(s) = 1 + \int_0^\infty (e^{-ux}-1)d\hat G_u(x) = 1 - u\int_0^\infty e^{-ux}g_u(x)(1-G(x))\,dx, \tag{7.1}$$

where $|g_u(x) - 1|_\infty = o(1)$ as $u \to 0$.

If $\beta \in (0,1)$, then

$$\lambda(s) = 1 - \ell(1/u)u^\beta \int_0^\infty e^{-\sigma} g_u(\sigma/u)\{\ell(\sigma/u)\ell(1/u)^{-1}\}\sigma^{-\beta}\,du.$$

By the dominated convergence theorem, $\lambda(s) = 1 - \Gamma(1-\beta)\ell(1/u)u^\beta(1+o(1))$. The result follows since $u = -\log s = 1 - s + O((1-s)^2)$.

If $\beta = 1$, then picking up from (7.1),

$$\begin{aligned}\textstyle\int_0^\infty e^{-ux}g_u(x)(1-G(x))\,dx &= \textstyle\int_0^{1/u} g_u(x)(1-G(x))\,dx \\ &\quad + \textstyle\int_0^{1/u}(e^{-ux}-1)g_u(x)(1-G(x))\,dx + \int_{1/u}^\infty e^{-ux}g_u(x)(1-G(x))\,dx.\end{aligned}$$

The last two integrals are $O(\ell(1/u))$, and $\int_0^{1/u} g_u(x)(1-G(x))\,dx = (1+o(1))\int_0^{1/u}(1-G(x))\,dx \sim \tilde\ell(1/u)$ by definition of $\tilde\ell$. ∎

Theorem 7.2 *Let $v \in L^1(X)$ and $\beta \in (0,1]$. Then*

$$\lim_{n\to\infty} m(n)n^{-\beta}\sum_{j=1}^n L^j v = \beta^{-1}d_\beta \int_X v\,d\mu, \quad \textit{almost everywhere on } X.$$

Proof By [3, Proposition 3.7.6] (see also the proof of Proposition 10.2), it suffices to prove pointwise dual ergodicity on the tower Δ defined in Section 2.4. Let $L_\Delta : L^1(\Delta) \to L^1(\Delta)$ denote the transfer operator on Δ. Note that $T_n v = 1_Y L_\Delta^n(1_Y v)$ coincides with our usual T_n. By the Hurewicz ergodic theorem [26], it is enough to prove that $m(n)n^{-\beta}\sum_{j=0}^{n-1} L_\Delta^j v \to \beta^{-1}d_\beta \int_\Delta v\,d\mu$ almost everywhere on Δ for the particular choice $v = 1_Y$.

Let $y \in Y$. For $\beta \in (0,1)$, Proposition 7.1 gives $(T(s)v)(y) \sim \Gamma(1-\beta)\ell(\frac{1}{1-s})^{-1}(1-s)^{-\beta}\int_Y v\,d\mu$ as $s \to 1^-$. By (the discrete version of) the Karamata Tauberian Theorem [27] [16, p. 445], [41, Proposition 4.2], it follows that $\sum_{j=1}^n (T_j v)(y) \sim \beta^{-1}d_\beta \ell(n)^{-1}n^\beta \int_Y v\,d\mu$ as $n \to \infty$. Similarly for $\beta = 1$.

Finally, let $p = (y,j)$ be a general point in Δ. Then $(L_\Delta^n v)(p) = (L_\Delta^{n-j}v)(y,0) = (T_{n-j}v)(y)$ for all $n > j$. ∎

An immediate consequence is the Darling-Kac law [10]. Recall that a random variable $\mathcal{M}_\beta$ on $(0,\infty)$ has the *normalised Mittag-Leffler distribution of order* β if $E(e^{z\mathcal{M}_\beta}) = \sum_{p=0}^\infty \Gamma(1+\beta)^p z^p/\Gamma(1+p\beta)$ for all $z \in \mathbb{C}$.

Corollary 7.3 *Let $v \in L^1(X)$, $v \ge 0$, $\int_X v\,d\mu = 1$, and let $\beta \in (0,1]$. Then*

$$m(n)n^{-\beta}\sum_{j=1}^n v \circ f^j \to_d \mathcal{M}_\beta \ \ \textit{as } n \to \infty.$$

The convergence is in the sense of strong distributional convergence*: convergence in distribution under any probability measure absolutely continuous w.r.t. μ.*

Proof This follows from Theorem 7.2 by Aaronson [1], [3, Corollary 3.7.3]. ∎

To conclude the section, we mention a simple consequence of Theorem 2.1 which gives uniform convergence on Y in the pointwise dual ergodic theorem for $\beta > \frac{1}{2}$ for sufficiently regular observables.

Proposition 7.4 *If $\beta \in (\frac{1}{2}, 1]$, then $\lim_{n\to\infty} m(n)n^{-\beta}\sum_{j=1}^n T_j = \beta^{-1}d_\beta P$.*

Proof By Theorem 2.1, $T_n = m(n)^{-1}n^{-(1-\beta)}d_\beta P + S_n$ where $\|S_n\| = o(m(n)^{-1}n^{-(1-\beta)})$. Hence

$$m(n)n^{-\beta}\sum_{j=1}^n T_j = m(n)n^{-\beta}\sum_{j=1}^n m(j)^{-1}j^{-(1-\beta)}\,d_\beta P + m(n)n^{-\beta}\sum_{j=1}^n S_j. \tag{7.2}$$

By Proposition 2.6(a), $\sum_{j=1}^n m(j)^{-1}j^{-(1-\beta)} \sim \beta^{-1}m(n)^{-1}n^\beta$, so the first term on the RHS of (7.2) converges to the desired limit $\beta^{-1}d_\beta P$.

Let $\delta > 0$, and choose n_0 such that $\|S_n\| \le \delta m(n)^{-1}n^{-(1-\beta)}$ for $n > n_0$. Then $\sum_{j=1}^n \|S_j\| \le \sum_{j=1}^{n_0}\|S_j\| + \sum_{j=n_0+1}^n \delta m(j)^{-1}j^{-(1-\beta)}$. Applying Proposition 2.6(a) once more, we obtain $\limsup_{n\to\infty} m(n)n^{-\beta}\sum_{j=1}^n \|S_j\| \le \beta^{-1}\delta$. Since $\delta > 0$ is arbitrary, the second term on the RHS of (7.2) converges to zero. ∎

8 Results for $\beta \in (0, \frac{1}{2}]$

In this section, we prove Theorems 2.2 and 2.3.

Proof of Theorem 2.2 If $\beta \in (0, \frac{1}{2}]$, then the proof of Theorem 2.1 breaks down only in the estimation of I_3 in Lemma 5.1.

(a) When $\beta = \frac{1}{2}$, it is evident from the proof of Lemma 5.1 that $I_3 \ll \ell(n) n^{-\frac{1}{2}} \int_{1/n}^{\pi} \ell(1/\theta)^{-2} \theta^{-1}\, d\theta$. The remaining estimates are $O(\ell(n)^{-1} n^{-\frac{1}{2}})$ as before. By Proposition 2.6(b), $\ell(n)^2 \int_{1/n}^{\pi} \ell(1/\theta)^{-2}\theta^{-1}\, d\theta \to \infty$ as $n \to \infty$. Hence the estimate for I_3 is the dominant one.

(b) For $\beta \in (0, \frac{1}{2})$, $I_3 \ll \ell(n) n^{-\beta} \int_0^{\pi} \ell(1/\theta)^{-2} \theta^{-2\beta}\, d\theta \ll \ell(n) n^{-\beta}$. The remaining estimates are $O(\ell(n)^{-1} n^{-(1-\beta)})$ as before.

(c) If $Pv = 0$, then $\|T(\theta)v\| \ll \|v\|$. Hence the resolvent identity

$$\{T(\theta) - T(\theta - \pi/n)\}v = T(\theta)(R(\theta) - R(\theta - \pi/n))T(\theta - \pi/n)v$$

yields $\|\{T(\theta) - T(\theta - \pi/n)\}v\| \ll \ell(1/\theta)^{-1}\theta^{-\beta}\ell(n)n^{-\beta}$. It follows that

$$2T_n v = \int_0^{2\pi} \{T(\theta) - T(\theta - \pi/n)\} e^{-in\theta}\, d\theta \ll \ell(n) n^{-\beta} \|v\|,$$

as required. ∎

Next, we establish the lower bound in Theorem 2.3(b).

Proposition 8.1 *If $\beta \in (0,1)$, then $\liminf_{n\to\infty} \ell(n) n^{1-\beta} T_n v \ge d_\beta \int_Y v\, d\mu$ pointwise on Y for all $v \ge 0$.*

Proof For any $m \ge 1$, we can write

$$T = (I - R)^{-1} = I + R + \cdots + R^{m-1} + T^{(m)}, \quad T^{(m)} = R^m (I - R)^{-1}.$$

Since R is a positive operator, we deduce that $(T_n v)(y) \ge (T_n^{(m)} v)(y)$ for all $v \ge 0$, $y \in Y$, $m \ge 1$. Choosing $m = b_n \sim b\ell(n)^{-1} n^{\beta}$, $b > 0$, as in [17, Theorem 3.6.1], we obtain $\ell(n) n^{1-\beta} T_n^{(b_n)} \sim d_b P$, where $d_b \to d_\beta$ as $b \to 0$, and the result follows. ∎

The following result is well-known (see [7, Theorem 2.9.1], [17]) but stated in a slightly different form, so we provide the proof for completeness.

Proposition 8.2 *Let f_n be a sequence in $\mathbb{R}$ and let $A \in \mathbb{R}$, Suppose that $\beta \in (0,1)$, that $\ell(n)$ is slowly varying, and that*

(a) $\liminf_{n\to\infty} \ell(n) n^{1-\beta} f_n \ge A$,

(b) $\lim_{n\to\infty} \ell(n) n^{-\beta} \sum_{j=1}^{n} f_j = \beta^{-1} A$.

