New planforms in systems of partial differential equations with Euclidean symmetry *

I. Bosch Vivancos P. Chossat
I. Melbourne †
Institut Non Linéaire de Nice
CNRS — Université de Nice
1361, route des Lucioles
06560 Valbonne
France

September 19, 1994

Abstract

Dionne and Golubitsky [10] consider the classification of planforms bifurcating (simultaneously) in scalar PDEs that are equivariant with respect to the Euclidean group in the plane. In particular, those planforms corresponding to isotropy subgroups with one-dimensional fixed-point space are classified.

Many important Euclidean-equivariant systems of PDEs essentially reduce to a scalar PDE, but this is not always true for general systems. We extend the classification of [10] obtaining precisely three planforms that can arise for general systems and do not exist for scalar PDEs. In particular, there is a class of one-dimensional 'pseudoscalar' PDEs for which the new planforms bifurcate in place of three of the standard planforms from scalar PDEs. For example the

*Appeared: *Arch. Rational. Mech. Anal.* **131** (1995) 199–224

†Supported in part by NSF Grant DMS-9101836, by ONR Grant N00014-94-1-0317 and by the CNRS. Permanent address: Department of Mathematics, University of Houston, Houston, Texas 77204-3476, USA

usual rolls solutions are replaced by a nonstandard planform called anti-rolls. Scalar and pseudoscalar PDEs are distinguished by the representation of the Euclidean group.

1 Introduction

Systems of PDEs such as the Navier-Stokes equations, the Boussinesq equations (modelling the planar Bénard problem), the Kuramoto-Sivashinsky equation and reaction-diffusion equations have Euclidean symmetry when posed on the whole of $\mathbb{R}^n$. A common approach to such systems of PDEs is to search for spatially-periodic solutions (often called *planforms*), for an overview see Cross and Hohenberg [7]. In the specific case of the planar Bénard problem, see for example Busse [5], Schlüter et al [17] and Kirchgässner [13].

In such systems of PDEs, steady-state spatially-periodic solutions are found to bifurcate as a 'trivial' Euclidean-invariant solution loses (linear) stability. An important but intractable problem is to classify such bifurcating solutions. However a large subclass of these solutions has been classified in Dionne and Golubitsky [10] for $n = 2$. (For the case $n = 3$, see Dionne [9].) An important tool in equivariant bifurcation theory is the so-called equivariant branching lemma, see for example [12]. This lemma guarantees the existence of equilibria possessing certain symmetries (the isotropy subgroup of the solution) provided a certain algebraic criterion is satisfied. Thus the idea in [10] is to classify those solutions whose existence is guaranteed by the equivariant branching lemma. Borrowing some terminology from [11] we refer to such isotropy subgroups as *axial* and the corresponding spatially periodic solutions as *axial planforms*.

The problem of classifying axial planforms now becomes an algebraic one. Note that the resulting classification is partial since it is well known that in general there may exist additional solutions whose existence does not follow from the equivariant branching lemma. Nevertheless, the axial planforms include the usual rolls, rectangles, simple squares and simple hexagons. Moreover the methods in [10] lead to new solutions: squares, anti-squares and hexagons (see also [13]). (This terminology is due to [10]. Traditionally, simple squares for example were called squares. The discovery of more complicated square solutions necessitates the change in terminology.)

There are two main observations in this paper. The first observation is that there are Euclidean-equivariant systems of PDEs in the plane for which the classification in [10] is not appropriate. We prove that are precisely ten different axial planforms: the seven obtained by [10] and three new planforms which we call anti-rolls, simple anti-squares, and simple oriented hexagons. These three planforms are analogous to, but different from, rolls, simple squares and simple hexagons, see Figures 1, 2 and 3.

The second observation concerns the realization of these planforms in Euclidean-equivariant PDEs. There is a reduction process (which we do not attempt to make precise) whereby many PDEs, including those in the opening paragraph when $n = 2$, reduce to a one-dimensional *scalar* PDE in the plane. In particular, the seven axial planforms obtained by [10] exist simultaneously for any system of PDEs that reduces to a scalar PDE. However, some Euclidean-equivariant systems of PDEs reduce to a second class of one-dimensional PDEs which we call *pseudoscalar*. Again there exist seven axial planforms for such PDEs, but the standard rolls, simple squares and simple hexagons are replaced by the three new planforms.

In general, a Euclidean-equivariant system of PDEs need not reduce either to a scalar or a pseudoscalar PDE and we may then obtain a different combination of axial planforms bifurcating simultaneously. A description of the possible combinations is beyond the scope of this paper. However we prove, for arbitrary Euclidean-equivariant systems of PDEs, that no further axial planforms are possible other than the ten planforms that have been mentioned.

Both scalar and pseudoscalar PDEs are one-dimensional Euclidean-equivariant PDEs and are posed on a function space consisting of functions $u : \mathbb{R}^2 \to \mathbb{R}$. The distinction lies in the fact that the PDEs are equivariant with respect to different representations of the Euclidean group. In the scalar case, an isometry ϕ acts on a function u by transforming the domain variables in $\mathbb{R}^2$ in the standard way. In symbols

$$u(x) \to u(\phi^{-1}(x)).$$

The pseudoscalar action is the same when ϕ is a rotation or a translation but if ϕ is a reflection the action is given by

$$u(x) \to -u(\phi^{-1}(x)).$$

At first glance, this action might seem quite artificial. However on a purely theoretical level, the scalar and pseudoscalar actions are on an equal footing.

Moreover, it transpires that the two-dimensional Navier-Stokes equations reduces to a pseudoscalar PDE rather than a scalar PDE.

Dionne and Golubitsky [10, Theorem 2.3] prove that under reasonable hypotheses, scalar Euclidean-equivariant PDEs in the plane (and more generally, systems of PDEs that reduce to such a scalar PDE) undergoing steady-state bifurcation from a trivial solution admit simultaneously branches of axial planforms corresponding to each of the following:

1. Rolls, symmetric pitchfork
2. Rectangles (a continuum), symmetric pitchfork
3. Simple squares, symmetric pitchfork
4. Simple hexagons, transcritical
5. Squares (countably many), symmetric pitchfork
6. Anti-squares (countably many), symmetric pitchfork
7. Hexagons (countably many), transcritical

We show that, under the same hypotheses as in [10], steady-state bifurcation for pseudoscalar PDEs in the plane leads to simultaneous bifurcation of the following axial planforms:

1. Anti-rolls, symmetric pitchfork
2. Rectangles (a continuum), symmetric pitchfork
3. Simple anti-squares, symmetric pitchfork
4. Simple oriented hexagons, symmetric pitchfork
5. Squares (countably many), symmetric pitchfork
6. Anti-squares (countably many), symmetric pitchfork
7. Hexagons (countably many), asymmetric pitchfork

It is interesting to observe that in the pseudoscalar case there are no longer any axial planforms bifurcating transcritically. (See Subsection 3.3 for an explanation of terminology such as symmetric and asymmetric pitchfork. Roughly speaking, a symmetric pitchfork consists of two half-branches of equilibria that are related by symmetry. An asymmetric pitchfork consists of two half-branches bifurcating in the same direction but not related by symmetry.)

At present there appears to be no physical examples of primary steady-state bifurcation to nonstandard planforms. We note that the issues discussed in this paper find a natural context in secondary bifurcations in magnetic dynamo problems, see Bosch Vivancos et al [3] (indeed it was this work that motivated this paper). However pseudoscalar PDEs do arise naturally in theoretical physics. Moreover, there are much-studied PDEs in which the basic solution, called the Kolmogorov flow, is a nonstandard planform. We note that the distinction between standard and nonstandard planforms as depicted in Figures 1, 2 and 3 is quite natural when seen from the viewpoint of fluid dynamics. These points are discussed in Section 2. Also we show how pseudoscalar PDEs give rise to the new planforms, contrasting these results with the standard results for scalar PDEs and their planforms.

The remainder of the paper is concerned with extending the classification in [10] and so obtaining a complete description of axial planforms for Euclidean-equivariant systems of PDEs in the plane. In Section 3 we describe the formulation of the classification as an algebraic problem, largely following [10]. Then in Section 4 we recall the classification in [10] for scalar PDEs while in Section 5 we obtain the analogous classification for pseudoscalar PDEs. Finally, in Section 6, we prove that for general Euclidean-equivariant systems of PDEs there are no further axial planforms other than those that arise for scalar and pseudoscalar PDEs.

2 New planforms in the plane

In this section we give a concrete illustration of how nonstandard planforms arise in Euclidean-equivariant systems of PDEs in the plane. In order to fix ideas we begin with a simple scalar PDE known as the Swift-Hohenberg equation [7] and recall how this leads to standard planforms such as rolls solutions. Then we make a slight modification to the Swift-Hohenberg equation but leaving the linear terms unchanged. The new equation is called a *pseudoscalar* equation and the branching for such equations is analogous to that for scalar equations — except that the symmetries of the bifurcating solutions are changed. For example, the standard planform rolls is replaced by a new nonstandard planform that we call *anti-rolls*.

We go on to discuss how pseudoscalar equations arise naturally in theoretical physics. In particular, the standard reduction of the two-dimensional Navier-Stokes equations leads not to a scalar PDE but to a pseudoscalar PDE. Moreover, the Kolmogorov flows associated to certain symmetry-breaking forcing terms are nonstandard planforms.

(a) A scalar PDE and rolls solutions

The Swift-Hohenberg equation is given by

$$\partial_t u = \lambda u - (\Delta + 1)^2 u + \beta u^2 - u^3, \tag{2.1}$$

where $u : \mathbb{R}^2 \to \mathbb{R}$. Here $\lambda \in \mathbb{R}$ is the bifurcation parameter and $\beta \in \mathbb{R}$. We denote points in the plane by $x = (x_1, x_2) \in \mathbb{R}^2$. Let $\mathbf{E}(2)$ denote the Euclidean group acting on the plane by translation, rotation and reflection. The PDE is equivariant under the action of $\mathbf{E}(2)$ defined by

$$\phi \cdot u(x) = u(\phi^{-1}x), \quad \phi \in \mathbf{E}(2). \tag{2.2}$$

Indeed equivariance with respect to this action of $\mathbf{E}(2)$ may be taken as the definition of a *scalar* $\mathbf{E}(2)$-equivariant PDE. We note that often β is taken to be zero. We demand that $\beta \neq 0$ so that the PDE is not equivariant under the transformation $u \to -u$.

