RAPID MIXING FOR THE LORENZ ATTRACTOR AND STATISTICAL LIMIT LAWS FOR THEIR TIME-1 MAPS

V. ARAÚJO AND I. MELBOURNE AND P. VARANDAS

Abstract. We prove that every geometric Lorenz attractor satisfying a strong dissipativity condition has superpolynomial decay of correlations with respect to the unique SRB measure. Moreover, we prove the Central Limit Theorem and Almost Sure Invariance Principle for the time-1 map of the flow of such attractors. In particular, our results apply to the classical Lorenz attractor.

1. Introduction

The statistical point of view on Dynamical Systems is one of the most useful tools available for the study of the asymptotic behavior of transformations or flows. Statistical properties are often easier to study than pointwise behavior, since the future behavior of an initial data point can be unpredictable, but statistical properties are often regular and with simpler description.

One of the main concepts introduced is the notion of *physical* (or *Sinai-Ruelle-Bowen* (SRB)) measure for a flow (or transformation). An invariant probability measure μ for a flow Z_t is a physical probability measure if the subset of points z satisfying for all continuous functions w

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t w\big(Z_s(z)\big)\,ds=\int w\,d\mu,$$

has positive volume in the ambient space. These time averages are in principle physically observable if the flow models a real world phenomenon admitting some measurable features.

In 1963, the meteorologist Edward Lorenz published in the Journal of Atmospheric Sciences [20] an example of a polynomial system of differential equations

$$\begin{aligned}\dot{x}&=10(y-x)\\ \dot{y}&=28x-y-xz\\ \dot{z}&=xy-\tfrac{8}{3}z\end{aligned}\tag{1.1}$$

as a very simplified model for thermal fluid convection, motivated by an attempt to understand the foundations of weather forecast.

Numerical simulations performed by Lorenz for an open neighborhood of the chosen parameters suggested that almost all points in phase space tend to a *chaotic attractor*, whose well known picture can be easily found in the literature.

The mathematical study of these equations began with the geometric Lorenz flows, introduced independently by Afraĭmovič *et al.* [1] and Guckenheimer & Williams [16, 35] as an abstraction of the numerically observed features of solutions to (1.1). The geometric flows were shown to possess a "strange" attractor with sensitive dependence on initial conditions. It is well known, see e.g. [6], that geometric Lorenz attractors have a unique SRB (or physical) measure. Tucker [31] showed that the attractor of the classical Lorenz equations (1.1) is in fact a geometric Lorenz attractor

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(see Remark 2.3 below). For more on the rich history of the study of this system of equations, the reader can consult [33, 5].

An invariant probability measure μ for a flow is mixing if

$$\mu(Z_t(A) \cap B) \to \mu(A)\mu(B)$$

as $t \to \infty$ for all measurable sets A, B. Mixing for the SRB measure of geometric Lorenz attractors was proved in [21] and, by [31], this includes the classical Lorenz attractor [20].

Results on the speed of convergence in the limit above, that is, of rates of mixing for the Lorenz attractor were obtained only recently: a first result on robust exponential decay of correlations was proved in [7] for a nonempty open subset of geometric Lorenz attractors. However, this open set does not contain the classical Lorenz attractor. Also, it follows straightforwardly from [23] that a C^2-open and C^∞-dense set of geometric Lorenz flows have superpolynomial decay of correlations (in the sense of [14]). It is likely, but unproven, that this open and dense set includes the classical Lorenz attractor.

1.1. Statement of results.

In this paper, we introduce an additional open assumption, *strong dissipativity*, that is satisfied by the classical Lorenz attractor, under which we can prove superpolynomial decay of correlations.

We consider C^∞ vector fields G on $\mathbb{R}^3$ possessing an equilibrium p which is *Lorenz-like*: the eigenvalues of DG_p are real and satisfy

$$\lambda_{ss} < \lambda_s < 0 < -\lambda_s < \lambda_u. \tag{1.2}$$

We say that G is *strongly dissipative* if the divergence of the vector field G is strictly negative: there exists a constant $\delta > 0$ such that $(\operatorname{div} G)(x) \le -\delta$ for all $x \in U$, and moreover the eigenvalues of the singularity at p satisfy the additional constraint $\lambda_u + \lambda_{ss} < \lambda_s$. For the classical Lorenz equations (1.1), we have

$$\operatorname{div} G \equiv -\tfrac{41}{3}, \quad \lambda_s = -\tfrac{8}{3}, \quad \lambda_u \approx 11.83, \quad \lambda_{ss} \approx -22.83,$$

so the conditions (1.2) and strong dissipativity are satisfied.

Let $\mathcal{U}$ denote the open set of C^∞ vector fields having a strongly dissipative geometric Lorenz attractor Λ; see Section 2 for precise definitions. Given $G \in \mathcal{U}$, let Z_t denote the flow generated by G and let μ denote the unique SRB measure supported on Λ.

Theorem A. *Let $G \in \mathcal{U}$. Then for all $\gamma > 0$, there exists $C > 0$ and $k \ge 1$ such that for all C^k observables $v, w : \mathbb{R}^3 \to \mathbb{R}$ and all $t > 0$,*

$$\Big| \int v\, w \circ Z_t \, d\mu - \int v\, d\mu \int w\, d\mu \Big| \le C \|v\|_{C^k} \|w\|_{C^k} t^{-\gamma}.$$

By [19], geometric Lorenz flows satisfy the Central Limit Theorem (CLT) for Hölder observables. A stronger property is the CLT for the time-1 map $Z = Z_1$ which is only partially hyperbolic. By Theorem A, Z has superpolynomial decay of correlations. Following [24], we use this information to prove the CLT for time-1 maps of geometric Lorenz flows thereby verifying Conjecture 4 in [7].

Theorem B. *Let $G \in \mathcal{U}$. Then there exists $k \ge 1$ such that for all C^k observables $v : \mathbb{R}^3 \to \mathbb{R}$ there exists $\sigma \ge 0$ such that*

$$\frac{1}{\sqrt{n}} \left[\sum_{j=0}^{n-1} v \circ Z^j \; - \; n \int v \, d\mu \right] \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$$

where the convergence is in distribution.

Moreover, if $\sigma^2 = 0$, then for every periodic point $q \in \Lambda$, there exists $T > 0$ (independent of v) such that $\int_0^T v(Z_t q)\, dt = 0$.

Remark 1.1. Since there are infinitely many distinct periodic solutions in Λ, it follows from the final statement of Theorem B that the family of C^k observables $v : \mathbb{R}^3 \to \mathbb{R}$ for which $\sigma^2 = 0$ forms an infinite codimension family in the space of all C^k observables.

By [19], geometric Lorenz flows satisfy also an Almost Sure Invariance Principle (ASIP) for vector-valued observables $v : \mathbb{R}^3 \to \mathbb{R}^d$. Such a result is currently unavailable for the time-1 map Z, but we are able to prove a scalar ASIP.

Theorem C. *Let $G \in \mathcal{U}$. There exists $k \geq 1$ such that for all C^k observables $v : \mathbb{R}^3 \to \mathbb{R}$ the ASIP holds for the time-1 map: passing to an enriched probability space, there exists a sequence $X_0, X_1, \ldots$ of iid normal random variables with mean zero and variance σ^2 (as in Theorem B), such that*

$$\sum_{j=0}^{n-1} v \circ Z^j = n \int v \, d\mu + \sum_{j=0}^{n-1} X_j + O(n^{1/4} (\log n)^{1/2} (\log \log n)^{1/4}), \ a.e.$$

Remark 1.2. The ASIP implies the CLT and also the functional CLT (weak invariance principle), and the law of the iterated logarithm together with its functional version, as well as numerous other results. See [27] for a comprehensive list.