Then there exists a set E of density zero such that $\lim_{n\to\infty,\, n\notin E} \ell(n)n^{1-\beta} f_n = A$.

In particular, $\liminf_{n\to\infty} \ell(n)n^{1-\beta} f_n = A$.

Proof Our proof is modelled on [32, p. 65, Lemma 6.2].

By Proposition 2.6(a), $\sum_{j=1}^n \ell(j)^{-1}j^{-(1-\beta)} \sim \beta^{-1}\ell(n)^{-1}n^{\beta}$. Let $\hat{f}_n = f_n - \ell(n)^{-1}n^{(\beta-1)}A$. Then (b) is equivalent to $\lim_{n\to\infty} \ell(n)n^{-\beta}\sum_{j=1}^n \hat{f}_j = 0$. Hence we may suppose without loss that $A = 0$. In addition, there is a monotone increasing function $g(n)$ such that $\ell(n)n^{1-\beta} \sim g(n)$ (see for example [7, Theorem 1.5.3]). Hence we may suppose that $\ell(n)n^{1-\beta}$ is increasing.

Define the nested sequence of sets $E_q = \{n \geq 1 : \ell(n)n^{1-\beta} f_n > 1/q\}$. We claim that each E_q has density zero. Let $\delta > 0$. By (a), there exists $n_0 \geq 1$ such that $\ell(n)n^{1-\beta} f_n > -\delta$ for all $n \geq n_0$. Hence

$$\begin{aligned}
\frac{1}{n}\sum_{j=1}^n 1_{E_q}(j) &\leq \ell(n)n^{-\beta}\sum_{j=1}^n \{\ell(j)j^{(1-\beta)}\}^{-1}1_{E_q}(j) \leq q\ell(n)n^{-\beta}\sum_{j=1}^n f_j 1_{E_q}(j) \\
&= q\ell(n)n^{-\beta}\Big(\sum_{j=1}^n f_j - \sum_{j=1}^{n_0} f_j 1_{E_q^c}(j) - \sum_{n_0+1}^{n} f_j 1_{E_q^c}(j)\Big) \\
&\leq q\ell(n)n^{-\beta}\sum_{j=1}^n f_j + q\ell(n)n^{-\beta}\sum_{j=1}^{n_0} |f_j| + q\ell(n)n^{-\beta}\sum_{n_0+1}^{n} \ell(j)^{-1}j^{\beta-1}\delta \\
&= q\ell(n)n^{-\beta}\sum_{j=1}^n f_j + O(\ell(n)n^{-\beta}) + O(\delta).
\end{aligned}$$

By (b), $\limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^n 1_{E_q}(j) = O(\delta)$ and the claim follows since δ is arbitrary.

By the claim, there exist $1 = i_0 < i_1 < i_2 < \cdots$ such that $\frac{1}{n}\sum_{j=1}^n 1_{E_q}(j) < 1/q$ for $n \geq i_{q-1}$, $q \geq 2$. Let $E = \bigcup_{q=1}^\infty E_q \cap (i_{q-1}, i_q)$. If $n \in E$ and $n \leq i_q$, then $n \in E_q$. Hence for $i_{q-1} \leq n \leq i_q$, we have $\frac{1}{n}\sum_{j=1}^n 1_E(j) \leq \frac{1}{n}\sum_{j=1}^n 1_{E_q}(j) \leq 1/q$, verifying that E has density zero.

On the other hand, if $n \notin E$ and $i_{q-1} < n < i_q$, then $n \notin E_q$ and so $\ell(n)n^{1-\beta} f_n \leq 1/q$. Hence $\limsup_{n\to\infty,\, n\notin E} \ell(n)n^{1-\beta} f_n \leq 0$. Combined with assumption (a), we deduce that $\lim_{n\to\infty,\, n\notin E} \ell(n)n^{1-\beta} f_n = 0$ and the last statement follows immediately. ∎

Proof of Theorem 2.3 Part (b) is stated for $v \geq 0$ and in part (a) we can break v into positive and negative parts. Hence without loss we may suppose that $v \geq 0$.

By Proposition 8.1 and Theorem 7.2, we have verified the hypotheses of Proposition 8.2 with $f_n = (T_n v)(y)$ and $A = d_\beta \int_Y v\, d\mu$. The result is immediate. ∎

9 Second order asymptotics

In this section we prove results on second order asymptotics and higher order asymptotic expansions under assumptions on the asymptotics of $\mu(\varphi > n)$. Throughout, we suppose that $\ell(n)$ is asymptotically constant (and that $\beta > \frac{1}{2}$). In Subsection 9.1, we consider the case when $\beta \in (\frac{1}{2}, 1)$. The case $\beta = 1$ is covered in Subsection 9.2. Error terms in the Dynkin-Lamperti arcsine law are obtained in Subsection 9.3.

9.1 Second order asymptotics for $\beta \in (\frac{1}{2}, 1)$

We assume that $\mu(\varphi > n) = c(n^{-\beta} + H(n))$, where $H(n) = O(n^{-2\beta})$ and $c > 0$. (It is easy to relax this to the more general hypothesis that $H(n) = O(n^{-q})$, $q > 1$. However the formulas become more complicated and our assumption is satisfied by (1.1).)

Recall that $c_H = \int_0^\infty H_1(x)\,dx$ where $H_1(x) = [x]^{-\beta} - x^{-\beta} + H([x])$. Define $\xi_p^{\pm} = \int_0^\infty e^{\pm i\sigma}\sigma^{-p}\,d\sigma$, $0 < p < 1$, so $c_\beta = -i\xi_\beta^+$, and recall that $e_0 = ic_H/c_\beta$.

Set $d'_{\beta,j} = e_0^j \xi^-_{(j+1)\beta - j}/c_\beta = ie_0^j\xi^-_{(j+1)\beta-j}/\xi^+_\beta$ and $d_{\beta,j} = \frac{1}{\pi}\operatorname{Re} d'_{\beta,j}$.

We note that $d_{\beta,0} = d_\beta = \frac{1}{\pi}\sin\beta\pi > 0$, and that either $d_{\beta,j} = 0$ for all $j \geq 1$ or $d_{\beta,j} \neq 0$ for all $j \geq 1$. Moreover, the latter situation is typical.

Theorem 9.1 *Suppose that $\beta \in (\frac{1}{2}, 1)$ and that $\mu(\varphi > n) = c(n^{-\beta} + H(n))$, where $H(n) = O(n^{-2\beta})$ and $c > 0$. Let $\gamma = \min\{1-\beta, \beta - \frac{1}{2}\}$. Then*

$$n^{1-\beta}T_n = c^{-1}d_\beta P + O(n^{-\gamma}).$$

Moreover, if $\beta \in (\frac{3}{4}, 1)$, then $\lim_{n\to\infty} n^{1-\beta}\{n^{1-\beta}T_n - c^{-1}d_\beta P\} = d_{\beta,1}P$.

Remark 9.2 For β close to 1, we obtain higher order asymptotic expansions. There exist constants $d_{\beta,j} \in \mathbb{R}$, $j \geq 0$ with $d_{\beta,0} = d_\beta$, such that for each $q = 0, 1, 2, \ldots$,

$$n^{1-\beta}T_n = c^{-1}\Big\{\sum_{j=0}^{q} d_{\beta,j}n^{-j(1-\beta)} + O(n^{-(q+1)(1-\beta)})\Big\}P + O(n^{-(\beta-\frac{1}{2})}).$$

Thus, $n^{1-\beta}T_n = \begin{cases} c^{-1}d_\beta P + O(n^{-(\beta-\frac{1}{2})}), & \beta \in (\frac{1}{2}, \frac{3}{4}] \\ c^{-1}d_\beta P + c^{-1}d_{\beta,1}n^{-(1-\beta)}P + O(n^{-(\beta-\frac{1}{2})}), & \beta \in (\frac{3}{4}, \frac{5}{6}] \end{cases}$ and so on.

Corollary 9.3 $n^{-\beta}\sum_{j=1}^n T_j = c^{-1}\beta^{-1}d_\beta P + O(n^{-\gamma})$ *uniformly on Y.*

Proof Specialising the proof of Proposition 7.4, we have

$$n^{-\beta}\sum_{j=1}^{n} T_j = n^{-\beta}\sum_{j=1}^{n} j^{-(1-\beta)}\,c^{-1}d_\beta P + n^{-\beta}\sum_{j=1}^{n} S_j,$$

where $S_j = O(j^{-(1-\beta+\gamma)})$. Now $\sum_{j=1}^n j^{-(1-\beta)} = \int_1^n x^{-(1-\beta)}\,dx + O(1) = \beta^{-1}n^\beta + O(1)$, and $\sum_{j=1}^n S_j = O(n^{\beta-\gamma})$. ∎

In the remainder of this subsection, we prove Theorem 9.1.

Proposition 9.4 $(1-\lambda(\theta))^{-1} = c^{-1}c_\beta^{-1}\sum_j e_0^j \theta^{-((j+1)\beta-j)} + O(1)$, *where the sum is over those* $j \ge 0$ *with* $(j+1)\beta - j > 0$.

Proof By Lemma 3.2, $1-\lambda(\theta) = cc_\beta\theta^\beta(1-e_0\theta^{1-\beta}+O(\theta^\beta))$. Now invert and note that $(1-e_0\theta^{1-\beta}+O(\theta^\beta))^{-1} = \sum_{j\ge 0} e_0^j\theta^{j(1-\beta)} + O(\theta^\beta)$. ∎

Proposition 9.5 *Let* $n \ge 1$, $0 < a < \epsilon n$. *Then*

$$n^{1-\beta}\int_0^{a/n}(1-\lambda(\theta))^{-1}e^{-in\theta}\,d\theta \\ = c^{-1}\sum_j d'_{\beta,j}n^{-j(1-\beta)} + O\Big(\sum_j n^{-j(1-\beta)}a^{-((j+1)\beta-j)}\Big) + O(an^{-\beta}),$$

and the sums are over those $j \ge 0$ *with* $(j+1)\beta - j > 0$.