A steady-state bifurcation occurs when the trivial solution $u = 0$ loses stability as λ varies. The linear stability of the trivial solution is computed by looking for solutions of the linear equation

$$L_\lambda u = \lambda u - (\Delta + 1)^2 u = 0,$$

in the form of Fourier modes, or *wave functions*, $u = e^{ik\cdot x}$. The vector $k \in \mathbb{R}^2$ is called the *wave vector* and $|k|$ is the wave number. Substituting this form of u into the equation $L_\lambda = 0$ yields the 'neutral stability curve'

$$\lambda - (|k|^2 - 1)^2 = 0.$$

In particular, the spectrum of L_λ consists of the real interval $(-\infty, \lambda)$. It follows from the principle of linear stability that the solution $u = 0$ is asymptotically stable for $\lambda < 0$ and unstable for $\lambda > 0$. Observe that the 'most unstable' wave functions are those with *critical* wave number $|k| = 1$. Hence there is a circle of critical wave vectors. The corresponding wave functions are also called critical.

We search for branches of planforms bifurcating from the trivial solution as λ passes through zero. Since wave functions with wave number bounded away from 1 are damped, it is usual to look for solutions consisting of sums of critical wave functions. The simplest of such *planforms* is the rolls solution which at leading order has the form

$$u(x) = a(e^{ix_1} + e^{-ix_1}), \quad a > 0.$$

Substituting u into the full equations and ignoring terms consisting of non-critical wave functions we obtain the branching equation

$$\lambda a - 3a^3 = 0.$$

This yields a (symmetric) pitchfork bifurcation of rolls solutions,

$$u(x) = \pm\sqrt{\lambda/3}(e^{ix_1} + e^{-ix_1}) + O(\lambda^{3/2}). \tag{2.3}$$

(We note that this formal analysis can be completely justified by a center manifold or Liapunov-Schmidt reduction.)

(b) A pseudoscalar equation and anti-rolls solutions

Now we make an at first sight insignificant change to the Swift-Hohenberg equation: we replace the quadratic term βu^2 by $\beta Q(u)$ where

$$Q(u) = \partial_{x_1}(\Delta u \partial_{x_2} u) - \partial_{x_2}(\Delta u \partial_{x_1} u).$$

(In the next subsection we shall see that this quadratic term is not as unnatural as it first appears.) Hence we obtain the modified PDE

$$\partial_t u = \lambda u - (\Delta + 1)^2 u + \beta Q(u) - u^3. \tag{2.4}$$

The first thing to notice is that since the linear terms are unchanged, the linear stability analysis is unaltered. Moreover the solutions that we previously called rolls bifurcate exactly as before.

In fact things have changed but in a subtle manner. The PDE is no longer equivariant with respect to the action of $\mathbf{E}(2)$ given in equation (2.2). It is clear that equivariance under translations in $\mathbf{E}(2)$ is preserved. Rotation-equivariance is also preserved. However if κ is a reflection, for example the reflection acting on $x \in \mathbb{R}^2$ as $(x_1, x_2) \mapsto (-x_1, x_2)$, then the PDE (2.4) is equivariant with respect to the action $u(x) \mapsto -u(\kappa x)$. To sum up, the PDE is $\mathbf{E}(2)$-equivariant if we define the action of $\mathbf{E}(2)$ as follows:

$$\phi \cdot u(x) = \begin{cases} u(\phi^{-1}x), & \phi \in \mathbf{E}(2) \text{ a translation or a rotation.} \\ -u(\phi^{-1}x), & \phi \in \mathbf{E}(2) \text{ a reflection.} \end{cases} \tag{2.5}$$

We say that a PDE is *pseudoscalar* if it is $\mathbf{E}(2)$-equivariant with the action of $\mathbf{E}(2)$ as defined in (2.5).

Now consider the symmetry properties of the solution branch (2.3) viewed as a rolls solution for the scalar PDE (2.1). Rolls are invariant under discrete translation by multiples of 2π parallel to the x_1-axis. The corresponding subgroup of $\mathbf{E}(2)$ is isomorphic to the integers $\mathbb{Z}$. In addition, there is the continuous group (isomorphic to $\mathbb{R}$) of all translations parallel to the x_2-axis. Hence, the translation symmetry of rolls is given by the subgroup $\mathbb{Z}\times\mathbb{R}$. The subgroup of rotations and reflections that leaves the rolls solution invariant is isomorphic to $\mathbb{D}_2$, rotation through π and reflection in each of the coordinate axes. The full symmetry group of rolls can be represented by the subgroup of $\mathbf{E}(2)$

$$\mathbb{D}_2 \dot{+} (\mathbb{Z} \times \mathbb{R}).$$

(We shall say more about subgroups of $\mathbf{E}(2)$ in Section 3.1.)

Now consider the same solution but as a planform for the pseudoscalar PDE (2.4). Invariance under translations and rotations is unchanged, but there are no reflections that leave the planform invariant. It is necessary to combine the reflections in $\mathbb{D}_2$ with *a translation by* π *parallel to the* x_1*-axis*. The symmetries are therefore given by

$$\mathbb{D}_2^- \dot{+} (\mathbb{Z} \times \mathbb{R}),$$

where $\mathbb{D}_2^-$ is a 'twisted' version of $\mathbb{D}_2$. We call these planforms *anti-rolls*.

It is not completely straightforward to see how the differences in the symmetry of rolls and anti-rolls are manifested in physical space. This is due to the fact that at first order the rolls and anti-rolls that we have computed are identical. This is no longer the case for systems of PDEs. Systems are discussed later on in this section and the corresponding visualizations of rolls and anti-rolls are strikingly different, see Figure 1.

There is a simple reason why we cannot distinguish between rolls and anti-rolls at first order in scalar and pseudoscalar PDEs. Odd order terms of a PDE commute with the transformation $u \mapsto -u$. Given this additional symmetry, there is no longer any distinction between the actions (2.2) and (2.5). In particular, any linear scalar PDE is automatically pseudoscalar and vice versa. (It should now be clear why we demand $\beta \neq 0$ in (2.1).)

If instead of rolls we look for (simple) square solutions, we obtain the branching equation

$$u(x) = \pm\sqrt{\lambda/3}(e^{ix_1} + e^{-ix_1} + e^{ix_2} + e^{-ix_2}) + O(\lambda^{3/2}).$$

As before, we obtain a standard planform *simple squares* and a nonstandard planform *simple anti-squares* with symmetries $\mathbb{D}_4 \dotplus \mathbb{Z}^2$ and $\mathbb{D}_4^- \dotplus \mathbb{Z}^2$ respectively, see Figure 2. The discrete subgroup (or lattice) of translations $\mathbb{Z}^2$ consists of translation by 2π parallel to each of the coordinate axes.

The situation for hexagons is slightly different. In the scalar PDE (2.1) the branching equation is determined at quadratic order and so we obtain a transcritical branch

$$u(x) = (-\lambda/2\beta)(e^{ix_2} + e^{i(\sqrt{3}x_1 - x_2)/2} + e^{i(-\sqrt{3}x_1 - x_2)/2} + \text{c.c.}) + O(\lambda^2),$$

where c.c. denotes complex conjugates. However, a calculation shows that in the pseudoscalar PDE (2.4) the branching is determined at cubic order (even though there are nontrivial quadratic terms) and we have the supercritical branch

$$u(x) = \pm\sqrt{\lambda/3}(e^{ix_2} + e^{i(\sqrt{3}x_1 - x_2)/2} + e^{i(-\sqrt{3}x_1 - x_2)/2} + \text{c.c.}) + O(\lambda^{3/2}).$$

The standard planform *simple hexagons* has symmetries $\mathbb{D}_6 \dotplus \mathbb{Z}^2$. However the nonstandard planform does not have symmetry $\mathbb{D}_6^- \dotplus \mathbb{Z}^2$ as might be expected, but $\mathbb{Z}_6 \dotplus \mathbb{Z}^2$. It is no longer possible to cancel out the extra minus sign coming from the pseudoscalar action of the reflections. We call the nonstandard planform in this case *simple oriented hexagons*, see Figure 3.

Remark 2.1 The quadratic term $Q(u)$ in the pseudoscalar PDE (2.4) can be written as

$$Q(u) = \frac{\partial(\Delta u)}{\partial x_1}\frac{\partial u}{\partial x_2} - \frac{\partial(\Delta u)}{\partial x_2}\frac{\partial u}{\partial x_1}.$$

If u is a sum of critical wave functions then $\Delta u = -u$ and it follows that $Q(u) = 0$. In particular, all planforms undergo pitchfork bifurcations.

It remains to show that these results do not depend on the particular quadratic term that we have used. In fact a tedious calculation shows that $Q(u)$ is a combination of terms such as

$$Q(u) = \frac{\partial(\Delta^r u)}{\partial x_1}\frac{\partial(\Delta^s u)}{\partial x_2} - \frac{\partial(\Delta^r u)}{\partial x_2}\frac{\partial(\Delta^s u)}{\partial x_1},$$

where $r > s \geq 0$. (We have taken the simplest case $r = 1$, $s = 0$.) The same argument as before shows that $Q(u) = 0$ if u is a sum of critical wave functions.

(c) Pseudoscalar equations in physics

Partial differential equations for *pseudoscalar* fields occur naturally in physics. For example, it is a well known fact that the two-dimensional Navier-Stokes equations reduce to a single PDE for a quantity called the *stream function.* It transpires that this PDE is pseudoscalar.