1.2. **Comments and organization of the paper.** In Section 2, we recall basic properties of geometric Lorenz attractors. In Section 3, we define the temporal distortion function and prove a result about the dimension of its range. This is the main new ingredient in the proof of Theorem A in Section 4.

In Section 5, we prove a general result on the ASIP for time-1 maps of nonuniformly expanding semiflows. In Section 6, we prove that the ASIP is typically nondegenerate. In Section 7, we prove Theorems B and C.

It is natural to extend all these results to more general singular-hyperbolic attractors (formerly referred to as Lorenz-like flows), that is, transitive attracting sets of three-dimensional flows having finitely many Lorenz-like singularities and a volume hyperbolic structure; see e.g. [5] for the precise definitions. Indeed, analogously to the geometric Lorenz case, it is possible to reduce the dynamics of these attractors to a piecewise expanding $C^{1+\epsilon}$ one-dimensional map; see e.g. [5, Chapter 6] or [4, Section 4] for a detailed presentation.

Conjecture 1. Let $\mathcal{U}$ denote the open set of C^∞ vector fields having a singular-hyperbolic attractor on a given compact three-dimensional manifold. Then the results stated in Theorems A, B and C are true for all $G \in \mathcal{U}$.

There exists a natural generalization of singular-hyperbolicity for higher-dimensional attractors, known as sectional-hyperbolicity; see e.g. [5, Sections 5.2 & 8.2] and also [25]. In this setting both the stable and the unstable manifolds of points in the attractor need not be codimension one embedded submanifolds, which makes analysis of these singular flows challenging.

Conjecture 2. Let $\mathcal{U}$ denote the open set of C^∞ vector fields having a sectional-hyperbolic attractor in a given compact finite dimensional manifold. Then the results stated in Theorems A, B and C are true for all $G \in \mathcal{U}$.

Notation. Throughout, C is used to denote a constant whose value may change from line to line.

2. Geometric aspects of Lorenz attractors

2.1. **Geometric Lorenz attractors.** We define here the open set $\mathcal{U}$ of C^∞ vector fields exhibiting strongly dissipative geometric Lorenz attractors and we describe the basic structure of such attractors; see e.g. [5].

Let G be a strongly dissipative C^∞ vector field on $\mathbb{R}^3$ possessing a Lorenz-like equilibrium, which we suppose without loss to be at 0. We assume that the flow Z_t is $C^{1+\epsilon}$ linearizable in a neighborhood of 0 which, by a suitable choice of coordinates, can be assumed to contain the cube $[-1, 1]^3$. (It follows from [17, Theorem 12.1] that smooth linearizability holds for an open and dense set of such vector fields.) Choose coordinates x_1, x_2, x_3 corresponding to the eigenspaces of λ_u, λ_{ss}, λ_s respectively. We define the cross-section $X = \{(x_1, x_2, 1) : |x_1|, |x_2| \leq 1\}$ and the Poincaré map $f : X \to X$. For $x \in X$ we write $f(x) = Z_{r(x)}(x)$ where $r : X \to \mathbb{R}^+$ is the Poincaré first return time to X, also referred to as the roof function.

We assume that there exists a global exponentially contracting f-invariant *stable foliation*. That is, there is a compact neighborhood $N \subset X$ of $(0,0,1)$ satisfying $f(N \setminus \{x_1 = 0\}) \subset X$ and a partition $\mathcal{W}^s_f$ of N consisting of C^∞ one-dimensional disks called *stable leaves* (including the "singular leaf" $\{x_1 = 0\}$). Let $W^s_f(x)$ denote the stable leaf containing x. Then it is required that $f(W^s_f(x)) \subset W^s_f(f(x))$ for all $x \in N$ and that there exist constants $C > 0$, $\lambda_0 \in (0,1)$ such that $|f^n(x) - f^n(x')| \le C\lambda_0^n$ for all x, x' in the same leaf and all $n \ge 1$.

Moreover, we assume that $\mathcal{W}^s_f$ is a $C^{1+\epsilon}$ foliation, meaning that N can be chosen so that there is a $C^{1+\epsilon}$ change of coordinates from the interior of N onto $(-1,1) \times (-1,1)$ transforming stable leaves into vertical lines.

Shrinking N if necessary, we can arrange that each stable leaf intersects $\bar{X} = \{(x_1, 0, 1) : |x_1| \le 1\} \cong [-1,1]$ in a single point. Define the $C^{1+\epsilon}$ projection $\pi : X \to \bar{X}$ given by holonomy along the stable leaves (so $\pi(x) = W^s_f(x) \cap \bar{X}$). Quotienting along stable leaves, we obtain a $C^{1+\epsilon}$ one-dimensional map $\bar{f} : \bar{X} \to \bar{X}$ with a singularity at 0: $\bar{f}(x_1) = \pi(f(x_1, 0, 1))$.

Lemma 2.1 (Proposition 2.6 in [19])**.** *Let $\eta = -\lambda_s/\lambda_u \in (0,1)$.*

1. *$\bar{f}'$ is Hölder on $\bar{X} \setminus \{0\}$: $\bar{f}'(x) = |x|^{\eta-1} g(x)$ with $g \in C^{\eta\epsilon}(\bar{X}), g > 0$;*
2. *the roof function has a logarithmic singularity at 0: $r = h_1 + h_2$ with $h_1(x) = -\lambda_u^{-1} \log |\pi(x)|$ and $h_2 \in C^\epsilon(X)$.*

In addition, we assume that $\bar{f}$ is *uniformly expanding*: there are constants $\lambda_1 > 1$ and $c > 0$ such that $|(\bar{f}^n)'(x)| \ge c\lambda_1^n$ for all $x \in \bar{X}$ and $n > 1$.

As in [21], we assume further that $\bar{f}$ is *locally eventually onto (l.e.o.)*; namely that for any open set $U \subset \bar{X} \setminus \{0\}$, there exists $k \ge 0$ such that $f^k U$ contains $(0,1)$. (More generally, it suffices that almost every point in $\bar{X}$ has dense preimages in $\bar{X}$. However, the l.e.o. property is standard in the literature and holds for the classical Lorenz attractor [31].)

Considering $U = \bigcup_{x \in X} Z_{[0,r(x)]}(x)$ we obtain a closed neighborhood of $[-1,1]^3$ and, in what follows, we denote by $\Lambda = \bigcap_{t>0} Z_t(U)$ the *geometric Lorenz attractor* of the vector field G. It can be shown that Λ is compact, volume hyperbolic, has a dense regular orbit and has zero volume (Lebesgue measure in $\mathbb{R}^3$); see e.g. [5, 2].

2.2. **Volume hyperbolicity, dissipativity and consequences.** We recall that, in our three-dimensional setting, volume hyperbolicity means that there exists a DZ_t-invariant singular-hyperbolic splitting of the tangent bundle over Λ. That is, there is a vector bundle splitting $T_\Lambda \mathbb{R}^3 = E \oplus F$ with $\dim E = 1$, $\dim F = 2$, and there are constants $c > 0$, $\lambda \in (0,1)$, such that for all $x \in \Lambda$, $t > 0$,

- *the splitting is dominated*: $\|DZ_t \mid E_x\| \cdot \|DZ_{-t} \mid F_{X_t(x)}\| < c\,\lambda^t$;
- *E is uniformly contracting*: $\|DZ_t \mid E_x\| < c\,\lambda^t$;
- *the area along F is uniformly expanded*: $|\det DZ_t \mid F_x| \ge c\lambda^{-t}$.