Proof By Proposition 9.4,

$$\int_0^{a/n}(1-\lambda(\theta))^{-1}e^{-in\theta}\,d\theta = c^{-1}c_\beta^{-1}\sum_j e_0^j\int_0^{a/n}\theta^{-((j+1)\beta-j)}e^{-in\theta}\,d\theta + O(a/n) \\ = c^{-1}c_\beta^{-1}\sum_j e_0^j n^{-(j+1)(1-\beta)}\int_0^a \sigma^{-((j+1)\beta-j)}e^{-i\sigma}\,d\sigma + O(a/n).$$

Hence

$$n^{1-\beta}\int_0^{a/n}(1-\lambda(\theta))^{-1}e^{-in\theta}\,d\theta \\ = c^{-1}\sum_j d'_{\beta,j}n^{-j(1-\beta)} - c^{-1}c_\beta^{-1}\sum_j e_0^j n^{-j(1-\beta)}\int_a^\infty \sigma^{-((j+1)\beta-j)}e^{-i\sigma}\,d\sigma + O(an^{-\beta}),$$

yielding the required result. ∎

Proof of Theorem 9.1 By Lemma 5.1, $n^{1-\beta}\int_{a/n}^\pi T(\theta)e^{-in\theta}\,d\theta \ll a^{-(2\beta-1)}$. For $\theta \in [0,a/n] \subset [0,\epsilon]$, $T(\theta) = (1-\lambda(\theta))^{-1}P + O(1)$ by Lemma 3.1(b). Hence,

$$n^{1-\beta}\int_0^\pi T(\theta)e^{-in\theta}\,d\theta = n^{1-\beta}\int_0^{a/n}(1-\lambda(\theta))^{-1}e^{-in\theta}\,d\theta\,P + O(an^{-\beta}) + O(a^{-(2\beta-1)}).$$

Taking $a = n^{1/2}$ we obtain the error term $O(n^{-(\beta-\frac{1}{2})})$ and the result follows from Proposition 9.5 and Corollary 4.2. ∎

9.2 Second order asymptotics for $\beta = 1$

Theorem 9.6 *Suppose that* $\mu(\varphi > n) = c(n^{-1} + H(n))$ *where* $c > 0$ *and* $H(n) = O(n^{-q})$, $q > 1$. *Let* $H_1(x) = [x]^{-1} - x^{-1} + H([x])$, $x \ge 1$ *and* $H_1(x) = \frac{1}{c}$, $x \in [0,1)$. *Then*

$$(\log n)T_n = c^{-1}\big\{1 - \textstyle\int_0^\infty H_1(x)\,dx\,(\log n)^{-1} + O((\log n)^{-2})\big\}P + O((\log n)^{\frac12}n^{-\frac12}).$$

Corollary 9.7 $(\log n)n^{-1}\sum_{j=1}^n T_j = c^{-1}P + O((\log n)^{-1})$ *uniformly on* Y.

Proof As in the proof of Corollary 9.3, we have

$$(\log n)n^{-1}\sum_{j=1}^n T_j = (\log n)n^{-1}\sum_{j=1}^n (\log j)^{-1}c^{-1}P + (\log n)n^{-1}\sum_{j=1}^n S_j,$$

where $\|S_j\| = O((\log j)^{-2})$. Integration by parts yields $\sum_{j=1}^n(\log j)^{-1} = n(\log n)^{-1} + \int_2^n(\log x)^{-2}\,dx + O(1) = n(\log n)^{-1} + O(n(\log n)^{-2})$ while $\sum_{j=1}^n S_j \ll n(\log n)^{-2}$. ∎

In the remainder of this subsection, we prove Theorem 9.6.

Proposition 9.8 *Let* $c_H = \int_0^1(\cos\sigma - 1)\sigma^{-1}\,d\sigma + \int_1^\infty \cos\sigma\,\sigma^{-1}\,d\sigma + \int_0^\infty H_1(x)\,dx$. *Then*

$$\mathrm{Re}\{(1-\lambda(\theta))^{-1}\} = c^{-1}\tfrac{\pi}{2}\theta^{-1}(\log\tfrac{1}{\theta})^{-2} - c^{-1}c_H\pi\theta^{-1}(\log\tfrac{1}{\theta})^{-3} + O(\theta^{-1}(\log\tfrac{1}{\theta})^{-4}).$$

Proof Without loss of generality, we may suppose that $q \in (1,2)$. Recall that $G(x) \equiv 0$ for $x \in [0,1)$ and $1 - G(x) = c(x^{-1} + H_1(x))$ for $x \ge 1$ where $H_1(x) = O(x^{-q})$. In particular, $H_1 \in L^1$. Write

$$I_C = \textstyle\int_0^\infty \cos\theta x(1-G(x))\,dx = c\int_1^\infty \cos\theta x\,x^{-1}\,dx + c\int_0^\infty \cos\theta x\,H_1(x)\,dx.$$

Now,

$$\begin{aligned}\textstyle\int_1^\infty \cos\theta x\,x^{-1}\,dx &= \textstyle\int_1^{1/\theta} x^{-1}\,dx + \int_1^{1/\theta}(\cos\theta x - 1)x^{-1}\,dx + \int_{1/\theta}^\infty \cos\theta x\,x^{-1}\,dx \\ &= \log\tfrac{1}{\theta} + \textstyle\int_0^1(\cos\sigma - 1)\sigma^{-1}\,d\sigma + \int_1^\infty \cos\sigma\,\sigma^{-1}\,d\sigma + O(\theta),\end{aligned}$$

and

$$\begin{aligned}\textstyle\int_0^\infty \cos\theta x\,H_1(x)\,dx = \int_0^\infty H_1(x)\,dx + \int_0^{1/\theta}(\cos\theta x - 1)H_1(x)\,dx \\ + \textstyle\int_{1/\theta}^\infty(\cos\theta x - 1)H_1(x)\,dx = \int_0^\infty H_1(x)\,dx + O(\theta^{q-1}).\end{aligned}$$

Hence $I_C = c\log\frac{1}{\theta} + cc_H + O(\theta^{q-1})$. Also,

$$\begin{aligned}I_S &= \textstyle\int_0^\infty \sin\theta x(1-G(x))\,dx = c\int_1^\infty \sin\theta x\,x^{-1}\,dx + c\int_0^\infty \sin\theta x H_1(x)\,dx \\ &= c\textstyle\int_\theta^\infty \sin\sigma\,\sigma^{-1}\,d\sigma + O(\theta^{q-1}) = \frac{c\pi}{2} + O(\theta^{q-1}).\end{aligned}$$

Hence

$$\mathrm{Re}(1-\lambda(\theta)) = \tfrac{c\pi}{2}\theta(1+O(\theta^{q-1})), \quad \mathrm{Im}\,\lambda(\theta) = c\theta(\log\tfrac{1}{\theta})(1+c_H(\log\tfrac{1}{\theta})^{-1}+O((\log\tfrac{1}{\theta})^{-2})),$$

and the result follows. ∎

Lemma 9.9 *If $a = O(n^{1-\delta})$ for some $\delta > 0$, then*

(i) $(\log n)\int_{a/n}^{\pi} \cos n\theta\, T(\theta)\, d\theta \ll a^{-1}$.

(ii) $(\log n)\int_0^{a/n} \cos n\theta\{T(\theta) - (1-\lambda(\theta))^{-1}P\}\, d\theta \ll an^{-1}\log n$.

Proof (i) Since $T(\theta) = (1-\lambda(\theta))^{-1}P + O(1)$, we have $\mathrm{Re}\, T(\theta) \ll \theta^{-1}(\log\theta)^{-2}$ for $\theta \in [0,\epsilon]$ and $T(\theta) \ll 1$ for $\theta \in [\epsilon, \pi]$. The integral splits up into three parts as in Lemma 5.1. As usual $I_1 \ll n^{-1}$. Next,

$$\begin{aligned} I_2 &\ll \int_{a/n}^{(a+\pi)/n} \mathrm{Re}\, T(\theta)\, d\theta \ll \int_{a/n}^{(a+\pi)/n} \theta^{-1}(\log\theta)^{-2}\, d\theta \\ &\ll (\log(a/n))^{-1} - (\log((a+\pi)/n))^{-1} = \frac{\log((a+\pi)/n) - \log(a/n)}{\log(a/n)\log((a+\pi)/n)}. \end{aligned}$$

Since $a = O(n^{1-\delta})$, we deduce that $I_2 \ll \log(1+\pi/a)\,(\log n)^{-2} \ll a^{-1}(\log n)^{-2}$.

Finally, $R(\theta+h) - R(\theta) \ll h^{-1}\log h$ by Proposition 2.7, so

$$I_3 \ll \int_{(a+\pi)/n}^{\pi} \|T(\theta)\|\,\|T(\theta-\pi/n)\|\,\|R(\theta)-R(\theta-\pi/n)\|\, d\theta \ll (\log n)n^{-1}(1+A)$$

where

$$\begin{aligned} A &= \int_{(a+\pi)/n}^{\epsilon} \|T(\theta)\|\,\|T(\theta-\pi/n)\|\, d\theta \ll \int_{a/n}^{\epsilon}(\theta\log\theta)^{-2}\, d\theta \\ &\ll \int_{a/n}^{b}(\theta\log\theta)^{-2}\, d\theta + \int_b^{\epsilon}(\theta\log\theta)^{-2}\, d\theta \ll (\log b)^{-2}\int_{a/n}^{b}\theta^{-2}\, d\theta + (b\log b)^{-2}\int_b^{\epsilon} 1\, d\theta \\ &\ll (\log b)^{-2} n/a + (b\log b)^{-2}. \end{aligned}$$

Taking $b = n^{-\frac{1}{2}\delta}$ say, we obtain $I_3 \ll (\log n)^{-1}(a^{-1} + n^{-(1-\delta)}) \ll (\log n)^{-1}a^{-1}$.
(ii) This is immediate since $T(\theta) = (1-\lambda(\theta))^{-1}P + O(1)$. ∎

Proof of Theorem 9.6 We use Proposition 9.8 to estimate $\int_0^{a/n}\cos n\theta\ \mathrm{Re}\{(1-\lambda(\theta))^{-1}\}\, d\theta$, discarding all terms that are $O((\log n)^{-3})$, bearing in mind our eventual choice $a = n^{\frac{1}{2}}$.