The Navier-Stokes equations (in the plane) have the form

$$\begin{aligned} \partial_t \mathbf{V} + (\mathbf{V} \cdot \nabla)\mathbf{V} &= \nu \Delta \mathbf{V} + \nabla P + \mathbf{F} \\ \nabla \cdot \mathbf{V} &= 0 \end{aligned} \tag{2.6}$$

where $\mathbf{V} : \mathbb{R}^2 \to \mathbb{R}^2$ is the velocity of an incompressible fluid, $P : \mathbb{R}^2 \to \mathbb{R}$ is the pressure, $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ is a force which may depend on $\mathbf{V}$, and ν is the kinematic viscosity of the fluid. With the possible exception of $\mathbf{F}$, this system is equivariant under the standard action of $\mathbf{E}(2)$ on *vector fields*

$$\begin{aligned} \phi \cdot \mathbf{V}(x) &= \phi \mathbf{V}(\phi^{-1}x), \qquad \phi \in \mathbf{E}(2) \text{ a rotation or a reflection.} \\ \phi \cdot \mathbf{V}(x) &= \mathbf{V}(\phi^{-1}x), \qquad \phi \in \mathbf{E}(2) \text{ a translation.} \end{aligned} \tag{2.7}$$

We shall assume that $\mathbf{F}$, and hence the system (2.6), is equivariant with respect to this action of $\mathbf{E}(2)$. (Strictly speaking, the PDE (2.6) is an equation in $\mathbf{V}$ and P and we should include the scalar action of $\mathbf{E}(2)$ on P as in (2.2). However this is of no consequence since P is eliminated shortly.)

The stream function ψ is defined by the equation

$$\Delta \psi = \mathrm{curl}(\mathbf{V}) = \frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2}.$$

Conversely, given ψ we can recover a divergence-free vector field $\mathbf{V}$ with stream function ψ from the equations

$$V_1 = -\frac{\partial \psi}{\partial x_2}, \quad V_2 = \frac{\partial \psi}{\partial x_1}.$$

A calculation shows that if $\mathbf{V}$ transforms under $\mathbf{E}(2)$ as a vector field, then $\mathrm{curl}(\mathbf{V})$ transforms as a pseudoscalar. More precisely,

$$\phi \cdot \mathrm{curl}(\mathbf{V}) = \mathrm{curl}(\phi \cdot \mathbf{V}), \quad \phi \in \mathbf{E}(2),$$

where the actions of $\mathbf{E}(2)$ on the left-hand and right-hand sides are those in (2.5) and (2.7) respectively. Hence it follows from elementary vector calculus that the system (2.6) reduces to the single pseudoscalar PDE

$$\partial_t \Delta \psi + \mathrm{curl}[\Delta \psi \nabla \psi] = \nu \Delta^2 \psi + \mathrm{curl}(\mathbf{F}). \tag{2.8}$$

Note that the quadratic term curl$[\Delta\psi\nabla\psi]$ is precisely the term $Q(u)$ in the pseudoscalar equation (2.4). (It is traditional to use ψ rather than u for the stream function curl$(\mathbf{V})$.)

We can proceed as in the previous subsection to compute branches of anti-rolls, simple anti-squares and simple oriented hexagons. (But note by Remark 2.1 that the forcing term $\mathbf{F}$ must include cubic terms for nondegenerate branching to occur.) Again these nonstandard planforms are indistinguishable from the standard planforms for scalar PDEs at first order. However the difference becomes significant if we return to the vector field $\mathbf{V}$ from which ψ was derived. For example, the rolls solution $\psi(x) = a(e^{ix_1} + e^{-ix_1})$ becomes $\mathbf{V}(x) = a(0, ie^{ix_1} - ie^{-ix_1})$. Applying the translation $x_1 \mapsto x_1 - \pi/2$ we can work with the more convenient representative $\mathbf{V}(x) = a(0, e^{ix_1} + e^{-ix_1})$. The visualization of rolls and anti-rolls in Figure 1 is obtained by plotting the planar vector fields

$$\mathbf{V}(x) = (e^{ix_1} + e^{-ix_1}, 0), \qquad \mathbf{V}(x) = (0, e^{ix_1} + e^{-ix_1}).$$

Only the second of these vector fields corresponds to a solution of the two-dimensional Navier-Stokes equations. The first vector field can be viewed as the horizontal section of a solution of the *three-dimensional* Navier-Stokes equations for a fluid in a horizontally unbounded domain with prescribed conditions at the upper and lower boundaries (as for example in Rayleigh-Bénard convection). The planforms in Figure 1 and also in Figures 2 and 3 thus have a natural interpretation in terms of the flow of an incompressible fluid — the standard planforms correspond to the flow of a 'real' three-dimensional fluid. In two dimensions, there is not room to flow in this way, and so a two-dimensional fluid flow would correspond to a nonstandard planform. On the other hand, a three-dimensional fluid could in principle correspond to a nonstandard planform but clearly does not behave that way (at least for planar convection problems)!

Although physical fluid flows must correspond to standard rather than nonstandard planforms, there are a priori no such restrictions on physical objects that are not fluids. Thus pseudoscalar equations and nonstandard planforms may well arise in applications that do not come from fluid dynamics.

Similar observations apply in problems with spherical symmetry. In such problems, steady-state bifurcation leads generically to a $(2\ell+1)$-dimensional $\mathbf{O}(3)$-equivariant system of ODEs where ℓ is any nonnegative integer. The action of $\mathbf{SO}(3)$ is irreducible and is the action on spherical harmonics of degree ℓ. In addition, the element $-I$ in $\mathbf{O}(3)$ acts as plus or minus the identity. See Chossat et al [6] for details. Many physical problems lead to the so-called 'natural' representations where $-I$ acts as $(-I)^{\ell}$. However, it can be shown that the two-dimensional Navier-Stokes equations reduces instead to the 'unnatural' representations where $-I$ acts as $(-I)^{\ell+1}$. Again we expect that unnatural representations of $\mathbf{O}(3)$ may arise in applications that do not come from fluid dynamics.

(d) The Kolmogorov flow

We have shown that the two-dimensional Navier-Stokes equations reduces to a pseudoscalar equation provided the forcing term $\mathbf{F}$ is appropriately equivariant. Although this situation does not appear to have been considered in the literature, there has been much interest in the case when $\mathbf{F}$ is partially equivariant, but breaks most of the symmetry, see for example [14], [1] and the references therein.

Consider the force $\mathbf{F}(x) = (\sin x_2, 0)$ (after normalization). The corresponding one-dimensional PDE has a 'trivial' solution with stream function $\psi(x) = -\nu^{-1}\cos x_2$. This solution is called the *Kolmogorov flow* associated with this particular forcing. The velocity representation is $V(x) = (-\nu^{-1}\sin x_2, 0)$ so the solution is nothing other than anti-rolls. Similarly, if we take the force $\mathbf{F}(x) = (\sin x_2, -\sin x_1)$ then the associated Kolmogorov flow has the symmetry of simple anti-squares.

These examples show that nonstandard planforms in two-dimensional systems have already been studied in the physics literature. However the fact that the associated Kolmogorov flows in these cases are anti-rolls and simple anti-squares rather than rolls and simple squares appears to have been overlooked. This is probably due to the fact that the distinction between standard and nonstandard planforms is only at high order in the reduced PDE for the stream function.

3 The algebraic formulation

By restricting to axial planforms, Dionne and Golubitsky [10] were able to reduce the problem of classifying planforms to an algebraic problem. In this section, we describe this reduction, largely following [10]. In Subsection 3.1

we describe the class of representations of $\mathbf{E}(n)$ that we shall consider. Also we consider the structure of the isotropy subgroups for these representations, particularly for spatially periodic solutions. The analytic side of the problem is considered in Subsection 3.2 and we describe the procedure for the classification of axial planforms. Branching of planforms occurs via three distinct bifurcations: transcritical, symmetric pitchfork and asymmetric pitchfork. This terminology is explained in Subsection 3.3.

It is worth pointing out where our exposition differs from that of [10]. It is well known that in searching for spatially periodic solutions with a given spatial periodicity, it is is possible to reduce the noncompact group of symmetries $\mathbf{E}(n)$ to a compact group Γ. This is crucial for technical reasons (such as applying the implicit function theorem) and we consider the reduced problem in Subsection 3.2. However in general we work inside of $\mathbf{E}(n)$ as much as possible. There are several advantages to this approach:

(i) The symmetries of spatially periodic solutions in physical space correspond to noncompact isotropy subgroups of $\mathbf{E}(n)$. Moreover, the connection with crystallographic groups becomes apparent. Another point is that the procedure followed by [10] cannot distinguish antirolls from rolls. (Here we are resisting the temptation to reduce from $\mathbf{E}(2)$ to $\mathbf{E}(1)$.)

(ii) The Γ-equivariant vector fields for the reduced problem need not extend to $\mathbf{E}(n)$-equivariant vector fields. A graphic illustration when $n = 2$ is provided by (nonsimple) hexagons which generically bifurcate transcritically in scalar PDEs but sub/supercritically in pseudoscalar PDEs. There is a unique Γ-equivariant quadratic which extends for the scalar action of $\mathbf{E}(2)$ but not for the pseudoscalar action.

3.1 Isotropy subgroups and spatial periodicity

Consider the Euclidean group $\mathbf{E}(n)$ acting on $\mathbb{R}^n$ in the usual way. We can write $\mathbf{E}(n)$ as a semidirect product $\mathbf{E}(n) = \mathbf{O}(n) \dot{+} \mathbf{T}(n)$ where $\mathbf{O}(n)$ is the orthogonal group and $\mathbf{T}(n) \cong \mathbb{R}^n$ is the normal subgroup consisting of translations. Let $\Pi : \mathbf{E}(n) \to \mathbf{O}(n)$ denote the natural projection.

A lattice in $\mathbb{R}^n$ is defined to be a nontrivial discrete subgroup of $\mathbb{R}^n$, hence isomorphic to $\mathbb{Z}^p$ for some p. The lattice is then said to be p-dimensional. For our purposes it is more convenient to work with arbitrary closed subgroup of

$\mathbb{R}^n$ and to use the symbol $\mathcal{L}$ to denote a closed subgroup. By the next well known result, $\mathcal{L}$ differs from a lattice only by a factor of $\mathbb{R}^q$.