The existence of the stable foliation $\mathcal{W}^s_f$ of any small cross-section to the flow of G (such as X) is a consequence of volume hyperbolicity for three-dimensional smooth flows; see e.g. [5, Chapter 3, Section 3].

An important consequence of domination, uniform contraction along the stable direction E and strong dissipativity for the attractor Λ is the existence of a $C^{1+\epsilon}$ global exponentially contracting Z_t-invariant foliation $\mathcal{F}^{ss}$, defined in a neighborhood (which we may take to be U) of Λ.

Lemma 2.2. *The strong stable foliation $\mathcal{F}^{ss}$ is $C^{1+\epsilon}$ for some $\epsilon > 0$.*

Proof. We apply [18, Theorem 6.2] adapted to our setting, since only domination and uniform contraction is used in its proof. Indeed, a sufficient condition to obtain $C^{1+\epsilon}$ regularity for the strong stable foliation is that for some $t > 0$,

$$\|DZ_t \mid E_x\| \cdot \|DZ_t \mid F_x\|^{1+\epsilon} \cdot \|DZ_{-t} \mid F_{Z_t(x)}\| < 1 \tag{2.1}$$

for all $x \in \Lambda$. (We note that the statement in [18] covers only the case $\epsilon = 0$, but it is standard that their result extends to the case $\epsilon > 0$.)

For each $t \in \mathbb{R}$ we define $\eta_t : \Lambda \to \mathbb{R}$,

$$\eta_t(x) = \log\Big\{\|DZ_t|E_x\| \cdot \|DZ_t|F_x\|^{1+\epsilon}\|DZ_{-t}|F_{Z_t x}\|\Big\}.$$

Note that $\{\eta_t,\ t \in \mathbb{R}\}$ is a continuous family of continuous functions each of which is subadditive, that is, $\eta_{s+t}(x) \le \eta_s(x) + \eta_t(Z_s(x))$.

Let $\mathcal{M}$ denote the set of flow-invariant ergodic probability measures on Λ. We claim that for $\epsilon > 0$ sufficiently small, and for each $m \in \mathcal{M}$, the limit $\lim_{t\to\infty} \frac{1}{t}\eta(x)$ exists and is negative for m-almost every $x \in \Lambda$. It then follows from [8, Proposition 3.4] that there exists constants $C, \beta > 0$ such that $\exp \eta_t(x) \le Ce^{-\beta t}$ for all $t > 0$, $x \in \Lambda$. In particular, for t sufficiently large, $\exp \eta_t(x) < 1$ for all $x \in \Lambda$. Hence condition (2.1) is satisfied for such ϵ and t and the result follows.

It remains to verify the claim. Let m_0 denote the Dirac delta concentrated at 0 and let $\mathcal{M}_1 = \mathcal{M} \setminus \{m_0\}$. We deal with the cases $m \in \mathcal{M}_1$ and $m = m_0$ separately.

Each $m \in \mathcal{M}_1$ has a zero Lyapunov exponent in the flow direction and two further Lyapunov exponents $\lambda_E(m) < 0$ and $\lambda_F(m) > 0$ associated with the vector bundles E and F respectively. Fix $m \in \mathcal{M}_1$. For m-a.e. $x \in \Lambda$ we have

$$\lim_{t\to\infty} \frac{1}{t} \log|\det Z_t(x)| = \lambda_E(m) + \lambda_F(m), \tag{2.2}$$

$$\lim_{t\to\infty} \frac{1}{t} \log\|DZ_t|E_x\| = \lambda_E(m), \tag{2.3}$$

$$\lim_{t\to\infty} \frac{1}{t} \log\|DZ_t|F_x\| = \lambda_F(m), \tag{2.4}$$

$$\lim_{t\to\infty} \frac{1}{t} \log\|DZ_{-t}|F_{Z_t x}\| = 0. \tag{2.5}$$

On the other hand, it follows from dissipativity that $\limsup_{t\to\infty} \frac{1}{t}\log|\det Z_t(x)| \le -\delta$ for all x. Hence we deduce from (2.2) that $\lambda_E(m) + \lambda_F(m) \le -\delta$. Moreover, $\lambda_F(m) \le \sup_\Lambda \|DG\|$, so for $\epsilon > 0$ sufficiently small (uniformly in m) $\lambda_E(m) + (1+\epsilon)\lambda_F(m) < 0$. Using (2.3), (2.4) and (2.5) together with the definition of η_t, it follows that $\lim_{t\to\infty} \frac{1}{t}\eta_t(x) < 0$ for m-almost every $x \in \Lambda$.

It remains to consider the Dirac measure m_0. By strong dissipativity, for ϵ sufficiently small, $\frac{1}{t}\eta_t(0) = \lambda_{ss} + (1+\epsilon)\lambda_u - \lambda_s < 0$ for all t as required. □

By Lemma 2.2, we may consider the cross-section $X = \bigcup\{F^{ss}(x) : x \in \bar{X}\}$ in the place of the original cross-section X. Since the strong stable foliation is $C^{1+\epsilon}$ and the cross-section X is foliated by stable leaves over the smooth disk $\overline{X}$, it follows that X is a $C^{1+\epsilon}$ embedded surface in $\mathbb{R}^3$. All the properties described so far are retained, with the useful advantage that $W^s_f(x) = F^{ss}(x)$ for all $x \in X$ and

- **(C):** the first return time $r : X \to \mathbb{R}^+$ of any given point in $X \setminus \{x_1 = 0\}$ (where $\{x_1 = 0\}$ now represents the leaf of $\mathcal{W}^s_f$ through the point $0 \in \bar{X}$) to X is constant on the leaves of $\mathcal{W}^s_f$, that is, $r(x) = r(\pi(x))$ for all $x \in X \setminus \{x_1 = 0\}$. Since the cross-section X is a $C^{1+\epsilon}$ embedded surface in $\mathbb{R}^3$, the roof function $r : X \to \mathbb{R}^+$ retains the properties mentioned in Lemma 2.1(2); in particular r is a $C^{1+\epsilon}$ function with a logarithmic singularity at $\{x_1 = 0\}$.

We keep the notation $\pi : X \to \bar{X}$ for the holonomy along the leaves of $\mathcal{W}^s_f$ to $\bar{X}$ and also $\bar{f}$ for the one-dimensional $C^{1+\epsilon}$ quotient map of $f : X \setminus \{x_1 = 0\} \to X$ over $\mathcal{W}^s_f$.

Another consequence of volume hyperbolicity is that there exists a field of cones $\tilde{C}_b(x) = \{(u, v) \in E_x \times F_x : b\|v\| \ge \|u\|\}$ having width $b > 0$ containing the F subbundle over Λ which admit a continuous DZ_t-invariant extension $\hat{C}_b(x)$ to a neighborhood of Λ. For the geometric Lorenz flow we can assume without loss that this neighborhood coincides with U.

The invariance means that $DZ_t \cdot \hat{C}_b(x) \subset \hat{C}_b(Z_t(x))$ for x in an open neighborhood U of Λ and $t > 0$, where $b > 0$ is small enough. Then the cones $C_b(x) = \hat{C}_b(x) \cap T_xX$ on T_xX are also $D\bar{f}$-invariant and defined on the whole of $X \cap U$.