First we note that for $j \geq 2$,

$$\int_0^{a/n}\theta^{-1}(\log\tfrac{1}{\theta})^{-j}\, d\theta = \tfrac{1}{j-1}(\log\tfrac{n}{a})^{-(j-1)} \ll (\log n)^{-(j-1)}. \tag{9.1}$$

In particular, taking $j = 4$ disposes of the $O(\theta^{-1}(\log\frac{1}{\theta})^{-4})$ term.

Next we consider the $\theta^{-1}(\log\frac{1}{\theta})^{-3}$ term. Using properties of oscillatory integrals,

$$\int_{1/n}^{a/n}\cos n\theta\,\theta^{-1}(\log\tfrac{1}{\theta})^{-3}\,d\theta = \int_1^a \cos\sigma\,\sigma^{-1}(\log\tfrac{n}{\sigma})^{-3}\,d\sigma \ll (\log n)^{-3},$$

and

$$\int_0^{1/n}(\cos n\theta - 1)\theta^{-1}(\log\tfrac{1}{\theta})^{-3}\,d\theta = \int_0^1(\cos\sigma - 1)\sigma^{-1}(\log\tfrac{n}{\sigma})^{-3}\,d\sigma \ll (\log n)^{-3}.$$

Taking $j = 3$ and $a = 1$ in equation (9.1), we deduce that

$$\int_0^{a/n}\cos n\theta\,\theta^{-1}(\log\tfrac{1}{\theta})^{-3}\,d\theta = \tfrac{1}{2}(\log n)^{-2} + O((\log n)^{-3}).$$

To deal with the $\theta^{-1}(\log\frac{1}{\theta})^{-2}$ term, we use the identity $\dfrac{\log n}{\log\frac{n}{\sigma}} = 1 + \dfrac{\log\sigma}{\log\frac{n}{\sigma}}$. So

$$\begin{aligned}\int_{1/n}^{a/n}\cos n\theta\,\theta^{-1}(\log\tfrac{1}{\theta})^{-2}\,d\theta &= (\log n)^{-2}\int_1^a\cos\sigma\,\sigma^{-1}\{\log n/\log\tfrac{n}{\sigma}\}^2\,d\sigma\\ &= (\log n)^{-2}\int_1^a\cos\sigma\,\sigma^{-1}\,d\sigma + O((\log n)^{-3}) = (\log n)^{-2}\int_1^\infty\cos\sigma\,\sigma^{-1}\,d\sigma + O((\log n)^{-3}),\end{aligned}$$

and

$$\begin{aligned}\int_0^{1/n}(\cos n\theta - 1)\theta^{-1}(\log\tfrac{1}{\theta})^{-2}\,d\theta &= (\log n)^{-2}\int_0^1(\cos\sigma - 1)\sigma^{-1}\{\log n/\log\tfrac{n}{\sigma}\}^2\,d\sigma\\ &= (\log n)^{-2}\int_0^1(\cos\sigma - 1)\sigma^{-1}\,d\sigma + O((\log n)^{-3}).\end{aligned}$$

Taking $j = 2$ and $a = 1$ in equation (9.1), we deduce that

$$\int_0^{a/n}\cos n\theta\,\theta^{-1}(\log\tfrac{1}{\theta})^{-2}\,d\theta = (\log n)^{-1} + A(\log n)^{-2} + O((\log n)^{-3}),$$

where $A = \int_0^1(\cos\sigma - 1)\sigma^{-1}\,d\sigma + \int_1^\infty\cos\sigma\,\sigma^{-1}\,d\sigma$.

Combining these results, we obtain

$$\tfrac{2}{\pi}\int_0^{a/n}\cos n\theta\ \mathrm{Re}\{(1-\lambda(\theta))^{-1}\}\,d\theta = c^{-1} - c^{-1}\int_1^\infty H_1(x)\,dx\,(\log n)^{-1} + O((\log n)^{-2}),$$

which combined with Lemma 9.9 (taking $a = n^{\frac{1}{2}}$) gives the required result. ∎

9.3 Convergence rates in the arcsine law

As mentioned in Remark 2.5, a consequence of Theorem 2.1 is that the Dynkin-Lamperti arcsine law for waiting times holds when $\beta > \frac{1}{2}$. In fact, the arcsine law holds for AFN maps for all β [45]. See also [39, 41] for more general transformations. Here we show that our results on second order asymptotics yield a convergence rate.

For $x \in \bigcup_{j=0}^n f^{-j}Y$, $n \geq 1$, let

$$Z_n(x) = \max\{0 \leq j \leq n : f^j x \in Y\},$$

denote the time of the last visit of the orbit of x to Y during the time interval $[0,n]$.

Let ζ_β denote a random variable distributed according to the $\mathcal{B}(1-\beta,\beta)$ distribution:

$$\mathbb{P}(\zeta_\beta \le t) = d_\beta \int_0^t \frac{1}{u^{1-\beta}} \frac{1}{(1-u)^\beta}\, du, \quad t \in [0,1],$$

where $d_\beta = \frac{1}{\pi}\sin\beta\pi$.

Corollary 9.10 *Suppose that $\beta \in (\frac{1}{2},1)$ and that $\mu(\varphi > n) = cn^{-\beta} + O(n^{-2\beta})$, where $c > 0$. Let $\gamma = \min\{1-\beta, \beta-\frac{1}{2}\}$.*

Let ν be an absolutely continuous probability measure on Y with density $g \in \mathcal{B}$. Then there is a constant $C > 0$ independent of ν such that

$$\big|\nu\{\tfrac{1}{n}Z_n \le t\} - \mathbb{P}(\zeta_\beta \le t)\big| \le C\|g\|n^{-\gamma}.$$

Proof Following Thaler [40], we notice that

$$\nu\{\tfrac{1}{n}Z_n \le t\} = \sum_{0 \le j \le nt} \nu(f^{-j}\{\varphi > n-j\}), \tag{9.2}$$

$$\begin{aligned}\nu(f^{-j}\{\varphi > n-j\}) &= \int_X 1_{\{\varphi > n-j\}} \circ f^j\, 1_Y g\, d\mu = \int_X 1_{\{\varphi > n-j\}}\, L^j(1_Y g)\, d\mu \\ &= \int_Y 1_{\{\varphi > n-j\}}\, T_j g\, d\mu.\end{aligned}$$

By Theorem 9.1, $T_j g = c^{-1} d_\beta j^{-(1-\beta)}\big(1 + O(j^{-(1-\beta)}) + O(\|g\|j^{-(\beta-\frac{1}{2})})\big)$ uniformly on Y. Combined with the assumption on $\mu(\varphi > n)$, we obtain

$$\begin{aligned}&\nu(f^{-j}\{\varphi > n-j\}) \\ &\quad = d_\beta j^{-(1-\beta)}(n-j)^{-\beta}\big(1 + O(j^{-(1-\beta)}) + O(\|g\|j^{-(\beta-\frac{1}{2})})\big)\big(1 + O((n-j)^{-\beta})\big).\end{aligned}$$

Since functions of the form $s^{-a}(n-s)^{-b}$ have only one turning point, replacing the sum in (9.2) by an integral introduces only three errors all of order $\|g\|n^{-1}$ and so $\mu\{\frac{1}{n}Z_n \le t\} = d_\beta I + O(\|g\|n^{-1})$, where

$$\begin{aligned}I &= \int_0^{nt} s^{-(1-\beta)}(n-s)^{-\beta}\big(1 + O(s^{-(1-\beta)}) + O(\|g\|s^{-(\beta-\frac{1}{2})})\big)\big(1 + O((n-s)^{-\beta})\big)\, ds \\ &= \int_0^t u^{-(1-\beta)}(1-u)^{-\beta}\, du + O(n^{-(1-\beta)}) + O(\|g\|n^{-(\beta-\frac{1}{2})}),\end{aligned}$$

as required. ∎

Remark 9.11 (a) The proof shows that for any $q \geq 0$,

$$\nu(\{\tfrac{1}{n}Z_n \leq t\}) = \sum_{k=0}^{q} b_{\beta,k}\mathbb{P}(\zeta_{\beta,k+1} \leq t)n^{-k(1-\beta)} + O(n^{-(q+1)(1-\beta)}) + O(\|g\|n^{-(\beta-\frac{1}{2})}).$$

where $\zeta_{\beta,k}$ is the random variable with density proportional to $u^{-k(1-\beta)}(1-u)^{-\beta}$ and $b_{\beta,k} = d_{\beta,k}/\int_0^1 u^{-(k+1)(1-\beta)}(1-u)^{-\beta}\,du$. Here $b_{\beta,0} = 1$ and $\zeta_{\beta,1} = \zeta_\beta$.

Thus for β close to 1, we obtain asymptotic expansions to arbitrarily high order, and the error rate $n^{-\gamma}$ is optimal for $\beta \geq \frac{3}{4}$.

(b) For $x \in X$, let $Y_n(x) = \min\{k > n : f^k x \in Y\}$. Then $Y_k > n$ if and only if $Z_n \leq k$ so that the arcsine law is equivalent to strong distributional convergence of $\frac{1}{n}Y_n$ to ζ_β^{-1} (see for example [39]). It is easily verified that the convergence rate in Corollary 9.10 holds also for $\frac{1}{n}Y_n$.