Proposition 3.1 *Any closed subgroup $\mathcal{L} \subset \mathbb{R}^n$ has the form $\mathcal{L} \cong \mathbb{Z}^p \times \mathbb{R}^q$. The quotient $\mathbb{R}^n/\mathcal{L}$ is compact if and only if $p + q = n$ in which case the quotient is an r-torus T^r with $r = n - q$.*

If $\mathcal{L}$ is a closed subgroup, we can define its *dual*

$$\mathcal{L}^* = \{k \in \mathbb{R}^n,\ k \cdot \ell \in \mathbb{Z} \text{ for all } \ell \in \mathcal{L}\},$$

(where we use the standard inner product on $\mathbb{R}^n$). It is clear that $\mathcal{L}^*$ is also a closed subgroup, in particular if $\mathcal{L} \cong \mathbb{Z}^p \times \mathbb{R}^q$ then $\mathcal{L}^* \cong \mathbb{Z}^p \times \mathbb{R}^{n-p-q}$. If $\mathcal{L}$ is an n-dimensional lattice, then so is $\mathcal{L}^*$. In this case $\mathcal{L}^*$ is called the *dual lattice*. More generally, $\mathcal{L}^*$ is a lattice precisely when $\mathbb{R}^n/\mathcal{L}$ is compact.

Identifying $\mathbf{T}(n)$ with $\mathbb{R}^n$, we let $\mathbf{O}(n)$ act in the usual way on $\mathbf{T}(n)$ (indeed this is the action in the semidirect product description of $\mathbf{E}(n)$). Then we can define the *holohedry* of a closed subgroup $\mathcal{L} \subset \mathbf{T}(n)$

$$H = \{A \in \mathbf{O}(n),\ A\ell \in \mathcal{L} \text{ for all } \ell \in \mathcal{L}\}.$$

This generalizes the usual notion of holohedry of a lattice. Note that H is a closed (but not necessarily finite) subgroup of $\mathbf{O}(n)$ and that the holohedries of $\mathcal{L}$ and $\mathcal{L}^*$ coincide (since $\mathbf{O}(n)$ acts orthogonally). If $\mathcal{L}$ is an n-dimensional or $(n-1)$-dimensional lattice, then H is finite.

Now suppose that $u : \mathbb{R}^n \to \mathbb{R}^m$ is a continuous function. We define the *spatial periodicity* of u to be

$$\mathcal{L}_u = \{t \in \mathbf{T}(n),\ u(x+t) = u(x) \text{ for all } x \in \mathbb{R}^n\}.$$

Clearly, $\mathcal{L}_u$ is a closed subgroup of $\mathbf{T}(n)$. Denote the holohedry of $\mathcal{L}_u$ by H_u.

Let $\mathcal{Z}$ be the vector space of continuous functions $u : \mathbb{R}^n \to \mathbb{R}^m$. There are various ways that $\mathbf{E}(n)$ can act on $\mathcal{Z}$ depending on the value of m. We shall restrict to a class of representations that includes those encountered in applications.

First, let $\rho : \mathbf{O}(n) \to \mathrm{GL}(\mathbb{R}^m)$ denote a representation of $\mathbf{O}(n)$ on $\mathbb{R}^m$ and let ρ_A denote the image of $A \in \mathbf{O}(n)$ under ρ. Then for $\phi \in \mathbf{E}(n)$, set

$$(\phi \cdot u)(x) = \rho_A \cdot u(\phi^{-1}(x)),$$

where $A = \Pi(\phi)$. This defines an action of $\mathbf{E}(n)$ on $\mathcal{Z}$. Note that the action on the domain $\mathbb{R}^n$ is just the standard one and that $\mathbf{T}(n)$ acts trivially on the range $\mathbb{R}^m$. The most commonly encountered actions on the range $\mathbb{R}^m$ are as follows:

> $m = 1$ and $\rho_A = I$ in scalar PDEs such as reaction diffusion equations and the Kuramoto-Sivashinsky equation,
>
> $m = n$ and $\rho_A = A$ in vector field PDEs such as the Navier-Stokes equations.

The pseudoscalar case corresponds to $m = 1$ and $\rho_A = \det A$.

The *isotropy subgroup* of $u \in \mathcal{Z}$ is

$$I_u = \{\phi \in \mathbf{E}(n),\ \phi \cdot u = u\}.$$

Theorem 3.2 *(a)* $I_u \cap \mathbf{T}(n) = \mathcal{L}_u$,

(b) $\Pi(I_u) \subset H_u$, *and so* $I_u \subset H_u \dot{+} \mathbf{T}(n)$.

Proof Part (a) is immediate from the definitions. To prove part (b), suppose that $\phi \in I_u$. We can write $\phi = (A, t)$ where $A \in \mathbf{O}(n)$ and $t \in \mathbf{T}(n)$. It is more convenient to write $\phi^{-1} = (A, t)$. We show that $A \in H_u$. Since H_u is a group it then follows that $\Pi(\phi) = A^{-1} \in H_u$ as required.

Since $\phi \cdot u = u$ we compute that for each $\ell \in \mathcal{L}_u$,

$$\begin{aligned} \rho_A^{-1} u(Ax + t) &= u(x) \\ &= u(x + \ell) \\ &= \rho_A^{-1} u(A(x + \ell) + t) \\ &= \rho_A^{-1} u(Ax + t + A\ell). \end{aligned}$$

Setting $y = Ax + t$, we have $u(y + A\ell) = u(y)$ and since y is arbitrary, $A\ell \in \mathcal{L}_u$. Hence $A \in H_u$. ∎

When $\mathcal{L}_u$ is an n-dimensional lattice, that is $\mathcal{L}_u \cong \mathbb{Z}^n$, Theorem 3.2 is simply stating that I_u is a crystallographic group (see Armstrong [2] and Miller [15] for information on crystallographic groups). The group $J = \Pi(I_u)$ is called the *crystal class* of I_u. We are interested more generally in the case where $\mathcal{L}_u \cong \mathbb{Z}^p \times \mathbb{R}^q$ and $p + q = n$ (so that $\mathbf{T}(n)/\mathcal{L}_u$ is compact). In this case, we say that u is *spatially periodic*.

In the crystallographic classification, it is necessary to identify certain crystallographic groups — if only to make the classification finite! Such identifications are too severe for our purposes, for example it makes sense to talk about a continuum of rectangle planforms. Nevertheless it is convenient to make two identifications. We classify only up to (i) conjugacy of I and (ii) scaling of $\mathcal{L}$. The first identification is standard: we classify conjugacy classes of isotropy subgroups. This in particular implies that two closed subgroups $\mathcal{L}, \mathcal{L}' \subset \mathbf{T}(n)$ are identified if one can be transformed into the other by an element of $\mathbf{O}(n)$. However, we identify these subgroups also if one is a scaled version of the other: there is a real number $\mu \neq 0$ such that

$$\mathcal{L}' = \{\mu\ell,\, \ell \in \mathcal{L}\}.$$

In this case, we write $\mathcal{L}' = \mu\mathcal{L}$.

In general, suppose that $I, I' \subset \mathbf{E}(n)$ are closed subgroups and $I = H \dot{+} \mathcal{L}$, $I' = H' \dot{+} \mathcal{L}'$. We say that I and I' are equivalent if I is conjugate to $H' \dot{+} \mu\mathcal{L}'$ for some $\mu \neq 0$.

3.2 Fixed-point spaces and the equivariant branching lemma

Here we follow for the most part [10, Subsection 1(a)]. Write the PDE for steady solutions in (nonlinear) operator form between two suitably chosen function spaces $\mathcal{X}$ and $\mathcal{Y}$:

$$F : \mathcal{X} \times \mathbb{R} \to \mathcal{Y}, \quad F(u, \lambda) = 0,$$

where $\lambda \in \mathbb{R}$ is a bifurcation parameter and $u : \mathbb{R}^n \to \mathbb{R}^m$ (unlike [10] we do not specify $m = 1$). Since the underlying PDE is Euclidean-equivariant, we expect that the action of $\mathbf{E}(n)$ on $\mathcal{Z}$ described in the previous subsection induces actions on $\mathcal{X}$ and $\mathcal{Y}$ and that F is equivariant with respect to these actions.

Suppose that there is a trivial solution $u = 0$ (so $F(0, \lambda) \equiv 0$). We investigate the bifurcation of solutions from the trivial solution as λ passes through 0 say. To overcome difficulties related to the noncompactness of $\mathbf{E}(n)$, we restrict attention to spatially periodic solutions — this leads naturally to a compact group of symmetries.

To find spatially periodic solutions, fix a lattice $\mathcal{L} \subset \mathbf{T}(n)$. More generally, we take $\mathcal{L}$ to be any closed subgroup of $\mathbf{T}(n)$ such that $\mathbf{T}(n)/\mathcal{L} \cong T^r$ is a torus (so $\mathcal{L} \cong \mathbb{Z}^p \times \mathbb{R}^q$ where $p+q=n$, and $r=n-q$). Let $\mathcal{X}_\mathcal{L}$ consist of the functions $u \in \mathcal{X}$ that have isotropy at least $\mathcal{L}$. In other words,

$$\mathcal{X}_\mathcal{L} = \{u \in \mathcal{X},\, u(x+\ell) = u(x) \text{ for all } \ell \in \mathcal{L}\}.$$

Then Euclidean equivariance of F implies that F restricts to an operator

$$F : \mathcal{X}_\mathcal{L} \times \mathbb{R} \to \mathcal{Y}_\mathcal{L}. \tag{3.9}$$

The largest subgroup of $\mathbf{E}(n)$ that preserves $\mathcal{X}_\mathcal{L}$ is $H \dot{+} \mathbf{T}(n)$ where H is the holohedry of $\mathcal{L}$. Since the normal subgroup $\mathcal{L}$ acts trivially, we quotient out to arrive at the compact group $\Gamma = H \dot{+} T^r$. Note that F in equation (3.9) is Γ-equivariant.