We say that a C^1 curve γ in X is a *u-curve* if $\gamma'(s) \subset C_b(\gamma(s))$ for all parameter values s. The $D\bar{f}$-invariance of the field of cones C_b ensures that the image by f of every u-curve is sent into another u-curve. Moreover, the tangent direction to the stable leaves $T_x\mathcal{W}^s_f(x)$ is not contained in the $C_b(x)$ cone and makes an angle bounded away from zero with any vector inside $C_b(x)$, for all $x \in X$, by the volume hyperbolicity assumption; see [6, 5].

Remark 2.3. Tucker [31] showed that the classical Lorenz equations have a robust nontrivial attractor Λ containing the equilibrium at the origin. It follows from Morales *et al.* [26] that Λ is a singular hyperbolic attractor that (in their words) "resembles a geometric Lorenz attractor". In particular, it is immediate that all of the properties listed above are satisfied except possibly for (i) strong dissipativity, (ii) the l.e.o property, and (iii) smoothness (class $C^{1+\epsilon}$) of the contracting foliations $\mathcal{F}^{ss}$ and $\mathcal{W}^s_f$ for the flow and Poincaré map respectively. We note that property (i) is immediate for the classical Lorenz equations and condition (ii) was verified in [31]. Regarding (iii), it is claimed in [31, Section 2.4] that $\mathcal{W}^s_f$ is a smooth foliation but no details are provided.

Smoothness of the contracting foliations $\mathcal{F}^{ss}$ and $\mathcal{W}^s_f$ is not part of the definition of singular hyperbolic attractor, and hence is not discussed in [26]. However, proofs of existence of an SRB measure with good statistical properties rely heavily on the smoothness of $\mathcal{W}^s_f$. Although this foliation is of codimension one, the fact that the Poincaré map f is singular means that an extra argument is required; see for example Robinson [28] and Rychlik [29, Section 4]. In particular, our results apply to the open set of flows considered by [28, 29], but these do not include the classical Lorenz equations. However, as shown above in Lemma 2.2, the properties established by [26, 31] combined with strong dissipativity guarantee smoothness of $\mathcal{F}^{ss}$, and hence of $\mathcal{W}^s_f$, for the classical parameters and nearby parameters.

2.3. **Inducing and quotienting.** The geometric Lorenz attractor can be written as a suspension flow $S_t : X^r \to X^r$ given by $S_t(x, s) = (x, s + t)$ on the space

$$X^r = \{(x, t) \in X \times \mathbb{R} : 0 \le t \le r(x)\}/\sim$$

where $(x, r(x)) \sim (f(x), 0)$. Indeed, we can take the conjugacy as $\Phi : X^r \to U, (x, s) \mapsto Z_s(x)$ (which is smooth) and the roof function $r : X \to \mathbb{R}^+$ has a logarithmic singularity at all points of $X \cap \{x_1 = 0\}$, is smooth elsewhere and f is a non-uniformly hyperbolic map with invariant stable foliation $\mathcal{W}^s_f$. We denote also by 0 the point $\pi(\{x_1 = 0\})$ in what follows and since $r \circ \pi = r$ we also write r for the restriction $r : \bar{X} \to \mathbb{R}^+$.

In particular, since $\bar{f} : \bar{X} \to \bar{X}$ is a $C^{1+\epsilon}$-piecewise nonuniformly expanding map, there exists a subset $\bar{Y} \subset \bar{X}$ and an inducing time $\tau : \bar{Y} \to \mathbb{Z}^+$ such that $\bar{F} = \bar{f}^\tau : \bar{Y} \to \bar{Y}$ is a $C^{1+\epsilon}$-piecewise expanding Markov map with partition α_0; see e.g. [7, Theorem 4.3].

Since $\bar{f}$ is l.e.o., for any specified point $x \in \bar{X}$ we can choose $\bar{Y}$ to be an arbitrarily small open interval containing x; see e.g. [3], where an inductive construction procedure is described showing that we can build a full branch Markov map $\bar{F} : \bar{Y} \to \bar{Y}$ as long as $\bar{Y}$ is a neighborhood of a point with dense preimages.

During the paper, we will consider various inducing schemes. All of these are full branch on an interval except for the one constructed in Section 3.1 which is the combination of two such inducing schemes.

Let $\alpha_0^n = \bigvee_{i=0}^{n-1}(\bar{F}^i)^{-1}(\alpha_0)$ denote the nth refinement of α_0, and set $\tau_n(y) = \sum_{j=0}^{n-1} \tau(\bar{f}^j(y))$, so that $\bar{F}^n(y) = \bar{f}^{\tau_n(y)}(y)$.

The most important features of $\bar{f}$ are the following backward contraction and bounded distortion properties. There exist constants $c_0 > 0, \lambda \in (0, 1)$ such that for each $n \ge 1$

backward contraction: $\bar{F}^n \mid_{\alpha_0^n(y)}: \alpha_0^n(y) \to \bar{Y}$ is a $C^{1+\epsilon}$ diffeomorphism and if $y' \in \alpha_0^n(y)$, then

$$|\bar{f}^i(y') - \bar{f}^i(y)| \le c_0 \lambda^{\tau_n(y)-i} |\bar{F}^n(y') - \bar{F}^n(y)|, \quad i = 0, \ldots, \tau_n(y) - 1, \tag{2.6}$$

Moreover there is *slow recurrence to the singular point*

$$|\bar{f}^i(y)| \ge \sqrt{\lambda}^{\tau_n(y)-i}, \quad i = 0, \ldots, \tau_n(y) - 1; \tag{2.7}$$

bounded distortion: if $y' \in \alpha_0^n(y)$, then

$$\left| \frac{D\bar{F}^n(y)}{D\bar{F}^n(y')} - 1 \right| \le c_0 |\bar{F}^n(y) - \bar{F}^n(y')|. \tag{2.8}$$

We note that the induced map can be obtained by the methods presented in [3] and conditions (2.6) and (2.7) follow from the definition of hyperbolic times (c.f. Definition 10 in [3] with $b = 1/2$).

Next we construct a piecewise uniformly hyperbolic map $F : Y \to Y$ with infinitely many branches, which covers $\bar{F}$, as follows: Define $Y = \bigcup\{W_f^s(y) : y \in \bar{Y}\}$ to be the union of the stable leaves through $\bar{Y}$ and define the Poincaré return map $F(y) = f^{\tau(\pi(y))}(y)$ for $y \in Y$. We let α denote the measurable partition of Y whose elements are $\bigcup\{W_f^s(x) : x \in a\}$ with $a \in \alpha_0$. Also, we extend $\tau : \bar{Y} \to \mathbb{Z}^+$ to a function on Y by setting $\tau(y) = \tau(\pi y)$.

Let $\mu_{\bar{Y}}$ be the unique $\bar{F}$-invariant absolutely continuous probability measure on $\bar{Y}$. It is well-known that $r \in L^1(\mu_{\bar{Y}})$. It is then standard that there exist unique invariant measures μ_Y for $F : Y \to Y$, μ_X for $f : X \to X$ and $\mu_{\bar{X}}$ for $\bar{f} : \bar{X} \to \bar{X}$ satisfying $\pi_*(\mu_Y) = \mu_{\bar{Y}}$, $\pi_*(\mu_X) = \mu_{\bar{X}}$ and also $\mu_X = \sum_{n\ge1}\sum_{j=0}^{n-1} f_*^j(\mu_Y|\{\tau\circ\pi = n\})$ and $\mu_{\bar{X}} = \sum_{n\ge1}\sum_{j=0}^{n-1} \bar{f}_*^j(\mu_{\bar{Y}}|\{\tau = n\})$. We have $\mu_Y \ll \mu_X$ and $\mu_Y(Y) = 1$, hence $\mu_X(Y) > 0$; see e.g. [7, Section 3] for more details.