10 Convergence results for L^n

Sections 2 to 9 were concerned with the analysis of the sequence of renewal operators T_n given by $T_n v = 1_Y L^n(1_Y v)$. An important issue is to study the iterates L^n themselves. In Subsection 10.1, we show how convergence on Y implies convergence almost everywhere on X. In Subsection 10.2, we consider observables not supported on Y.

10.1 Convergence on X

Theorem 2.1 gives (uniform) convergence results on Y for observables $v \in \mathcal{B}$. Recall that Y can be regarded as a first return set for both the underlying system $f : X \to X$ and the tower map $f_\Delta : \Delta \to \Delta$ introduced in Subsection 2.4. We now show that observables $v \in \mathcal{B}$ enjoy pointwise convergence everywhere on Δ and almost everywhere on X.

Proposition 10.1 *Let $v \in L^1(\Delta)$, $w_n \in \mathbb{R}$, $A \in \mathbb{R}$. Suppose that $w_n L_\Delta^n v \to A$ pointwise on Y. Then $w_n L_\Delta^n v \to A$ pointwise on Δ.*

Proof Let $p = (y,j) \in \Delta$. Then $f_\Delta^{-j}p$ consists of the single preimage $(y,0) \cong y$, and $w_n(L_\Delta^n v)(p) = w_n(L_\Delta^{n-j}v)(y) \to A$. ∎

Let $\pi : \Delta \to X$ be the projection $\pi(y,j) = f^j y$. Let $1 \leq p,q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Define $\pi^* : L^p(X) \to L^p(\Delta)$, $\pi^* v = v \circ \pi$, and by duality define $\hat\pi : L^q(\Delta) \to L^q(X)$, $\int_\Delta \hat\pi v\, w\, d\mu_X = \int_X v\, \pi^* w\, d\mu_\Delta$. As usual, $\|\pi^*\|_p = \|\hat\pi\|_q = 1$ and we have the standard properties $\hat\pi 1 = 1$, $\hat\pi\pi^* = I$, $\hat\pi L_\Delta = L\hat\pi$, $\hat\pi(1_{\pi^{-1}E}v) = 1_E\hat\pi v$.

Proposition 10.2 *Let $v \in L^1(\Delta)$, $w_n \in \mathbb{R}$, $A \in \mathbb{R}$. Suppose that $w_n L_\Delta^n v \to A$ almost everywhere on Δ. Then $w_n L^n \hat\pi v \to A$ almost everywhere on X.*

Proof Suppose for contradiction that $E \subset X$ is a set of positive finite measure such that everywhere on E, $w_n L^n \hat{\pi} v$ fails to converge to A. By assumption, $w_n L_\Delta^n v \to A$ almost everywhere on $\pi^{-1}E$. By Egorov's Theorem, there is a subset $C \subset \pi^{-1}E$ of positive measure such that $w_n L_\Delta^n v \to A$ uniformly on C. Indeed $\mu_\Delta(\pi^{-1}E \setminus C)$ is arbitrarily small, and since π has only countably many branches it follows from an $\epsilon/2^k$ argument that we can choose $C = \pi^{-1}E'$ where E' is a positive measure subset of E.

In particular, $\|1_{\pi^{-1}E'}(w_n L_\Delta^n v - A)\|_{L^\infty(\Delta)} \to 0$ and since $\hat{\pi} : L^\infty(\Delta) \to L^\infty(X)$ is bounded, $\|\hat{\pi}\{1_{\pi^{-1}E'}(w_n L_\Delta^n v - A)\}\|_{L^\infty(X)} \to 0$. But $\hat{\pi}\{1_{\pi^{-1}E'}(w_n L_\Delta^n v - A)\} = 1_{E'}\hat{\pi}(w_n L_\Delta^n v - A) = 1_{E'}(w_n L^n \hat{\pi} v - A)$ so we conclude that $w_n L^n \hat{\pi} v \to A$ on E' which is the desired contradiction. ∎

Corollary 10.3 *If $\beta \in (\frac{1}{2}, 1]$ and $v \in \mathcal{B}$, then $\lim_{n\to\infty} m(n) n^{1-\beta} L^n v = d_\beta \int_Y v \, d\mu$ uniformly on Y and almost everywhere on X.*

Proof Since v is supported on Y, we can regard v as an observable on Δ or on X and $\hat{\pi} v = v$. Also, $T_n v = 1_Y L^n v = 1_Y L_\Delta^n v$. Theorem 2.1 immediately implies uniform convergence on Y. By Propositions 10.1 and 10.2, $m(n) n^{1-\beta} L_\Delta^n v$ converges pointwise on Δ and $m(n) n^{1-\beta} L^n v$ converges almost everywhere on X. ∎

10.2 Convergence for general observables

In this subsection, we enlarge the class of observables so that they need not be supported on Y. Define $X_k = f^{-k}Y \setminus \bigcup_{j=0}^{k-1} f^{-j}Y$. Thus $z \in X_k$ if and only if $k \geq 0$ is least such that $f^k z \in Y$. (In particular, $X_0 = Y$.)

Given $v \in L^\infty(X)$, define $v_k = 1_{X_k} v$. Then $L^k v_k$ is supported in Y and $L^n v_k$ vanishes on Y for all $n < k$. If $n \geq k$, we have $1_Y L^n v_k = T_{n-k} L^k v_k$.

Write $v \in \mathcal{B}(X)$ if $v \in L^1(X)$ and $L^k v_k \in \mathcal{B}$ for each $k \geq 0$.

Theorem 10.4 *Let $\beta \in (\frac{1}{2}, 1]$. Suppose that $v \in \mathcal{B}(X)$ and $\sum \|L^k v_k\| < \infty$. Then $\lim_{n\to\infty} m(n) n^{1-\beta} L^n v = d_\beta \int_X v \, d\mu$ uniformly on Y and pointwise on X.*

Proof Let $w_n = d_\beta^{-1} m(n) n^{1-\beta}$ and $c_{j,n} = \frac{w_n}{w_{n-j}} - 1$. By Theorem 2.1, $T_n = w_n^{-1} P + S_n$ where $S_n = o(w_n^{-1})$. Hence on Y,

$$w_n L^n v - \textstyle\int v = w_n \displaystyle\sum_{j=0}^n T_{n-j} L^j v_j - \textstyle\int v \ll \Big| w_n \displaystyle\sum_{j=0}^n w_{n-j}^{-1} \textstyle\int L^j v_j \; - \; \int v \Big| + w_n \displaystyle\sum_{j=0}^n \|S_{n-j}\| \|L^j v_j\|$$

$$\ll \sum_{j=0}^n c_{j,n} \textstyle\int |v_j| + w_n \displaystyle\sum_{j=0}^n \|S_{n-j}\| \|L^j v_j\| + \sum_{j>n} \textstyle\int |v_j|.$$

It is immediate that the third term converges to zero. Also $\int |v_j|$ is summable and $\lim_{n\to\infty} c_{j,n} = 0$ for each j, so the first term converges to zero. Similarly, the second term converges to zero since $\lim_{n\to\infty} \|S_{n-j}\| w_n = 0$ for each j. This completes the proof of uniform convergence on Y.

To prove pointwise convergence on X, define $u = \pi^* v : \Delta \to \mathbb{R}$ and note that $\hat\pi u = v$. Also define $\Delta_k = f_\Delta^{-k} Y \setminus \bigcup_{j=0}^{k-1} f_\Delta^{-j} Y$. Then $\Delta_0 = Y$ and for $k \ge 1$, Δ_k consists of those points $(y, j) \in \Delta$ with $j = \varphi(y) - k > 0$. Note also that $\pi^{-1} X_k \subset \Delta_k$ (since $(y, j) \in \pi^{-1} X_k$ if and only if $f^j y \in X_k$, that is $\varphi(y) = j + k$). In particular, $u_k = \pi^* v_k$ is supported in Δ_k. Hence $L_\Delta^k u_k$ is supported in Y and $L_\Delta^k u_k = \hat\pi L_\Delta^k u_k = L^k \hat\pi u_k = L^k v_k$. In particular, $\sum \|L_\Delta^k u\| < \infty$ and the argument above shows that $w_n L_\Delta^n u \to \int_\Delta u \, d\mu_\Delta$ uniformly on Y. By Proposition 10.1, pointwise convergence extends to Δ. By Proposition 10.2, pointwise convergence for u drops down to pointwise convergence for v. ∎

Note that Theorem 10.4 includes the case where v is supported on $\bigcup_{j=0}^k X_j$ for some k, and hence significantly extends Theorem 2.1.

In the next result, we extend Theorem 2.3, and we drop the requirement that $\sum \|L^k v_k\| < \infty$ in Theorem 10.4.