If $u \in \mathcal{X}_\mathcal{L}$ we can define the isotropy subgroup $\Sigma_u \subset \Gamma$ to be

$$\Sigma_u = \{\gamma \in \Gamma,\, \gamma \cdot u = u\}.$$

We have already defined the isotropy subgroup I_u of u in $\mathbf{E}(n)$. It follows from Theorem 3.2 that

$$\Sigma_u = I_u/\mathcal{L}$$

and that I_u can be recovered from Σ_u. In the case $\mathcal{L} = \mathcal{L}_u$ we have

$$\Sigma_u \cap T^r = 1. \tag{3.10}$$

Such a subgroup $\Sigma_u \subset \Gamma$ is said to be *translation-free* in [10].

Suppose that there is a steady-state bifurcation at $\lambda = 0$ in (3.9), that is

$$U \equiv \ker(dF)_{0,0} \neq \{0\}.$$

The kernel U is Γ-invariant and branches of planforms may be found using the equivariant branching lemma as follows. Fix a subgroup $\Sigma \subset \Gamma$ and compute $\dim \mathrm{Fix}_U(\Sigma)$ where the *fixed-point subspace* is defined by

$$\mathrm{Fix}_U(\Sigma) = \{u \in U,\, \sigma u = u \text{ for all } \sigma \in \Sigma\}.$$

In this notation, $\mathcal{X}_\mathcal{L} = \mathrm{Fix}_\mathcal{X}(\mathcal{L})$. The equivariant branching lemma states that if $\dim \mathrm{Fix}_U(\Sigma) = 1$ then generically there is a unique branch of steady-state solutions to (3.9) with isotropy Σ.

Since Γ is compact, we can write $U = U_1 \oplus \cdots \oplus U_p$ as a direct sum of Γ-irreducible spaces. Then $\mathrm{Fix}_U(\Sigma) = \mathrm{Fix}_{U_1}(\Sigma) \oplus \cdots \oplus \mathrm{Fix}_{U_p}(\Sigma)$. When $\dim \mathrm{Fix}_U(\Sigma) = 1$ it follows that $\dim \mathrm{Fix}_{U_j}(\Sigma) = 1$ for some U_j. In particular, U_j is absolutely irreducible (the linear Γ-commuting maps are real scalar multiples of the identity). Hence it is sufficient to consider absolutely irreducible representations V of Γ for which

$$\dim \mathrm{Fix}_V(\Sigma) = 1. \tag{3.11}$$

Our aim in this paper is to classify the set of equivalence classes of axial isotropy subgroups $I \subset \mathbf{E}(2)$, namely those isotropy subgroups corresponding to spatially periodic steady-state solutions whose existence can generically be deduced from the equivariant branching lemma. We shall proceed in the following manner which is justified by the previous discussion.

(a) List (up to equivalence) the closed subgroups $\mathcal{L} \subset \mathbf{T}(2)$ with $T^r = \mathbf{T}(2)/\mathcal{L}$ compact, $r \geq 1$ and in each case set $\Gamma = H \dot{+} T^r$ where H is the holohedry of $\mathcal{L}$.

(b) Enumerate the absolutely irreducible representations V for Γ that occur in scalar and pseudoscalar PDEs.

(c) Classify (up to conjugacy) those isotropy subgroups $\Sigma \subset \Gamma$ that are translation-free and axial (that is, they satisfy conditions (3.10) and (3.11)).

(d) Prove that there are no further absolutely irreducible representations for Γ other than those considered in (b).

The required (equivalence classes of) isotropy subgroups I can then be recovered from the isotropy subgroups Σ classified in (c).

In the remainder of this subsection, we carry out step (a). Steps (b) and (c) were performed for scalar PDEs in [10]. We recall these results in Section 4. Then in Section 5 we carry out the corresponding procedure for pseudoscalar PDEs. Step (d) is dealt with in Section 6.

As promised, we end this subsection with step (a). Up to a notion of equivalence that is weaker than ours, there are five two-dimensional lattices, see Armstrong [2]. In addition, we must consider the closed subgroup $\mathbb{Z} \times \mathbb{R}$ (which we shall refer to as the roll lattice). This leads to the six closed subgroups listed in Table 1 (cf [10, Table 1]). Since we are more restrictive in our definition of equivalence, we have continuous families of rhombic, rectangular and oblique lattices.

3.3 Types of branching

Suppose that Γ is a compact Lie group acting on $\mathbb{R}^n$ and that Σ is an axial isotropy subgroup. Generically a branch of equilibria with isotropy Σ will undergo either a transcritical bifurcation or a pitchfork (supercritical or subcritical) bifurcation. The transcritical case occurs when there is a Γ-equivariant quadratic map $Q : \mathbb{R}^n \to \mathbb{R}^n$ such that $Q|_{\mathrm{Fix}(\Sigma)} \neq 0$. Otherwise, the branch is a pitchfork. In this case, the branch consists of two 'half-branches' of equilibria. These two halves may be related by equivariance, in which case we say that the branch is a symmetric pitchfork. If the two halves are unrelated by symmetry, the branch is an asymmetric pitchfork.

A sufficient and necessary, criterion for a symmetric pitchfork is that there is an element $\gamma \in \Gamma$ that acts as -1 on $\mathrm{Fix}(\Sigma)$. More technically, the normalizer $N(\Sigma)$ of Σ in Γ acts on $\mathrm{Fix}(\Sigma)$. The quotient group $N(\Sigma)/\Sigma$ acts faithfully on $\mathrm{Fix}(\Sigma)$ and since $\dim \mathrm{Fix}(\Sigma) = 1$ either $N(\Sigma)/\Sigma \cong 1$ or $N(\Sigma)/\Sigma \cong \mathbb{Z}_2$. Symmetric pitchforks correspond to the case $N(\Sigma)/\Sigma \cong \mathbb{Z}_2$.

The above discussion is formalized in the following definition.

Definition 3.3 Suppose that Σ is an axial isotropy subgroup of Γ.

(i) If there is a Γ-equivariant quadratic $Q : \mathbb{R}^n \to \mathbb{R}^n$ such that $Q|_{\mathrm{Fix}(\Sigma)} \neq 0$ then Σ is *transcritical*.

(ii) If $N(\Sigma)/\Sigma \cong \mathbb{Z}_2$ then Σ is a *symmetric pitchfork*.

(iii) If Σ is not transcritical and is not a symmetric pitchfork, then Σ is an *asymmetric pitchfork*.

Remark 3.4 (a) Possibilities (i), (ii) and (iii) all arise for the standard action of $\mathbb{D}_m$ on $\mathbb{R}^2$, $m \geq 3$. If m is odd, there is a unique axial isotropy subgroup and this is transcritical for $m = 3$ and an asymmetric pitchfork for $m \geq 5$. If m is even, $\mathbb{D}_m$ contains $-I$ and all axial isotropy subgroups (there are two of them) are symmetric pitchforks.
(b) Suppose that Σ is an axial isotropy subgroup for two representations V_1 and V_2 of Γ. Then Σ is a symmetric pitchfork for V_1 if and only if it is a symmetric pitchfork for V_2 (since $N(\Sigma)/\Sigma$ is independent of the representation of Γ). In contrast, Σ may be transcritical for V_1 and an asymmetric pitchfork for V_2. An example is provided by (nonsimple) hexagons.
(c) The above discussion goes through for general actions of a group G on

a vector space X (provided the branch of equilibria actually exists). In particular, Definition 3.3 makes sense in this generality and we will use this terminology for infinite-dimensional $\mathbf{E}(2)$-equivariant problems.

4 Planforms in scalar PDEs

Recall that a Euclidean-equivariant PDE is said to be *scalar* if it is posed on a function space consisting of functions $u : \mathbb{R}^n \to \mathbb{R}$ and the action of $\mathbf{E}(n)$ on u is given by

$$(\phi \cdot u)(x) = u(\phi^{-1}(x)).$$

Dionne and Golubitsky [10] carried out the procedure described in the previous section for classifying axial planforms for scalar PDEs in the case $n = 2$. It is these results that we describe in this section. (The case $n = 3$ can be found in Dionne [9].)

Let $\mathcal{L}$ be one of the closed subgroups of $\mathbf{T}(2)$ listed in Table 1. Associated to $\mathcal{L}$ we have the dual $\mathcal{L}^*$, the holohedry H, the torus $T^r = \mathbf{T}(2)/\mathcal{L}$, $r = 1$ or $r = 2$, and the compact group $\Gamma = H \dot{+} T^r$. We give, for each $\mathcal{L}$, a list of Γ-irreducible representations V and the corresponding axial translation-free subgroups $\Sigma \subset \Gamma$. (It transpires that all Γ-irreducible representations that arise in the context of scalar and pseudoscalar PDEs are absolutely irreducible.)

To describe the irreducible representations we recall some notation from [10]. First, if u is a scalar function on $\mathbb{R}^2$ (real or complex valued), we let Γ act on u by $(\gamma \cdot u)(x) = u(\gamma^{-1}x)$. For $k \in \mathbb{R}^2$, define the *wave function* w_k with *wave vector* k to be the complex-valued function

$$w_k(x) = e^{2\pi i k \cdot x}.$$

Observe that $w_k(x + \ell) = w_k(x)$ for all $\ell \in \mathcal{L}$ if and only if $k \in \mathcal{L}^*$.

For $k \in \mathcal{L}^*$, $k \neq 0$ define the two-dimensional space

$$V_k = \{\mathrm{Re}(z w_k),\ z \in \mathbb{C}\} \cong \mathbb{C}.$$

Then T^r acts irreducibly on V_k. Moreover, $V_{-k} = V_k$, and the representations of T^r on V_k and $V_{k'}$ are distinct unless $k = \pm k'$.

Let S_c consist of the vectors in $\mathcal{L}^*$ of a fixed length $c > 0$. Since $\mathcal{L}^*$ is a lattice, S_c is finite (possibly empty). Define U to be the direct sum of the subspaces V_k with $k \in S_c$. Then U is a finite-dimensional vector space, and it is easily checked that $H \dot{+} \mathbf{T}(2)$ acts on U. Moreover, $\mathcal{L}$ acts trivially on U so that there is an induced action of the compact group $\Gamma = H \dot{+} T^r$ on U.