2.4. **Local product structure.** Here we obtain an almost everywhere defined local product structure for the induced hyperbolic map $F : Y \to Y$. We have already seen that the Poincaré map $f : X \to X$ has a stable foliation $\mathcal{W}_f^s$ with leaves that cross X, and hence the induced map $F : Y \to Y$ has stable manifolds $W_F^s(y) = W_f^s(y)$ that cross Y. In the next proposition, we construct local unstable manifolds for F of uniform size, defined almost everywhere.

Proposition 2.4. *For μ_Y-almost every $y \in Y$, there exists a local unstable manifold $W_F^u(y) \subset W_{loc,f}^u(y)$ that crosses Y. In particular, $\pi(W_F^u(y)) = \bar{Y}$ for μ_Y-almost every $y \in Y$.*

Proof. We begin with the local unstable manifolds W_{loc}^u for the flow Z_t. It follows from Pesin theory (see e.g. [13]) that almost every point p of Λ with respect to the SRB measure μ admits a local unstable manifold $W_{loc}^u(p)$ which is a $C^{1+\epsilon}$-curve containing p in its interior. By definition, $p' \in W_{loc}^u(p)$ if and only if $|Z_t(p') - Z_t(p)| \le C_p\lambda^{-t}$ for all $t < 0$ (recall that $\lambda \in (0,1)$ and also that the constant C_p depends on the leaf); see e.g. [6]. Since Λ is an attractor, unstable leaves are contained in Λ.

The smooth conjugacy Φ^{-1} sends these leaves into unstable leaves for the suspension flow S_t, which can be written locally as $\tilde{W}_{loc}^u(\Phi^{-1}(p)) = \{(\gamma(s), t(s)) : s \in [-1,1]\}$, where $\gamma : [-1,1] \to X$ and $t : [-1,1] \to \mathbb{R}$ are $C^{1+\epsilon}$ diffeomorphisms into their images. By its definition, the curve γ is the local unstable manifold $W_{loc,f}^u(x)$ through $x = \gamma(0)$ with respect to f, that is, $x' \in W_{loc,f}^u(x)$ if and only if $|f^n(x') - f^n(x)| \le C'_x\lambda^{-n}$, for all $n < 0$. We observe that the inverse images of x and x' are all well defined since these points belong to the attractor $\Phi^{-1}(\Lambda)$ which is S_t-invariant. Moreover, the curve γ is a graph $\gamma(s) = (x(s), y(x(s)))$ and γ' is contained in a cone $\{(x', y') \in \mathbb{R}^2 : |y'| < \xi|x'|\}$ for some $0 < \xi < 1$ by the domination condition on f, consequence of the existence of dominating splitting for the flow on the attractor.

We remark that $W_{loc,f}^u(x)$ is a u-curve which coincides with $W_F^u(x)$ if x also belongs to Y, and so the statements above hold for μ_Y almost every point $y \in Y$. Indeed, on the one hand, $W_{loc,f}^u(x)$ is formed by points whose preorbit is asymptotic to the preorbit of x, hence these preorbits contain the preorbits with respect to F, and so $W_{loc,f}^u(x) \subset W_F^u(x)$. On the other hand, this inclusion shows that $W_{loc,f}^u(x)$ and $W_F^u(x)$ coincide in a neighborhood of x inside $W_F^u(x)$. Since these unstable manifolds are contained in the attractor then, repeating the argument around each point $z \in W_{loc,f}^u(x)$, we see that the two manifolds coincide.

In addition, the stable leaves through points $y \in W_F^u(x)$ are transverse to $W_F^u(y)$ and the angle between $T_yW_F^s(y)$ and $T_yW_F^u(y)$ is bounded away from zero. Hence $\pi(W_F^u(y))$ is a neighborhood of $y_0 = \pi(y)$ in $\bar{Y}$ for μ_Y-a.e. y.

Let $\alpha_0^n(y_0)$ denote the element of the nth refinement α_0^n that contains y_0. This is well-defined for all $n \geq 1$ for $\mu_{\bar{Y}}$-a.e. $y_0 \in \bar{Y}$. Moreover, $\pi(W_F^u(y)) \cap \alpha_0^n(y_0)$ is a neighborhood of y_0 for all $n \geq 1$ for μ_Y-a.e. y. Since $\bar{F}$ is full-branch, $\bar{F}^n(\alpha_0^n(y_0)) = \bar{Y}$ for all n.

By the Poincaré Recurrence Theorem, we may assume without loss that y is recurrent: there exists $n_i \to \infty$ such that $F^{n_i}y \to y$. Therefore, for all large enough i we have $\bar{F}^{n_i}(y_0) \in \pi(W_F^u(y))$ and hence the iterate of a connected piece of the unstable manifold of y defined by

$$W_{n_i} = F^{n_i}\big((\pi \mid W_F^u(y))^{-1}\alpha_0^{n_i}(y_0)\big)$$

is a u-curve that crosses Y. The sequence W_{n_i} has a convergent subsequence to W by the Arzelá-Ascoli Theorem and by the recurrence assumption on y we have $y \in W$.

We claim that $W = W_{loc,f}^u(y)$, which completes the proof that μ_Y-almost every point has an unstable manifold crossing Y. The last statement of the proposition is a simple restatement of this conclusion.

To prove the claim, we consider $y' \in W$ and sequences $y_i, y_i' \in W_{n_i}$ such that $(y_i, y_i') \to (y, y')$. Fix $l \geq 1$ and choose $L \in \mathbb{Z}^+$ so that $\tau_{n_i}(y') > l$ for all $i \geq L$. By the definition of W_{n_i} and since $W_{loc,f}^u(y') = W_F^u(y')$, we have uniform backwards contraction. Thus

$$|y_i - y_i'| = |f^l(f^{\tau_{n_i}(y')-l}(z_i)) - f^l(f^{\tau_{n_i}(y')-l}(y'))| \geq \frac{\lambda^{-l}}{c_0'}|f^{\tau_{n_i}(y')-l}(z_i) - f^{\tau_{n_i}(y')-l}(y')| \tag{2.9}$$

where $z_i \in W_{loc,f}^u(y')$ is such that $y_i = F^{n_i}(z_i)$. Hence $|f^{-l}(y_i) - f^{-l}(y_i')| \leq c_0'\lambda^l|y_i - y_i'|$. To obtain c_0' we have used that all the iterates W_n of $W_{loc,f}^u(y')$ are u-curves and so their length is comparable to the length of their projection $\pi(W_n)$ on $\bar{X}$; and then take advantage of the backward contraction property (2.6) associated to the partition α_0 with the same contraction rate λ. Finally, since these constants are independent of i, letting $i \to \infty$ gives $|f^{-l}(y) - f^{-l}(y')| \leq c_0'\lambda^l|y - y'|$ for each given fixed $l \geq 1$. This completes the proof of the claim and finishes the proof of the proposition. ☐

Using this geometric structure we can also prove the following:

Proposition 2.5. *The induced map $F : Y \to Y$ has a* local product structure*: for any partition element $a \in \alpha$ there exists a measurable map $[\cdot,\cdot] : Y \times a \to a$ defined for all $y' \in a$ and μ_Y almost every $y \in Y$ such that*

$$[y, y'] \in W_F^u(y) \pitchfork W_F^s(y')$$

consists of a unique point. In addition, the map $[\cdot,\cdot]$ is constant along unstable manifolds in the first coordinate, and constant along stable manifolds in the second coordinate. Furthermore, $[\cdot,\cdot]$ is $C^{1+\epsilon}$ in the second coordinate.