Proposition 10.5 *Let $\beta \in (0, 1]$ and $v \in \mathcal{B}(X)$.*

- *(a) For each $y \in Y$, there is a zero density set $E \subset \mathbb{N}$ such that $\lim_{n\to\infty,\, n\notin E} m(n) n^{1-\beta} (L^n v)(y) = d_\beta \int_X v \, d\mu$.*
- *(b) If $v \ge 0$, then $\liminf_{n\to\infty} m(n) n^{1-\beta} L^n v = d_\beta \int_X v \, d\mu$ pointwise on Y.*

Proof Let $w_n = d_\beta^{-1} m(n) n^{1-\beta}$. By the argument in the proof of Theorem 2.3, it suffices to prove the $\ge$ inequality in part (b). Let $v \ge 0$ and define $v(k) = \sum_{j=0}^k v_j$. By Theorem 2.3(b), $\liminf_{n\to\infty} w_n 1_Y L^n v \ge \liminf_{n\to\infty} w_n 1_Y L^n v(k) = \sum_{j=0}^k \liminf_{n\to\infty} w_n 1_Y L^n v_j = \sum_{j=0}^k \int v_j = \int v(k)$. Since k is arbitrary, $\liminf_{n\to\infty} w_n 1_Y L^n v \ge \int v$ as required. ∎

10.3 Second order asymptotics on X

Under the assumptions of Theorem 9.1, we can investigate second order asymptotics in Theorem 10.4. For example, we have the following:

Theorem 10.6 *Suppose that $\beta \in (\frac12, 1)$ and that $\mu(\varphi > n) = cn^{-\beta} + O(n^{-2\beta})$ where $c > 0$. Suppose further that $v \in \mathcal{B}(X)$ and that (i) $\|L^k v_k\| = O(k^{-p})$, and (ii) $\int |v_k| \, d\mu = O(k^{-q})$, where $p > \frac32 - \beta$ and $q > 1$. Then*

$$n^{1-\beta} L^n v = c^{-1} d_\beta \int_X v \, d\mu + O(n^{-\gamma} \|v\|) \ \text{uniformly on } Y,$$

where $\gamma = \min\{1-\beta, \beta - \frac12, q - 1\}$ if $p > 1$, and $\gamma = \min\{1 - \beta, \beta + p - \frac32, q - 1\}$ if $\frac32 - \beta < p < 1$.

Proof The estimates follow from Theorem 9.1 and the proof of Theorem 10.4. ∎

11 Examples

In this section, we apply our results to specific examples. In Subsection 11.1, we describe a method for verifying our functional-analytic hypotheses (H1), (H2), that suffices for our purposes. In Subsection 11.2, we consider situations where the first return map $F : Y \to Y$ is Gibbs-Markov. This includes the nonuniformly expanding maps studied by Thaler [38] and parabolic rational maps of the complex plane [6]. In Subsection 11.3, we consider the full class of AFN maps [44]. In Subsection 11.4, we specialise to the case of Pomeau-Manneville intermittency maps (1.1).

11.1 Verification of hypotheses (H1) and (H2)

In our examples, (H1) can be verified in the process of verifying (H2), so we focus on the latter. The standard approach (cf. [18, Lemma 6.7] and [37, Section 5]) to (H2) proceeds via the following result.

Proposition 11.1 *Suppose that $F : Y \to Y$ is ergodic. Assume that (1) $R(z) : \mathcal{B} \to \mathcal{B}$ has essential spectral radius strictly less than 1 for every $z \in \bar{\mathbb{D}}$. (2) For each $\theta \in (0, 2\pi)$, there are no nontrivial L^2 solutions to the equation $v \circ F = e^{i\theta\varphi} v$ a.e. Then (H2) is satisfied.*

Proof By (1), it suffices to consider generalized eigenfunctions $v \in L^2$ for the operator $R(z)$. Suppose that $R(z)v = v$ where $v \in L^2$ is nonzero. Write $z = \rho e^{i\theta}$, $\rho \in [0, 1]$, $\theta \in [0, 2\pi)$. Then $|v|_2 = |R(z)v|_2 = |R(\rho^\varphi e^{i\theta\varphi} v)|_2 = |\rho^\varphi v|_2 \le |\rho^\varphi|_\infty |v|_2 \le \rho |v|_2$, so $\rho = 1$ and $R(e^{i\theta})v = v$. The L^2 adjoint of $U = R(e^{i\theta})$ is the operator $U^* v = e^{-i\theta\varphi} v \circ F$ and an elementary calculation shows that $|U^* v - v|_2^2 = |Uv|_2^2 - |v|_2^2 = 0$. Hence $v \circ F = e^{i\theta\varphi} v$. By (2), $\theta = 0$. Hence we have established (H2)(ii).

When $z = 1$, the eigenvalue 1 is isolated in the spectrum by (1) and the eigenvalue is simple by ergodicity of F, so (H2)(i) is valid. ∎

Definition 11.2 Suppose that Y is a topological space, that (Y, μ) is a probability space, and that $F : Y \to Y$ is a measure preserving transformation. Let $\varphi : Y \to \mathbb{R}$ be a measurable map. We say that (F, φ) satisfies property (*) if for every $\theta \in [0, 2\pi]$ and every measurable solution $v : Y \to S^1$ to the equation $v \circ F = e^{i\theta\varphi} v$ *a.e.*, there exists an open set $U \subset Y$ such that v is constant almost everywhere on U.

Lemma 11.3 *Suppose that f is topologically mixing with first return map $F = f^\varphi : Y \to Y$. Assume that (F, φ) satisfies property (*). Then there are no nontrivial measurable solutions $v : Y \to \mathbb{C}$ to the equation $v \circ F = e^{i\theta\varphi} v$ for all $\theta \in (0, 2\pi)$.*

Proof If v is a nontrivial solution, then by (*) there is an open set $U \subset Y$ on which v is almost everywhere constant. Now we follow the second half of the proof of [18, Lemma 6.7]. Since f is topologically mixing, there exists $N \ge 1$ such that

$f^nU \cap U \neq \emptyset$ for all $n \geq N$. In particular, for each $n \geq N$ we can choose $y \in U$ such that $f^n y \in U$ and $v(y) = v(f^n y) \neq 0$.

Let $0 = k_0 < k_1 < \cdots < k_p = n$ be the successive return times of y to Y. Then $T^n y = F^p y$ and $n = \varphi_p(y) = \sum_{j=0}^{p-1} \varphi(F^j y)$. Hence

$$e^{i\theta n} = \prod_{j=0}^{p-1} e^{i\theta\varphi(F^j y)} = \prod_{j=0}^{p-1} \frac{v(F^{j+1}y)}{v(f^j y)} = \frac{v(f^n y)}{v(y)} = 1.$$

Taking $n = N$ and $n = N + 1$, we deduce that $e^{i\theta} = 1$ which is a contradiction. ∎

11.2 Maps with Gibbs-Markov first return maps

A large class of examples covered by our methods are those with first return maps that are Gibbs-Markov. This includes parabolic rational maps of the complex plane (Aaronson *et al* [6]) and Thaler's class of interval maps with indifferent fixed points [38] (in particular the family (1.1)).

We recall the key definitions [3]. Let (X, μ) be a Lebesgue space with countable measurable partition α_X. Let $f : X \to X$ be an ergodic, conservative, measure-preserving, Markov map transforming each partition element bijectively onto a union of partition elements. Recall that f is *topologically mixing* if for all $a, b \in \alpha_X$ there exists $N \geq 1$ such that $b \subset f^n a$ for all $n \geq N$.

Let Y be a union of partition elements with $\mu(Y) \in (0, \infty)$. Define the first return time $\varphi : Y \to \mathbb{R}$ and first return map $F = f^\varphi : Y \to Y$. Let α be the partition of Y consisting of nonempty cylinders of the form $a \cap (\bigcap_{j=1}^{n-1} T^{-j}\xi_j) \cap T^{-n}\alpha$ where $a, \xi_j \in \alpha_X$, and $a \subset Y$, $\xi_j \subset X \setminus Y$. Fix $\tau \in (0, 1)$ and define $d_\tau(x, y) = \tau^{s(x,y)}$ where the *separation time* $s(x, y)$ is the greatest integer $n \geq 0$ such that $F^n x$ and $F^n y$ lie in the same partition element in α. It is assumed that the partition α separates orbits of F, so $s(x, y)$ is finite for all $x \neq y$. Then d_τ is a metric. Let $\mathrm{Lip}(Y)$ be the Banach space of d_τ-Lipschitz functions $v : Y \to \mathbb{R}$ with norm $\|v\| = |v|_\infty + \mathrm{Lip}(v)$.

Define the potential function $p = \log \frac{d\mu}{d\mu \circ F} : Y \to \mathbb{R}$. We require that p is uniformly piecewise Lipschitz: that is, $p|_a$ is d_τ-Lipschitz for each $a \in \alpha$ and the Lipschitz constants can be chosen independent of a. We also require the big images condition $\inf_a \mu(Fa) > 0$. A *Gibbs-Markov map* is a Markov map with uniformly piecewise Lipschitz potential and satisfying the big images property.

Proposition 11.4 *Suppose that (X, μ) is a Lebesgue space, that $f : X \to X$ is an ergodic, conservative, measure preserving, topologically mixing, Markov map, and that $Y \subset X$ is a union of partition elements with $\mu(Y) \in (0, \infty)$. Suppose further that the first return map $F = f^\varphi : Y \to Y$ is Gibbs-Markov. Then the Banach space $\mathcal{B} = \mathrm{Lip}(Y)$ satisfies hypotheses (H1) and (H2).*

Proof Since the details can be found in [18, 37], we only sketch the argument. By equation (8) in the proof of [18, Lemma 6.7], there is a constant $C > 0$ such that $\|R(z)^n v\| \le C(|v|_\infty + \tau^n \|v\|)$ for all $v \in \mathcal{B}$, $z \in \bar{\mathbb{D}}$, $n \ge 1$. Since the unit ball in $\mathcal{B}$ is compact in L^∞, it follows from [23] that the essential spectral radius of $R(z)$ is at most τ establishing property (1) of Proposition 11.1. Property (*) follows from [4, Theorem 3.1]: measurable solutions v are constant almost everywhere on each partition element of α (even $F\alpha$). Hence property (2) of Proposition 11.1 follows from Lemma 11.3. This completes the verification of (H2). (H1) is established during the proof of [18, Lemma 6.7], see [18, Lemma 6.4]. ∎

Corollary 11.5 *In the setting of Proposition 11.4, if in addition $\mu(X) = \infty$ and $\mu(y \in Y : \varphi(y) > n)$ is regularly varying with index $\beta \in (0, 1]$, then our main results (including Theorems 2.1, 2.2, 2.3) apply.* ∎

Example 11.6 (Parabolic rational maps of the complex plane) Let $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be a rational map of the Riemann sphere with spherical metric d. A period k point $z \in \mathbb{C}$ is *rationally indifferent* if $(f^k)'(z)$ is a root of unity. The map f is *parabolic* if J contains no critical points and contains at least one rationally indifferent periodic point [11]

Aaronson *et al.* [6, Section 9] establish a number of properties of parabolic rational maps. Such maps are topologically mixing, conservative and exact with respect to Lebesgue measure and possess a σ-finite invariant measure μ equivalent to Lebesgue. Moreover, there is a Gibbs-Markov first return map $F = f^\varphi : Y \to Y$ where $\mu(Y) \in (0, \infty)$. Criteria are given for $\mu(X)$ to be finite or infinite, and in the infinite case it is shown that $\mu(\varphi > n) \sim Cn^{-\beta}$ where $C > 0$ and $\beta \in (0, 1]$. For any $\eta > 0$, it is possible to choose $\tau \in (0, 1)$ so that $C^\eta(Y) \subset \mathrm{Lip}(Y)$. By Corollary 11.5, our main results apply to Hölder observables supported on Y.