It follows from the construction of U as a sum of distinct T^r-irreducibles V_k, that if $V \subset U$ is Γ-invariant, then

$$V = V_{K_1} \oplus \cdots \oplus V_{K_s} \cong \mathbb{C}^s \tag{4.1}$$

for some set of wave vectors $K_1, \ldots, K_s \in \mathcal{L}^*$. A computation shows that if $h \in H$, then $h \cdot V_k = V_{hk}$ and hence the space V in (4.1) is Γ-irreducible if and only if H acts transitively on the set of $2s$ wave vectors $\{\pm K_1, \ldots, \pm K_s\}$. The orbits of H in S_c are of size $|H|$ or $|H|/2$ (since $H \cap \mathbf{SO}(2)$ acts freely on S_c) and it follows that $\dim V = |H|$ or $\dim V = |H|/2$.

It is now a relatively easy matter to list the irreducible representations V of the form (4.1) for each $\mathcal{L}$ in Table 1 and for each $c > 0$. (In practice, the value of c is determined by the linear stability analysis around the trivial solution; in the examples in Section 2 we had $c = 1$. Then consider all lattices of the form $\mu\mathcal{L}$, where $\mu > 0$ and $\mathcal{L}$ is taken from Table 1. Our approach of letting c vary and fixing $\mu = 1$ is clearly equivalent, and leads to slightly simpler arithmetic.)

An important simplification follows from the observation in [10] that it is sufficient to consider only the *translation-free* irreducible representations.

Definition 4.1 A Γ-invariant subspace V as in (4.1) is *translation-free* if T^r acts faithfully on V, that is there are no translations in T^r that fix all points in V.

Clearly, if V is not translation-free, then Γ contains no translation-free isotropy subgroups. Hence we can eliminate such representations V. Translation-free representations are easily characterized and often easily recognized.

Proposition 4.2 *If V in (4.1) is Γ-invariant, then V is translation-free if and only if the wave vectors $K_1, \ldots, K_s$ generate $\mathcal{L}^*$.*

Proof Let $\mathcal{M}^*$ denote the lattice generated by $K_1, \ldots, K_s$. Then $\mathcal{M}^*$ is the dual of a closed subgroup $\mathcal{M} \subset \mathbf{T}(2)$. Moreover, if we compute the isotropy subgroup $I_v \subset \mathbf{E}(2)$ of a point $v \in V$, then we find that $\mathcal{M} \subset I_v$ for all v. It follows that $\Sigma_v = I_v/\mathcal{L}$ contains $\mathcal{M}/\mathcal{L}$ for each v. Hence there are nontrivial translations fixing every v if and only $\mathcal{L}$ is a proper subset of $\mathcal{M}$ or, equivalently, $\mathcal{M}^*$ is properly contained in $\mathcal{L}^*$. ■

The translation-free Γ-irreducible representations V of the form (4.1) are listed in Table 2. Then, for each such V we list in Table 3 the (conjugacy classes of) isotropy subgroups $\Sigma \subset \Gamma$ satisfying conditions (3.10) and (3.11).

Most of the work required to obtain Tables 2 and 3 appears in [10] (cf Tables 2 and 3 in [10]). For more details, see [8]. The only differences are that we include the roll lattice (as previously discussed) and we state the type of branching (transcritical, symmetric pitchfork or asymmetric pitchfork) in Table 3. To give a flavor of the required computations, we give the details for these additional features.

The dual $\mathcal{L}^*$ of the roll lattice is one-dimensional and (up to our notion of equivalence — conjugacy plus scaling) we can choose as generator $k_1 = (1,0)$. We have the compact group $\Gamma = \mathbb{D}_2 \dot{+} T^1$. Let $V = V_{K_1} \oplus \cdots \oplus V_{K_s}$ be Γ-irreducible where $K_1, \ldots, K_s \in \mathcal{L}^*$. Then $\mathbb{D}_2$ must act transitively on the vectors $\{\pm K_1, \ldots, \pm K_s\}$ so that $s = 1$ or $s = 2$. Now $\mathbb{D}_2$ is generated by the 2×2 matrices $F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $-I$. But F fixes every vector in $\mathcal{L}^*$ so that $s = 1$. Hence the Γ-irreducible representations are given by $V = V_{K_1}$ where $K_1 = \alpha k_1$ for some integer $\alpha \neq 0$. Since $V_{K_1} = V_{-K_1}$ we can take $\alpha > 0$.

Next, observe that the lattice generated by $K_1 = \alpha k_1$ is the scaled dual lattice $\alpha\mathcal{L}^*$. By Proposition 4.2, V_{K_1} is translation-free if and only if $\alpha = 1$. This accounts for the entry in Table 2.

We can write this unique translation-free Γ-irreducible representation explicitly in the form

$$V = \{z e^{2\pi i k_1 \cdot x} + \bar{z} e^{-2\pi i k_1 \cdot x},\ z \in \mathbb{C}\}.$$

The action of Γ on $V \cong \mathbb{C}$ is given by

$$\begin{aligned} -I \in \mathbb{D}_2 &\ :\ z \mapsto \bar{z} \\ F \in \mathbb{D}_2 &\ :\ z \mapsto z \\ t \in T^1 &\ :\ z \mapsto e^{2\pi i t} z,\ 0 \leq t < 1 \end{aligned}$$

Next we compute the isotropy subgroups $\Sigma \subset \Gamma$ satisfying (3.10) and (3.11). Since we work with conjugacy classes of isotropy subgroups, we need only compute the isotropy subgroup Σ of representatives of groups orbits of points $z \in V$. By applying a suitable transformation in T^1 we can take $z = a \in \mathbb{R}$. If $a = 0$, Σ is the whole of Γ (which is not translation-free) but if $a \neq 0$, we have $\Sigma = \mathbb{D}_2$ (which is translation-free). Moreover

$$\mathrm{Fix}_V(\mathbb{D}_2) = \{z \in V,\, z \text{ real}\},$$

and has dimension one. Thus $\mathbb{D}_2$ is axial and we obtain the entry in Table 3.

Finally, we verify the entries in the last column of Table 3. First we observe that for many of the translation-free Γ-irreducible representations V in Table 2, there is an element of Γ that acts as $-I_V$. For these representations it is immediate that all axial isotropy subgroups are symmetric pitchforks. For example, in the case of the square lattice we take the translation $x \mapsto (x_1 + 1/2, x_2 + 1/2)$. (It is precisely this translation that leads to the restriction $\alpha + \beta$ odd for translation-free representations, else the translation acts trivially on V.) The exception is the case of the hexagonal lattice. Moreover it is easily checked that the quadratic equivariant βu^2 in the scalar PDE (2.1) satisfies the condition in Definition 3.3(i) for the isotropy subgroups corresponding to the axial planforms simple hexagons and hexagons.

5 Planforms in pseudoscalar PDEs

In this section, we repeat the classification in Section 4 except that we now work with pseudoscalar PDEs. The action of $\mathbf{E}(2)$ on a function $u : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$(\phi \cdot u)(x) = (\det A) u(\phi^{-1}(x)),$$

where $A = \Pi(\phi)$.

The description of the translation-free Γ-(absolutely) irreducible representations V in Table 2 is unchanged, but the actual action of Γ on V can be different — leading to different planforms as shown in Table 4.

Again we give the details for the roll lattice. Everything is unchanged up the point where we describe the action of Γ on $V \cong \mathbb{C}$. The translations in T^1 and the rotation $-I \in \mathbb{D}_2$ act on $z \in V$ as before but the flip F transforms z into $-z$. Again taking $z = a \in \mathbb{R}$ we see that when $a \neq 0$ the isotropy subgroup Σ does not contain F, but rather F combined with the translation $t = 1/2$. Thus $\Sigma = \mathbb{D}_2^-$, the twisted group generated by

$(-I,0)$, $(F,1/2) \in \mathbb{D}_2 \dot{+} T^2$. The group $\mathbb{D}_2^-$ replaces the group $\mathbb{D}_2$ as the only isotropy subgroup satisfying the required conditions. Indeed $\mathbb{D}_2$ is no longer an isotropy subgroup.

Finally, we comment on the last column in Table 4. Except for planforms on the hexagonal lattice, all bifurcations are symmetric pitchforks as for the scalar case. In Remark 2.1 we observed that all branching was super/subcritical in pseudoscalar PDEs. It remains to determine whether simple oriented hexagons and hexagons are symmetric pitchforks or asymmetric pitchforks. Also in Section 2 we observed that reflections in $\mathbb{D}_6$ act as $-I$ on the planform simple oriented hexagons so this is a symmetric pitchfork. On the other hand, it follows from Remark 3.4(b) that hexagons undergo an asymmetric pitchfork bifurcation.

6 Completeness of the classification of planforms

Let $\mathcal{L}$ be a closed subgroup of $\mathbf{T}(n)$ with $\mathbf{T}(n)/\mathcal{L} = T^r$ compact and $r \geq 1$. Let H denote the holohedry of $\mathcal{L}$ and construct the compact group $\Gamma = H \dot{+} T^r$. We show that when $n=2$ all translation-free absolutely irreducible representations of Γ are accounted for by those arising in scalar and pseudoscalar PDEs.

There are at least three ways to proceed. The most direct approach is to start with the irreducible representations (from now on abbreviated to irreps) of T^r and to build up Γ-irreps as a sum of T^r-irreps. We note that even in the simplest case $\Gamma = \mathbf{O}(2) = \mathbb{Z}_2 \dot{+} T^1$ this turns out to be surprisingly tedious (though elementary). A second method is to compute the characters of the representations in Sections 4 and 5 and to use the orthogonality relations to show that we have a complete set of characters. However it soon becomes clear that this is more tedious than the first method.

We shall use a third approach which requires heavier machinery (in particular the Peter-Weyl Theorem [4]) but has the advantage that the dependence of the computations on the semi-direct product structure in Γ is kept to a minimum. One disadvantage is that we may lose information on the irreps of Γ that are not absolutely irreducible. Of course it is only the absolutely irreducible representations that we are interested in since these are the ones that support axial isotropy subgroups. (It seems likely that all irreps are absolutely irreducible.)