Proof. From Proposition 2.4 we have that for μ_Y almost every point y the local unstable manifold $W_F^u(y)$ crosses Y. From the definition of geometric Lorenz attractor, $W_F^s(y')$ crosses a transversely to $W_F^u(y)$, for every $y' \in Y$. Hence $[y, y']$ is well defined for μ_Y-almost every $y \in Y$ and every $y' \in Y$. Since a is a union of local stable manifolds, it is immediate that if $y' \in a$ then $[y, y'] \in a$.

We note that if $[y, y']$ is defined, then

$$w \in W_F^u(y) \mapsto [w, y'] = [y, y'] \quad \text{and} \quad w \in W_F^s(y') \mapsto [w, s] = [y, y']$$

which shows that $[y, y']$ is constant along unstable manifolds on the first coordinate and stable manifolds on the second coordinate. In addition, the stable manifolds $W_F^s(y')$ depend continuously in the $C^{1+\epsilon}$ topology on the base point y' (by the partial hyperbolicity of the attractor) and the unstable manifolds $W_F^u(y)$ depend measurably on y (by nonuniform hyperbolicity). Hence $[\cdot,\cdot]$ is a measurable map and is $C^{1+\epsilon}$ along the second coordinate. ☐

Proposition 2.6. *Suppose that $y, y' \in Y$ are such that $[y, y']$ is well-defined. Then, there is a sequence of periodic points $z_i \in Y$ for F such that (i) $z_i \to y$, (ii) $[z_i, y']$ is well-defined for all i, and (iii) $[z_i, y'] \to [y, y']$.*

Proof. We use Proposition 2.4: we can assume without loss that y is recurrent. Let us fix a neighborhood U of y given by $U_1 \times U_2$, where U_1 is an open subinterval of $[-1,1] \setminus \{0\}$ and U_2 is an open subset of $[-1,1]$. We fix a similar smaller neighborhood $V = V_1 \times V_2$ such that closure of V_j is contained in $U_j, j = 1,2$. We can regard V_1 as a neighborhood of πy.

In our setting, this ensures the existence of a sequence $n_i \to \infty$ such that $F^{n_i} y \to y$, $\pi\alpha^{n_i}(y)$ is a neighborhood of πy and $\pi F^{n_i}(\alpha^{n_i}(y)) = \bar{Y}$; see the proof of Proposition 2.4.

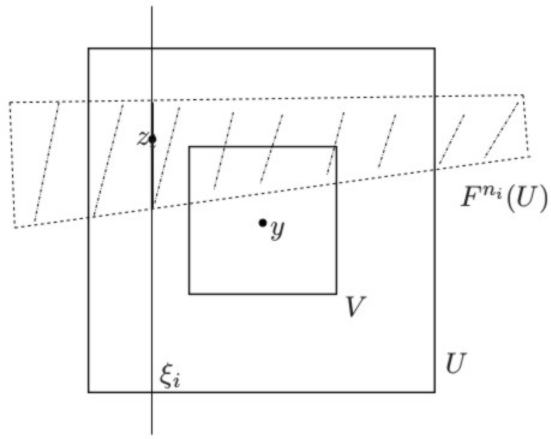

FIGURE 1. The density of periodic points for F.

Hence, there exists a stable leaf $\xi_i = \pi^{-1}(\bar{x}_i) \subset \alpha^{n_i}(y)$ for some $\bar{x}_i \in \pi\alpha^{n_i}(y)$ which is sent inside itself by F^{n_i}, by the uniform contraction of the stable leaves of $\mathcal{W}^s_F$. Since we can assume without loss that $F^{n_i} y \in V$, then taking n_i big enough, we claim that $F^{n_i}(\xi_i \cap U) \subset U$. Indeed, due to the domination assumption on f, the unstable manifold $W^u_F(F^{n_i} y)$ crosses Y and its angle with respect to the horizontal direction is uniformly bounded from above, so $W^u_F(F^{n_i} y) \pitchfork \xi_i \subset U$, and the claim follows by uniform contraction of the stable leaves; see Figure 1.

Thus we have a fixed point z_i of F^{n_i} in $\xi_i \cap U$. Since U belongs to a fundamental system of neighborhoods of y, this proves item (i) in the statement of the Proposition.

All periodic points of f are hyperbolic of saddle type, thus $W^u_F(z_i)$ is well-defined and $\pi W^u_F(z_i)$ is a neighborhood of $\pi z_i = \bar{x}_i$.

If $\pi W^u_F(z_i) \supset \pi\alpha^{n_i}(y)$, then clearly $W^u_F(z_i)$ crosses a and so $[z_i, y']$ is well-defined. We claim that this is always the case. For otherwise, if $\pi W^u_F(z_i) \subset \alpha^{n_i}(y)$, then since $\bar{F}^{n_i} \mid \pi\alpha^{n_i}(y) : \pi\alpha^{n_i}(y) \to \pi\alpha(y)$ is an expanding diffeomorphism and $W^u_F(z_i)$ is F^{n_i}-invariant, the length of $\bar{F}^{kn_i}(\pi W^u_F(z_i)), k \geq 1$ grows while this set is contained in $\alpha^{n_i}(y)$. Thus $\pi W^u_F(z_i)$ covers $\pi\alpha^{n_i}(y)$, as claimed.

Finally, for the continuity statement (iii), since $z_i \to y$ as $i \to \infty$ with z_i periodic points of F, it is enough to show that $W^u_F(z_i) \cap a \to W^u_F(y) \cap a$ as smooth curves that cross a. Since each curve $W^u_F(z_i) \cap a$ is a u-curve, then there exists a accumulation point γ which is also a u-curve (by the Arzelá-Ascoli Theorem). We show that γ contains $W^u_F(y) \cap a$.

Indeed, γ contains y. By uniform backward contraction, if $z^0_i, \tilde{z}^0_i \in W^u_F(z_i) \cap a$ converge to $z, \tilde{z} \in \gamma \cap a$, then we can argue similarly to (2.9) since there are $z^k_i, \tilde{z}^k_i \in W^u_F(z_i) \cap a$ so that $z^0_i = F^k(z^k_i)$ and $\tilde{z}^0_i = F^k(\tilde{z}^k_i), k \geq 1$. Hence, for a given fixed k we get

$$|z^0_i - \tilde{z}^0_i| = |F^k(z^k_i) - F^k(\tilde{z}^k_i)| \geq \frac{\lambda^{-k}}{c'_0}|z^k_i - \tilde{z}^k_i|,$$

and for limit points $z^k, \tilde{z}^k \in \gamma$ of $(z^k_i)_{i\geq 1}$ and $(\tilde{z}^k_i)_{i\geq 1}$ letting $i \to \infty$, we obtain (since F is smooth in a)

$$|z - \tilde{z}| = |F^k(z_k) - F^k(\tilde{z}_k)| \geq \frac{\lambda^{-k}}{c'_0}|z_k - \tilde{z}_k|.$$

Hence, because $k \geq 1$ was arbitrary, we see that $\gamma \subset W_F^u(y)$, as needed. The proof is complete. □

2.5. **The induced roof function** R. We define the *induced roof function* $R : Y \to \mathbb{R}^+$, $R(y) = \sum_{\ell=0}^{\tau(y)-1} r(f^\ell y)$. Since r, and hence R, is constant along stable leaves, we also denote by R the quotient induced roof function $R : \bar{Y} \to \mathbb{R}^+$.