Example 11.7 (Thaler maps) Thaler [38] considers a class of topologically mixing one-dimensional maps $f : X \to X$, $X = [0, 1]$ for which there is a countable measurable partition $\{B(k) : k \in I\}$ consisting of intervals, and a nonempty finite set $J \subset I$ such that each $B(j)$, $j \in J$, contains an indifferent fixed point x_j with $f'(x_j) = 1$. It is required that

(1) $f|_{B(k)}$ is twice differentiable and $\overline{fB(k)} = [0, 1]$ for all k.

(2) $|f'| \ge \rho(\epsilon) > 1$ on $\bigcup_{k \in I} B(k) \setminus \bigcup_{j \in J}(x_j - \epsilon, x_j + \epsilon)$ for each $\epsilon > 0$.

(3) For each $j \in J$ there exists $\eta > 0$ such that f' is decreasing on $(x_j - \eta, x_j) \cap B(j)$ and increasing on $(x_j, x_j + \eta) \cap B(j)$.

(4) $f''/(f')^2$ is bounded on $\bigcup_{k \in I} B(k)$.

Such a map f is conservative and exact with respect to Lebesgue measure, and admits an infinite σ-finite invariant measure μ equivalent to Lebesgue. As a special case of the construction in Zweimüller [45], f has a Gibbs-Markov first return map $F = f^\varphi : Y \to Y$, where $\mu(Y) \in (0, \infty)$. Furthermore, for every compact set $C \subset X'$ (the complement of the indifferent fixed points), the first return set Y can be chosen to contain C. As shown in [38], a sufficient condition for regularly varying return tail probabilities is that f has a "good" asymptotic expansion near each indifferent fixed point. For example, it suffices that $f(x) = x + a_j|x - x_j|^{p_j+1} + o(|x - x_j|^{p_j+1})$ as $x \to x_j$ for each $j \in J$, where $a_j \neq 0$, $p_j \geq 1$ and $p = \max_j p_j > 1$. In this case $\mu(\varphi > n) \sim Cn^{-1/p}$ where $C > 0$.

To summarise, suppose that f is a Thaler map and $\mu(\varphi > n) = \ell(n)n^{-\beta}$ is regularly varying with $\beta \in (0, 1]$. Set $w(n) = d_\beta^{-1} m(n) n^{1-\beta}$. By Corollary 11.5, our main results apply to Hölder continuous observables $v : X \to \mathbb{R}$ supported on a compact subset of X'. In particular, if $\beta \in (\frac{1}{2}, 1]$, we obtain uniform convergence on compact subsets of X'. Moreover, we can consider a more general class of observables extending the class considered in [38]:

Theorem 11.8 *Suppose that $f : X \to X$ is a Thaler map with regularly varying tails, $\beta \in (\frac{1}{2}, 1]$. Then $\lim_{n\to\infty} w(n)L^n v = \int_X v\, d\mu$ uniformly on compact subsets of X' for all v of the form $v = \xi u$ where ξ is μ-integrable and Hölder on X, and u is Riemann integrable.*

Proof Fix a first return set Y chosen so that $f(Y) = X$. Let $v : X \to \mathbb{R}$ be μ-integrable and Hölder. Define the sets X_k as in Subsection 10.2 and write $v = \sum v_k$ where $v_k = v|_{X_k}$. We claim that $\|L^k v_k\| \ll \mu(\varphi = k + 1)\|v_k\|_{C^\eta(X_k)} \leq \mu(\varphi = k + 1)\|v\|_{C^\eta}$. Hence $\|L^k v_k\|$ is summable and it follows from Theorem 10.4 that $\lim_{n\to\infty} w(n)L^n v = \int_X v\, d\mu$ uniformly on Y. Finally, given v of the form $v = \xi u$, we approximate u from above and below by Hölder functions $u^\pm$. Then v is approximated from above and below by observables $v^\pm = \xi u^\pm$ for which uniform convergence holds. Since $\int (v^+ - v^-)\, d\mu$ can be made arbitrarily small, the result follows.

It remains to verify the claim. Every point in X_k has a preimage in Y. To simplify notation, suppose that such preimages are unique (otherwise specify one of the preimages and omit the other preimages in the argument below). Write $(Lv)(x) = \sum_{fx'=x} g(x')v(x')$ and $(L^n v)(x) = \sum_{f^n x'=x} g_n(x')v(x')$ where $g_n(x) = g(x)g(fx)\cdots g(f^{n-1}x)$. Similarly, write $(Rv)(y) = \sum_{Fy'=y} G(y')v(y')$. Then

$$\begin{aligned}(L^k v_k)(x) &= \sum_{f^k x'=x} g_k(x')1_{X_k}(x')v(x') = \sum_{y\in\{\varphi=k+1\},\, f^{k+1}y=x} g(y)^{-1}g_{k+1}(y)v(fy) \\ &= \sum_{Fy=x} g(y)^{-1}1_{\{\varphi=k+1\}}(y)G(y)v(fy) = \sum_a g(y_a)^{-1}G(y_a)v(fy_a),\end{aligned}$$

where the summation is over those a with $\varphi|_a = k + 1$, and y_a is the unique point in a such that $Fy_a = x$. For Gibbs-Markov maps it is standard that $\|1_a G\| \ll \mu(a)$.

For the systems in [38], f' is bounded and f is uniformly expanding on Y so that $\|1_Y g^{-1}\| < \infty$ and $\|1_Y(v \circ f)\| \le C\|v_k\|_{C^\eta(X_k)}$. Hence $\|L^k v_k\| \ll \sum_a \mu(a)\|v\|_{C^\eta(X_k)} = \mu(\varphi = k+1)\|v\|_{C^\eta(X_k)}$ as required. ∎

Remark 11.9 As is evident from the proof, the condition that ξ is Hölder on X can be relaxed and it suffices that ξ is Hölder on each X_k and satisfies $\sum \mu(\varphi = k)\|1_{X_k}\xi\|_{C^\eta(X_K)} < \infty$ for some $\eta > 0$.

11.3 AFN maps

Zweimüller [44, 45] studied a class of non-Markovian interval maps $f : X \to X$, $X = [0, 1]$, with indifferent fixed points. It is assumed that there is a measurable partition ξ of X into open intervals such that f is C^2 and strictly monotone on each $Z \subset \xi$, and such that the following conditions are satisfied:

(A) *Adler's condition:* $f''/(f')^2$ is bounded on $\bigcup_{Z\in\xi} Z$,

(F) *Finite images:* $\{TZ : Z \in \xi\}$ is finite.

(N) *Nonuniform expansion:* There is a finite set $\zeta \subset \xi$ such that each interval $Z \in \zeta$ has an indifferent fixed point x_Z at one of its endpoints (so $fx_Z = x_Z$ and $f'(x_Z) = 1$) such that f has a C^1 extension to $Z \cup x_Z$ and T' is increasing (resp. decreasing) on Z if x_Z is the left (resp. right) end point of Z. Moreover, $|f'| \ge \rho(\epsilon) > 1$ on $X \setminus \bigcup_{Z\in\zeta}((x_Z - \epsilon, x_Z + \epsilon) \cap Z)$ for each $\epsilon > 0$.

Such a map is called an *AFN map.* A Thaler map (Example 11.7) is an AFN map with full branches. If condition (N) is replaced by

(U) *Uniform expansion:* $|f'| \ge \rho > 1$ on $\bigcup_{Z\in\xi} Z$,

then f is called an AFU map.

By the spectral decomposition theorem in [44], any AFN map decomposes into basic sets that are topologically mixing up to a finite cycle. From now on we suppose that $f : X \to X$ is a topologically mixing AFN map. Such a map is conservative and exact with respect to Lebesgue measure, and admits an equivalent σ-finite invariant measure μ. The measure is infinite if and only if X includes an indifferent fixed point, and we suppose that this is the case. Let $X' \subset X$ denote the complement of the indifferent fixed points.