We use the fact that real absolute irreps when complexified correspond to complex irreps. It follows from general theory (primarily the Peter-Weyl Theorem, orthogonality of characters and so on) that irreps of the compact group Γ are finite-dimensional and that the Hilbert space $V = L^2(\Gamma, \mathbb{C})$ decomposes into a 'direct sum' of complex Γ-irreps in which each irrep of dimension d occurs with multiplicity d (see for example [16, Corollary 5.7]).

We split the classification of (complex) Γ-irreps into two steps. First we compute the isotypic decomposition of V under T^r. Then we compute the irreps of Γ.

Isotypic decomposition of V under T^r Since T^r is abelian, the irreps of T^r are one-dimensional and can be described explicitly, see for example [12]. For $k \in \mathcal{L}^*$, let Z_k denote the vector space spanned by the map $w_k : \mathbb{R}^n \to \mathbb{C}$, $w_k(x) = e^{2\pi i k \cdot x}$. Then $Z_k \cong \mathbb{C}$ is an irrep and the T^r irreps are in one-to-one correspondence with vectors $k \in \mathcal{L}^*$.

Identify the holohedry H as a subgroup of the $n \times n$ orthogonal matrices $\mathbf{O}(n)$. Write elements of Γ in the form (h, t) where $h \in H$ and $t \in T^r$. Recall that the multiplication in Γ is defined by

$$(g, s) \cdot (h, t) = (gh, gt + s).$$

Here, we are using multiplicative notation in H and additive notation (mod $\mathcal{L}$) in T^r. Also gt means the $n \times n$ matrix g applied to the vector t. We can write each function $f \in V$ in a Fourier expansion

$$f(h, t) = \sum_{k \in \mathcal{L}^*} a_k(h) e^{2\pi i k \cdot t}$$

and the action of Γ on V is given by

$$((g, s) \cdot f)(h, t) = f(gh, gt + s).$$

For each $k \in \mathcal{L}^*$, define

$$V_k = \{f(h, t) = a(h) e^{2\pi i k \cdot t},\ a : H \to \mathbb{C}\} \cong \mathbb{C}^{|H|}.$$

Note that V_k is T^r-invariant and is a sum of $|H|$ T^r-irreps each of which is isomorphic to Z_k. Hence we have the isotypic decomposition $V = \oplus V_k$ under the action of T^r.

Irreducible representations of Γ If $k \in \mathcal{L}^*$ then we define $Q(k) \subset \mathcal{L}^*$ to be the H-group orbit $Q(k) = \{hk,\, h \in H\}$. If $Q = Q(k)$ for some k, set

$$V_Q = \oplus_{k \in Q} V_k.$$

Then $V = \oplus_Q V_Q$ is a decomposition of V into Γ-invariant subspaces.

Proposition 6.1 *Each Γ-isotypic component of V is contained in V_Q for some Q.*

Proof Suppose that $W \subset V$ is an irrep for Γ. We prove that (i) $W \subset V_Q$ for some Q, and (ii) V_Q contains all irreps that are isomorphic to W. The subspace W is T^r-invariant so we can choose a T^r-irrep $Z \subset W$. Then Z lies in a unique T^r-isotypic component V_k for some $k \in \mathcal{L}^*$. Also $V_k \subset V_{Q(k)}$ and so $Z \subset W \cap V_{Q(k)}$. Hence $W \cap V_{Q(k)}$ is a nontrivial Γ-invariant subspace of W. Since W is irreducible $W \subset V_{Q(k)}$ proving (i).

Now observe that if W' is isomorphic to W then W' contains a T^r-irrep Z' isomorphic to Z. Hence $Z' \subset V_k$ so that again we have $W' \subset V_{Q(k)}$ as required for (ii). ■

Proposition 6.2 *Let $Q = Q(k)$ for $k \in \mathcal{L}^*$ and define $q = |Q|$.*

(a) q divides $|H|$.

(b) $\dim V_Q = q|H|$.

(c) If $W \subset V_Q$ is Γ-invariant then q divides $\dim W$.

Proof Since Q is an H-orbit, q divides $|H|$. Also $\dim V_Q = q \dim V_k = q|H|$. It remains to prove part (c). For $k' \in Q$, define $W_{k'} = W \cap V_{k'}$. Then

$$hW_{k'} = W \cap hV_{k'} = W \cap V_{h^{-1}k'} = W_{h^{-1}k'}$$

for each $h \in H$. It follows that $\dim W_{k'} = \dim W_k$. Since $|Q| = q$ there are q distinct spaces $V_{k'}$ and so $\dim W = q \dim W_k$. ■

Corollary 6.3 *Suppose that $|H| = jq$ and $j = 1, 2$ or 3. Then V_Q contains precisely j nonisomorphic irreps each of dimension (and multiplicity) q.*

Proof By Proposition 6.2(c), any irrep $W \subset V_Q$ has dimension sq where s is a positive integer. The multiplicity of the irrep W in V is also sq. Moreover, by Proposition 6.1 all of these isomorphic irreps lie in V_Q and so account for $(sq)^2$ dimensions in V_Q. But by Proposition 6.2(b), $\dim V_Q = jq^2$. Therefore $s^2 \leq j$ and since $j < 4$ we have $s = 1$. It follows that the irrep W accounts for precisely q^2 of the jq^2 dimensions available in V_Q. We require j such irreps to account for all of the dimensions. ■

Now we specialize to the case $n = 2$ concentrating on the case when $\mathcal{L} \subset \mathbf{T}(2)$ is the square lattice. The remaining cases in Table 1 are similar, the arguments for the rectangular, rhombic and hexagonal lattices are identical while the oblique and roll cases are slightly simpler.

Proposition 6.4 *Suppose that $\mathcal{L}$ is the square lattice so that $\Gamma = \mathbb{D}_4 \dot{+} T^2$. The Γ-irreps can be enumerated as follows. The $\mathbb{D}_4$-group orbits $Q \subset \mathcal{L}^*$ have length $q = 1$, $q = 4$ and $q = 8$ The case $q = 1$ corresponds to the irreps of Γ where T^2 acts trivially. There are countably many orbits Q of length $q = 4$ and $q = 8$. Corresponding to each Q with $q = 8$ there is a single Γ-irrep of dimension 8. Corresponding to each Q with $q = 4$ there are two Γ-irreps of dimension 4.*

Proof When $q = 1$, $Q = \{0\}$ and T^2 acts trivially on the corresponding elements of V. (Indeed all that is left is the finite vector space $L^2(\mathbb{D}_4, \mathbb{C})$ and hence the irreps of $\mathbb{D}_4$.) Otherwise, it follows as in Section 4 that $q = 8$ or $q = 4$ (the s in Section 4 satisfies $s = q/2$). It is not difficult to see that each possibility is realized in a countable family. Corresponding to $q = 8$ and $q = 4$ we have $j = 1$ and $j = 2$ in Corollary 6.3 and the required result follows directly from the corollary. ■

We are primarily interested in finding those irreps that support translation-free subgroups (subgroups $\Sigma \subset \Gamma$ satisfying (3.10)). Hence, by the same argument used to prove Proposition 4.2 we can restrict to those *translation-free* orbits $Q \subset \mathcal{L}^*$ that have the property that Q generates $\mathcal{L}^*$. The translation-free orbits Q are precisely the ones of the form $Q = \{\pm K_1, \ldots, \pm K_s\}$ listed in Table 2.

Proposition 6.5 *The translation-free irreps in Proposition 6.4 (those corresponding to translation-free orbits Q) are in one-to-one correspondence with the translation-free irreps in Sections 4 and 5 for the square lattice.*

Proof The action of T^2 distinguishes each of the orbits Q and so to identify isomorphic irreps we can work one orbit Q at a time. The unique translation-free Q with $q = 4$ is $Q = Q_{k_1} = \{\pm k_1, \pm k_2\}$. We considered two irreps corresponding to Q, one in each of the Sections 4 and 5. Clearly these irreps are nonisomorphic (they have different isotropy subgroups) and so these correspond to the two irreps in Proposition 6.4. Similarly we have in Table 2 an irrep of dimension 8 for each translation-free Q with $q = 8$. Hence the irreps in Sections 4 and 5 account for all the translation-free irreps in Proposition 6.4. ■

Remark 6.6 It is not the case that all irreps of Γ arise in the context of scalar and pseudoscalar PDEs. However, it is easily checked that if an irrep is omitted, then T^2 acts trivially. Thus, all translation-free irreps and nearly all irreps that are not translation-free occur (but not all of these produce planforms).

Acknowledgments We are grateful to Marty Golubitsky for several helpful suggestions.

References

[1] D. Armbruster, B. Nicolaenko, N. Smaoui and P. Chossat. Analyzing bifurcations in the Kolmogorov flow equations. *Dynamics, Bifurcations, Symmetry* (P. Chossat ed.) Kluwer Acad. Publ., The Netherlands, 1994, 11-33.

[2] M. A. Armstrong. *Groups and Symmetry.* Undergrad. Texts in Maths. Springer-Verlag, New York 1988.

[3] I. Bosch Vivancos, P. Chossat and J. Oprea. Bifurcation of self-sustained magnetic fields in planar convective flows. Preprint INLN, 1994.

[4] T. Bröcker and T. tom Dieck. *Representations of Compact Lie Groups.* Grad. Texts in Math. **98** Springer, New York, 1985.

[5] F. H. Busse. Das Stabilitätsverhalten der Zellularkonvektion bei endlicher Amplitude. Thesis, University of Munich, 1962.

[6] P. Chossat, R. Lauterbach and I. Melbourne. Steady-state bifurcation with **O**(3)-symmetry, *Arch. Rat. Mech. Anal.* **113** (1990) 313-376.

[7] M. C. Cross and P. C. Hohenberg. Pattern formation outside of equilibrium. *Rev. of Mod. Phys.* **65** (1993) 851–1112.

[8] B. Dionne. Spatially periodic patterns in two and three dimensions. Thesis, University of Houston, 1990.