It follows in a completely analogous way to [7, Section 4.2.2] that $R : \bar{Y} \to \mathbb{R}^+$ satisfies

$$\sup_{h \in \mathcal{H}} \sup_{y \in \bar{Y}} |D(R \circ h)(y)| < \infty \tag{2.10}$$

where $\mathcal{H}$ is the set of all inverse branches of $\bar{F} : a_0 \to \bar{Y}, a_0 \in \alpha_0$.

Indeed, let $h \in \mathcal{H}$, $h : \bar{Y} \to a_0$ be an inverse branch of $\bar{F}$ with inducing time $l = \tau(a_0) \geq 1$ and let us fix $y \in a_0$. Then

$$|D(R \circ h)(y)| = |DR(h(y))| \cdot |Dh(y)| = \frac{|DR(h(y))|}{|D\bar{F}(h(y))|} = \left| \sum_{i=0}^{l-1} \frac{(Dr \circ \bar{f}^i) \cdot D\bar{f}^i}{D\bar{F}} \circ h(y) \right|.$$

Recall from the construction of the inducing partition using hyperbolic times that conditions (2.6) and (2.7) are satisfied (here we require these conditions only with $n = 1$). By (2.6),

$$\left| \frac{D\bar{f}^i}{D\bar{F}} \right| \circ h(y) \leq c_0 \lambda^{l-i}, \quad i = 0, \ldots, l-1.$$

Moreover, by (2.7), $|\bar{f}^i(h(y))| \geq \sqrt{\lambda}^{l-i}$ and so, by Lemma 2.1(2),

$$|(Dr \circ \bar{f}^i) \circ h(y)| \leq c_1 / \sqrt{\lambda}^{l-i}, \quad i = 0, \ldots, l-1,$$

for some $c_1 > 0$ depending only on f. Altogether this implies that $|D(R \circ h)(y)| \leq c_1 \sum_{i=0}^{\infty} \lambda^{i/2} < \infty$ establishing (2.10).

Proposition 2.7. *There exists $c > 0$ such that $\mu_Y(R > t) = O(e^{-ct})$.*

Proof. By [7, Section 4.2.1], $\mathrm{Leb}(R > t) = O(e^{-ct})$. The result follows since $d\mu_Y/d\mathrm{Leb}$ is bounded, see for example [7, Proposition 4.5]. (We caution that our $R : Y \to \mathbb{R}^+$ is denoted by $r : \Delta \to \mathbb{R}^+$ in [7]). □

Remark 2.8. Proposition 2.7 is not necessary for the results in this paper but simplifies the exposition, see Remark 4.5.

For $y, y' \in \bar{Y}$ define the *separation time* $s(y, y')$ to be the least integer $n \geq 0$ such that $\bar{F}^n(y), \bar{F}^n(y')$ are in distinct partition elements of α_0. For any given $\theta \in (0, 1)$ we define the *symbolic metric* $d_\theta(y, y') = \theta^{s(y,y')}$ on $\bar{Y}$. Let $|R|_\theta = \sup_{y \neq y'} |R(y) - R(y')|/d_\theta(y, y')$ denote the Lipschitz constant of the quotient induced roof function $R : \bar{Y} \to \mathbb{R}^+$ with respect to d_θ.

We write $r_k(y)$ for the sum $\sum_{i=0}^{k-1} r(f^i(y))$ in what follows.

Lemma 2.9. *There exists $C > 0$ such that for all $y, y' \in \bar{Y}$ with $s(y, y') \geq 1$ and $0 \leq k \leq \tau(y) = \tau(y')$ we have $|r_k(y) - r_k(y')| \leq C|\bar{F}(y) - \bar{F}(y')|^\epsilon$. As a consequence, there exists $\theta \in (0, 1)$ such that $|R|_\theta < \infty$ and also $|\bar{F}(y) - \bar{F}(y')| \leq C d_\theta(y, y')$.*

Proof. Let us consider $y, y' \in \bar{Y}$ such that $s(y, y') = n \geq 1$. Then $y' \in \alpha_0^n(y)$ and so $\tau(\bar{F}^i(y)) = \tau(\bar{F}^i(y')), i = 0, \ldots, n-1$. Thus, from the choice of the cross-section, ensuring that r is constant on stable leafs, together with Lemma 2.1(2), we can write

$$\begin{aligned} R(y) - R(y') &= \sum_{\ell=0}^{\tau(y)-1} [r(\bar{f}^\ell(y)) - r(\bar{f}^\ell(y'))] \\ &= \sum_{\ell=0}^{\tau(y)-1} \big[-\lambda_u^{-1}(\log|\bar{f}^\ell(y)| - \log|\bar{f}^\ell(y')|) + h(\bar{f}^\ell(y)) - h(\bar{f}^\ell(y')) \big]. \end{aligned}$$

Combining (2.6) together with (2.7) we obtain

$$\begin{aligned}|R(y)-R(y')| &\le C\sum_{\ell=0}^{\tau(y)-1}\left[\frac{|\bar f^\ell(y)-\bar f^\ell(y')|}{\max\{|\bar f^\ell(y)|,|\bar f^\ell(y')|\}}+\|h\|_\epsilon|\bar f^\ell(y)-\bar f^\ell(y')|^\epsilon\right]\\ &\le C\sum_{\ell=0}^{\tau(y)-1}\left[c_0\frac{\lambda^{\tau(y)-\ell}}{\sqrt{\lambda}^{\tau(y)-\ell}}|\bar F(y)-\bar F(y')|+\|h\|_\epsilon\lambda^{\epsilon(\tau(y)-\ell)}|\bar F(y)-\bar F(y')|^\epsilon\right]\\ &\le \kappa\lambda^{\epsilon/2}\cdot\frac{1-\lambda^{\frac{\epsilon}{2}\cdot\tau(y)}}{1-\lambda^{\epsilon/2}}|\bar F(y)-\bar F(y')|^\epsilon,\end{aligned}$$

for some constant $\kappa>0$ depending on the flow only. This implies in particular the first statement of the lemma.

Finally, because τ is at least 1, we also have

$$|\bar F(y)-\bar F(y')|^\epsilon \le c_0\lambda^{\epsilon\tau_{n-1}(\bar F(y))}|\bar F^n(y)-\bar F^n(y')|^\epsilon \le c_0\lambda^{\epsilon(n-1)}|\bar F^n(y)-\bar F^n(y')|^\epsilon$$

which, combined with the previous inequality, gives another constant $K>0$ depending only on the flow satisfying

$$|R(y)-R(y')|\le K\lambda^{n\epsilon/2}|\bar F^n(y)-\bar F^n(y')|^\epsilon.$$

We can find $\lambda^{\epsilon/2}<\theta<1$ and a constant $C_0>0$ so that $K\lambda^{n\epsilon/2}\le C_0\theta^n=C_0d_\theta(y,y')$ for all $n\ge 1$ and, since $|\bar F^n(y)-\bar F^n(y')|^\epsilon$ is bounded above, the proof is complete. □

2.6. **Expansion for the flow.** We are now ready to prove a useful consequence of backward contraction for the quotient map and expansion of the flow in the linearizable region. We keep the choice of θ from Lemma 2.9. Also, we define $d_\theta(y,y')$ for points $y,y'\in Y$ by setting $d_\theta(y,y')=d_\theta(\pi y,\pi y')$.