Proposition 11.10 *If $f : X \to X$ is a topologically mixing AFN map with $\mu(X) = \infty$, and C is a compact subset of X', then there exists a first return set Y with $\mu(Y) \in (0, \infty)$ such that Y contains C, and such that the first return map $F = f^\varphi : Y \to Y$ is AFU. Moreover, the Banach space $\mathcal{B} = BV(Y)$ consisting of bounded variation functions on Y satisfies hypotheses (H1) and (H2).*

Proof By [45, Lemma 8], the first return map F is AFU. By [36] and [44, Appendix]), $\mathcal{B} = BV(Y)$ is a suitable Banach space. In particular, $R(1) : \mathcal{B} \to \mathcal{B}$ has essential spectral radius less than 1. The argument in [36] is extended by [5, Proposition 4] who show that there exist constants $C > 0$, $\tau < (0,1)$ such that $\|R(e^{i\theta})^n v\| \le C(|v|_1 + \tau^n \|v\|)$ for all $v \in \mathcal{B}$, $\theta \in \mathbb{R}$, $n \ge 1$. It is easy to extend this argument to cover $R(z)$ for all $z \in \bar{\mathbb{D}}$. (It should be noted that in our setting, the proof in [5] is greatly simplified since in [5] φ is not assumed to be locally constant and F is not required to satisfy Adler's condition or finite images.) Since the unit ball in $\mathcal{B}$ is compact in L^1, property (1) of Proposition 11.1 again follows from [23]. Property (*) follows from [5, Theorems 1 and 2]: measurable solutions v are constant almost everywhere on "recurrent image sets" and there are plenty of such sets by [5, Theorem 3(4)], yielding property (2) of Preposition 11.1. Again, (H1) can be verified en route to the estimate for $\|R(z)^n v\|$ (the crucial estimate is stated in [5, p. 57, line 10] and is a simple consequence of the AFU structure). ∎

If in addition, $\mu(y \in Y : \varphi(y) > n) = \ell(n)n^{-\beta}$ is regularly varying with $\beta \in (0,1]$ (which includes the case when f has good asymptotic expansions near each indifferent fixed point as in Example 11.7), then again our main results apply. In particular, for $\beta \in (\frac{1}{2}, 1]$ it follows that for every BV observable $v : X \to \mathbb{R}$ supported on a compact subset of X', $\lim_{n\to\infty} w(n)L^n v = \int_X v\, d\mu$ uniformly on compact subsets of X', where $w_n = d_\beta^{-1} m(n) n^{1-\beta}$. Again, we can consider a much larger class of observables as in Theorem 1.1.

Proof of Theorem 1.1 Part (a) is identical to the proof of Theorem 11.8 with Hölder replaced by BV. By Theorem 10.4, it suffices to observe that $\sum \|L^k v_k\| \ll \sum \mu(\varphi = k)\|1_{X_k} v\|_{BV(X_k)} \le \|v\|_{BV}$. Parts (b) is proved using the argument in Proposition 10.5. To prove (c), by positivity of L^n we can suppose without loss that v is μ-integrable and BV, so $\|L^k v_k\| \ll \mu(\varphi = k)$ and hence $\|L^n v\| \le \sum_{k=0}^n \|T_{n-k}\|\|L^k v_k\| \ll [\{\|T_n\|\} \star \{\mu(\varphi = n)\}]_n \ll \ell(n)n^{-\beta}$. ∎

Remark 11.11 (i) The proof of Theorem 1.1 shows that the hypotheses are easily generalised. In (a), it suffices that ξ is μ-integrable, $\mu(\varphi = k)\|1_{X_k} v\|_{BV(X_k)}$ is summable, and u is Riemann-integrable. In (b), it suffices that v is μ-integrable and Riemann-integrable (and the result holds for all β). In (c), it suffices that $|v| \le v'$ where v' is μ-integrable and BV.

(ii) For Thaler's maps, which are AFN with Gibbs-Markov first return maps, we can work with Hölder or BV norms.

(iii) For the Pomeau-Manneville map we obtain the stronger estimate $\|L^k v_k\| \ll \mu(\varphi = k)\|1_{X_k} v\|_{BV(X_k)} \ll k^{-(\beta+1)}\|1_{X_k} v\|_{BV(X_k)}$, so in Theorem 1.1(a) it suffices (somewhat remarkably) that ξ is μ-integrable and BV on each X_k with the BV norms growing no faster than $k^{\beta-\epsilon}$.

11.4 Pomeau-Manneville maps

In this subsection, we verify that our results on second order asymptotics are applicable to certain Pomeau-Manneville intermittency maps, in particular the family (1.1) studied by Liverani *et al.* [31]. Write the invariant measure as $d\mu = h\,dm$ where m is Lebesgue measure and h is the density.

Proposition 11.12 *Suppose that* $f : X \to X$ *is given as in* (1.1) *with* $\beta = 1/\alpha \in (0, 1]$. *Then* $\mu(\varphi > n) = cn^{-\beta} + O(n^{-2\beta})$ *where* $c = \frac{1}{4}\beta^\beta h(\frac{1}{2})$.

Proof First, let $Y = [\frac{1}{2}, 1]$ with partition sets $Y_j = \{\varphi = j\}$. Let $x_n \in (0, \frac{1}{2}]$ be the sequence with $x_0 = \frac{1}{2}$ and $x_n = fx_{n+1}$, so x_n is decreasing and $x_n \to 0$. A standard argument shows that $x_n \sim \frac{1}{2}\beta^\beta n^{-\beta}$ (cf. [37, Corollary 1]) and moreover that $x_n = \frac{1}{2}\beta^\beta n^{-\beta} + O((\log n)n^{-(\beta+1)})$.

Write $Y_n = [y_{n-1}, y_{n-2}]$. Then $f([\frac{1}{2}, y_n]) = [0, x_n]$. In particular $m(\varphi > n) = \frac{1}{2}m([0, x_{n-1}]) = \frac{1}{2}x_{n-1} = \frac{1}{4}\beta^\beta n^{-\beta} + O((\log n)n^{-(\beta+1)})$.

The density h is globally Lipschitz on $(\epsilon, 1]$ for any $\epsilon > 0$ (see for example [25] or [31, Lemma 2.1]. Hence $\mu(\varphi > n) = m(\varphi > n)(h(\frac{1}{2}) + O(n^{-\beta}))$, and the result for $Y = [\frac{1}{2}, 1]$ follows. The same estimates are obtained by inducing on the set $Y = [x_q, 1]$ for any fixed $q \geq 0$. ∎

The next result is immediate by Theorems 9.1 and 9.6.

Corollary 11.13 *Suppose that* $f : X \to X$ *is given as in* (1.1) *with* $\beta \in (\frac{1}{2}, 1]$. *Suppose that* $v : [0, 1] \to \mathbb{R}$ *is Hölder or bounded variation supported on a compact subset of* $(0, 1]$. *Let* $m(n) = c$ *if* $\beta \in (\frac{1}{2}, 1)$ *and* $m(n) = c\log n$ *if* $\beta = 1$. *Let* $\gamma = \min\{1 - \beta, \beta - \frac{1}{2}\}$. *Then*

$$m(n)n^{1-\beta}L^n v = d_\beta \textstyle\int_X v\,d\mu + O(m(n)^{-1}n^{-\gamma}) \text{ uniformly on compact subsets of } (0, 1].$$

Moreover, if $\beta \in (\frac{3}{4}, 1]$, *then* $\lim_{n\to\infty} m(n)n^{1-\beta}\{m(n)n^{1-\beta}L^n v - d_\beta \int_X v\,d\mu\} = d_{\beta,1} \int_X v\,d\mu$, *uniformly on compact subsets of* $(0, 1]$, *where typically* $d_{\beta,1} \neq 0$. ∎

Finally, we mention a result on second order asymptotics for observables $v(x) = x^q$.

Theorem 11.14 *Suppose that* $f : X \to X$ *is given as in* (1.1) *with* $\beta \in (\frac{1}{2}, 1]$. *Let* $v(x) = x^q$ *where* $(1 + q)\beta > 1$. *Let* $\gamma = \min\{1 - \beta, \beta - \frac{1}{2}, (1 + q)\beta - 1\}$. *Then*

$$m(n)n^{1-\beta}L^n v = d_\beta \textstyle\int_X v\,d\mu + O(m(n)^{-1}n^{-\gamma}) \text{ uniformly on compact subsets of } (0, 1].$$

Moreover, if $\beta \in (\frac{3}{4}, 1]$ *or* $q\beta < \frac{1}{2}$, *then* $\lim_{n\to\infty} m(n)n^\gamma\{m(n)n^{1-\beta}L^n v - d_\beta \int_X v\,d\mu\} = d' \int_X v\,d\mu$, *uniformly on compact subsets of* $(0, 1]$, *where typically* $d' \neq 0$.

Proof We give the details for $\beta \in \frac{1}{2}, 1)$. By Theorem 9.1, $T_n = n^{-(1-\beta)}P + O(n^{-2(1-\beta)}P) + O(n^{-\frac{1}{2}})$. Following the proof of Theorem 10.4,

$$n^{1-\beta}L^n v - \textstyle\int v = n^{1-\beta}\displaystyle\sum_{j=0}^{n} T_{n-j}L^j v_j - \textstyle\int v = A + B + C + D,$$

$$A = \sum_{j=0}^{n}\{(1-j/n)^{-(1-\beta)} - 1\}\textstyle\int|v_j| \ll n^{-1}\displaystyle\sum_{j=0}^{n} j\textstyle\int|v_j|, \quad B = \displaystyle\sum_{j>n}\textstyle\int|v_j|,$$

$$C = O\Big(n^{1-\beta}\sum_{j=0}^{n}(n-j)^{-2(1-\beta)}\textstyle\int|v_j|\Big), \quad D = O\Big(n^{1-\beta}\displaystyle\sum_{j=0}^{n}(n-j)^{-\frac{1}{2}}\|L^j v_j\|\Big).$$

Since $x_n = \frac{1}{2}\beta^\beta n^{-\beta}(1 + O((\log n)n^{-1}))$, $h(x) = O(x^{-1/\beta})$ and h^{-1} is uniformly Lipschitz, it follows that $\int v_n\, d\mu = \int_{x_n}^{x_{n-1}} x^q h(x)\, dx \ll n^{-(1+q)\beta}$ accounting for A, B and C. By the proof of Theorem 1.1, $\|L^n v_n\|$ is summable taking care of D. This proves the first statement.

If $\beta \in (\frac{3}{4}, 1]$ or $q\beta < \frac{1}{2}$ then the $\beta - \frac{1}{2}$ component of γ is negligible and we obtain higher order expansions and so second order asymptotics. ∎

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