[9] B. Dionne. Planforms in three dimensions. *ZAMP* **44** (1993) 673-694.

[10] B. Dionne and M. Golubitsky. Planforms in two and three dimensions. *ZAMP* **43** (1992) 36-62.

[11] M. Golubitsky, B. Dionne and I. Stewart. Coupled cells, wreath products and direct products. *Dynamics, Bifurcations, Symmetry* (P. Chossat ed.) Kluwer Acad. Publ., The Netherlands, 1994, 127-138.

[12] M. Golubitsky, I. N. Stewart and D. G. Schaeffer. *Singularities and Groups in Bifurcation Theory*, Vol 2. Appl. Math. Sci. **69** Springer, New York, 1988.

[13] K. Kirchgässner. Exotische Lösungen des Bénardschen Problems. *Math. Meth. Appl. Sci. I* (1979) 453-467.

[14] L. D. Meshalkin and Ya. G. Sinai. Investigation of the stability of a stationary solution of a system of equations for the plane movement of an incompressible viscous liquid. *J. Appl. Math. and Mech.* (PMM) **25** (1961) 1700-1705.

[15] W. Miller Jr. *Symmetry Groups and their Applications.* Acad. Press, New York, 1972.

[16] A. Robert. *Introduction to the Representation Theory of Compact and Locally Compact Groups.* London Math. Soc. Lecture Note Series **80** Camb. Univ. Press, 1983.

[17] A. Schlüter, D. Lortz and F. Busse. On the stability of steady finite amplitude convection. *J. Fluid Mech.* **23** (1965) 129-144.

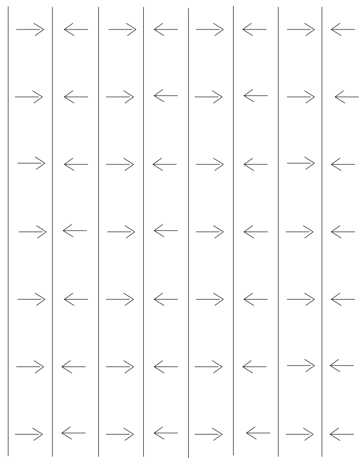
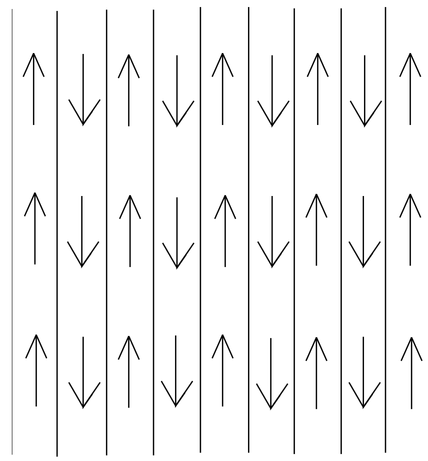


Figure 1: Rolls (schematic)



Anti-Rolls (schematic)

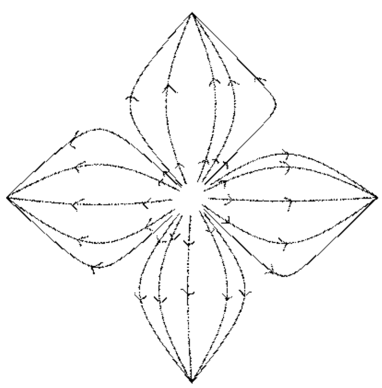
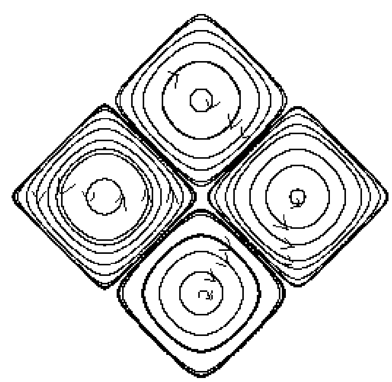


Figure 2: Simple squares



Simple anti-squares

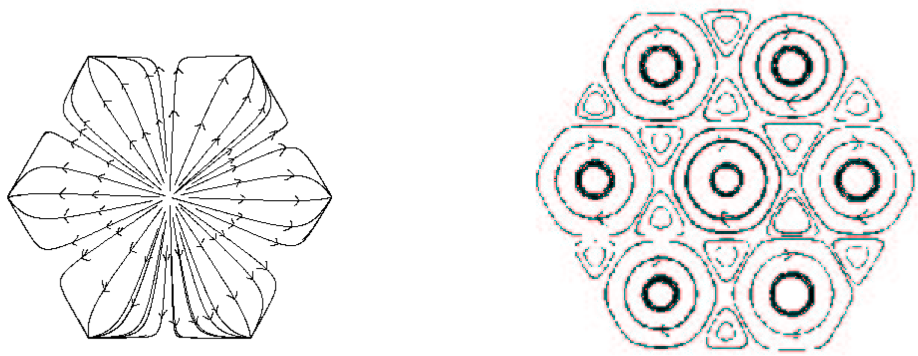


Figure 3: Simple hexagons Simple oriented hexagons

$\mathcal{L}$	Holohedry	Basis of $\mathcal{L}$	Basis of $\mathcal{L}^*$
Hexagonal	$\mathbb{D}_6$	$\ell_1 = (1/\sqrt{3}, 1)$ $\ell_2 = (2/\sqrt{3}, 0)$	$k_1 = (0, 1)$ $k_2 = (\sqrt{3}/2, -1/2)$
Square	$\mathbb{D}_4$	$\ell_1 = (1, 0)$ $\ell_2 = (0, 1)$	$k_1 = (1, 0)$ $k_2 = (0, 1)$
Rhombic	$\mathbb{D}_2$	$\ell_1 = (1, -\cot\theta)$ $\ell_2 = (0, \csc\theta)$ $0 < \theta < \pi/2,\ \theta \neq \pi/3$	$k_1 = (1, 0)$ $k_2 = (\cos\theta, \sin\theta)$
Rectangular	$\mathbb{D}_2$	$\ell_1 = (1, 0)$ $\ell_2 = (0, c)$ $0 < c < 1$	$k_1 = (1, 0)$ $k_2 = (0, 1/c)$
Oblique	$\mathbb{Z}_2$	$\lvert\ell_1\rvert \neq \lvert\ell_2\rvert$ $\ell_1 \cdot \ell_2 \neq 0$	
Roll	$\mathbb{D}_2$	$\ell_1 = (1, 0)$ $\{0\} \times \mathbb{R}$	$k_1 = (1, 0)$

Table 1: Closed subgroups $\mathcal{L} \subset \mathbf{T}(2)$ with $\mathbf{T}(2)/\mathcal{L} = T^r$. Here $r = 1$ for the roll lattice, and $r = 2$ in all other cases.

$\mathcal{L}$	Basis for $\mathcal{L}^*$	dim	$V_{K_1} \oplus \cdots \oplus V_{K_s}$
Roll $\mathbb{D}_2$	$k_1 = (1,0)$	2	$K_1 = k_1$
Rhombic $\mathbb{D}_2$	$k_1 = (1,0)$ $k_2 = (\cos\theta, \sin\theta)$ $0 < \theta < \pi/2,\ \theta \neq \pi/3$	4	$K_1 = k_1,\ K_2 = k_2$
Square $\mathbb{D}_4$	$k_1 = (1,0)$ $k_2 = (0,1)$	4	$K_1 = k_1,\ K_2 = k_2$
		8	$K_1 = \alpha k_1 + \beta k_2,\ K_2 = -\beta k_1 + \alpha k_2$ $K_3 = \beta k_1 + \alpha k_2,\ K_4 = -\alpha k_1 + \beta k_2$ $\alpha > \beta > 0,\ (2, \alpha + \beta) = 1$
Hexagonal $\mathbb{D}_6$	$k_1 = (0,1)$ $k_2 = (\sqrt{3}/2, -1/2)$	6	$K_1 = k_1 + k_2,\ K_2 = -k_2,\ K_3 = -k_1$
		12	$K_1 = \alpha k_1 + \beta k_2,\ K_2 = (\beta - \alpha)k_1 - \alpha k_2$ $K_3 = -\beta k_1 + (\alpha - \beta)k_2$ $K_4 = \alpha k_1 + (\alpha - \beta)k_2$ $K_5 = -\beta k_1 - \alpha k_2,\ K_6 = (\beta - \alpha)k_1 + \beta k_2$ $\alpha > \beta > \alpha/\beta > 0,\ (3, \alpha + \beta) = 1$

Table 2: Scalar translation-free Γ-irreducible representations. α and β are coprime integers

$\mathcal{L}$	$\dim V$	Σ	Planform	Branch
Roll	2	$\mathbb{D}_2$	Rolls	Symmetric pitchfork
Rhombic	4	$\mathbb{D}_2$	Rectangles	Symmetric pitchfork
Square	4	$\mathbb{D}_4$	Simple squares	Symmetric pitchfork
	8	$\mathbb{D}_4$	Squares	Symmetric pitchfork
		$\mathbb{D}_4^-$	Anti-squares	Symmetric pitchfork
Hexagonal	6	$\mathbb{D}_6$	Simple hexagons	Transcritical
	12	$\mathbb{D}_6$	Hexagons	Transcritical

Table 3: Axial planforms for scalar PDEs in the plane

$\mathcal{L}$	$\dim V$	Σ	Planform	Branch
Roll	2	$\mathbb{D}_2^-$	Anti-rolls	Symmetric pitchfork
Rhombic	4	$\mathbb{D}_2$	Rectangles	Symmetric pitchfork
Square	4	$\mathbb{D}_4^-$	Simple anti-squares	Symmetric pitchfork
	8	$\mathbb{D}_4$	Squares	Symmetric pitchfork
		$\mathbb{D}_4^-$	Anti-squares	Symmetric pitchfork
Hexagonal	6	$\mathbb{Z}_6$	Simple oriented hexagons	Symmetric pitchfork
	12	$\mathbb{D}_6$	Hexagons	Asymmetric pitchfork

Table 4: Axial planforms for pseudoscalar PDEs in the plane