Lemma 2.10. *There exist constants $C,\kappa>0$ such that for all $y,y'\in Y$ satisfying $d_\theta(y,y')<\kappa$ and all $u\in(0,\min\{R(y),R(y')\})$, we have $|Z_u(y)-Z_u(y')|\le Cd_\theta(y,y')$.*

Proof. Taking $\kappa<1$, we have $\tau(y)=\tau(y')$. There exist $k,k'\in\{1,\dots,\tau(y)\}$ such that

$$u\in[r_{k-1}(y),r_k(y)]\quad\text{and}\quad u\in[r_{k'-1}(y'),r_{k'}(y')].$$

From Lemma 2.9 we know that $|r_k(y)-r_k(y')|\le Cd_\theta(y,y')$. Hence for $d_\theta(y,y')$ small enough, $|r_k(y)-r_k(y')|<\inf r$ for all $0\le k\le\tau(y)$. It follows that $|k-k'|\le 1$, and we may suppose that $k\ge k'$. Hence, there are two cases to consider.

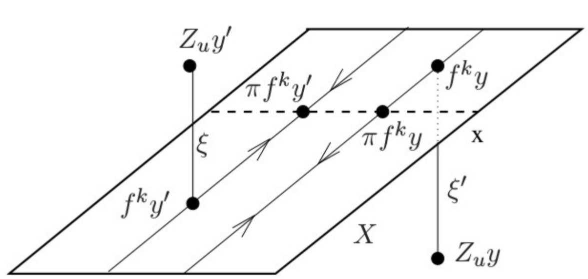

FIGURE 2. Estimating the distance between $Z_u y$ and $Z_u y'$.

The case $k = k' + 1$: We have $r_{k'}(y) \le u \le r_{k'}(y')$ and so the orbit of y has already had k' returns to the cross-section X, while the orbit of y' has only returned $k'-1$ times. We estimate the distance between the points with the distance between the $k'th$ returns of both orbits to X as follows. Setting $\xi = u - r_{k'}(y)$ and $\xi' = r_{k'}(y') - u$, we get

$$\begin{aligned}|Z_u y - Z_u y'| &\le |\xi + \xi'| + |f^k y - f^k y'| \\ &\le |r_{k'}(y) - r_{k'}(y')| + |f^k y - \bar{f}^k \pi y| + |\bar{f}^k \pi y - \bar{f}^k \pi y'| + |\bar{f}^k \pi y' - f^k y'| \\ &\le C d_\theta(y,y') + c\lambda^k(|y - \pi y| + |y' - \pi y'|) + c_0 \lambda^{\tau(y)-k} |\bar{F} y - \bar{F} y'|\end{aligned}$$

where we have used the uniform contraction of the stable foliation of the attractor, together with Lemma 2.9 and the uniform backward contraction of iterates of $\bar{f}$; see Figure 2. Again from Lemma 2.9 and the choice $\lambda^{\epsilon/2} < \theta < 1$ we obtain

$$|Z_u y - Z_u y'| \le (C + 2\ell + c_0 C) d_\theta(y,y')$$

where ℓ is the length of the largest stable leaf in the cross-section X.

The case $k = k'$: Now both points are past their $(k-1)$'th return and we again estimate the distance comparing with the distance of their $(k-1)$'th returns. Setting $\xi = u - r_{k-1}(y)$ and $\xi' = u - r_{k-1}(y')$ and assuming without loss that $\xi' \ge \xi$ we get

$$|Z_u y - Z_u y'| \le |\xi - \xi'| + |Z_\xi(f^{k-1} y) - Z_\xi(f^{k-1} y')|$$

and $|\xi - \xi'| \le C d_\delta(y,y')$ as before, while

$$\begin{aligned}|Z_\xi(f^{k-1}y) - Z_\xi(f^{k-1}y')| &\le |Z_\xi(f^{k-1}y) - Z_\xi(\pi f^{k-1}y)| \\ &\quad + |Z_\xi(\pi f^{k-1}y) - Z_\xi(\pi f^{k-1}y')| + |Z_\xi(\pi f^{k-1}y) - Z_\xi(f^{k-1}y')|.\end{aligned}$$

The uniform contraction along stable leaves and the relation $\pi f = \bar{f}\pi$ allows us to write

$$|Z_\xi(f^{k-1}y) - Z_\xi(f^{k-1}y')| \le c\lambda^\xi \cdot c\lambda^{k-1} \cdot 2\ell + |Z_\xi(\pi f^{k-1}y) - Z_\xi(\pi f^{k-1}y')|$$

and since $\lambda^{k-1} \le d_\theta(y,y')$, we are left to prove that

$$|Z_\xi(\pi f^{k-1}y) - Z_\xi(\pi f^{k-1}y')| \le C d_\theta(y,y'). \tag{2.11}$$

For this we use the construction of the geometric Lorenz attractor with the linearizable Lorenz-like singularity to explicitly calculate trajectories. In this way, we easily see that the distance between the pair of stable leaves $\zeta = \pi^{-1}(\pi f^{k-1} y)$ and $\zeta' = \pi^{-1}(\pi f^{k-1} y')$ (on X) is expanded by $e^{\lambda_1 t}$, that is,

$$d(Z_t\zeta, Z_t\zeta') \ge e^{\lambda_1 t} d(\zeta,\zeta'),$$

as long as $Z_s(\pi f^{k-1}y)$ and $Z_s(\pi f^{k-1}y')$ remain in the linearizable region around the singularity, for $0 \le s \le t$.

The flight time from X to X' (the boundary of the linearizable region) is given by $-\log|\pi f^{k-1} y|$ and $-\log|\pi f^{k-1}y'|$, and their difference is uniformly bounded since, by the backward contraction (2.6) and slow recurrence (2.7) properties,

$$\left|\log\frac{|\pi f^{k-1}y|}{|\pi f^{k-1}y'|}\right| \le \frac{|\bar{f}^{k-1}(\pi y) - \bar{f}^{k-1}(\pi y')|}{\max\{|\bar{f}^{k-1}(\pi y)|, |\bar{f}^{k-1}(\pi y')|\}} \le c_0 \frac{\lambda^{\tau(y)-k+1}}{\sqrt{\lambda}^{\tau(y)-k+1}} = c_0 \lambda^{(\tau(y)-k+1)/2}.$$

Hence, we have expansion of the distance between ζ, ζ' in the linear region, and the flow from X' back to X is performed in a uniformly bounded time for all points of the attractor. Thus, this last non-linear action of the flow distorts the distance by at most some constant factor (a bound on the norm of the derivative of the flow on a bounded interval of time). Therefore, we have shown that the left hand side of (2.11) is bounded by a constant factor of $|\bar{f}^k y - \bar{f}^k y'|$. This last difference is bounded by $c_0\lambda^{\tau(y)-k}|\bar{F}y - \bar{F}y'|$ which is bounded by the expression on the right hand side of (2.11). The proof is complete. □

2.7. **Suspension flow.** In Subsection 2.3, we saw that the geometric Lorenz flow can be modelled as a suspension flow $S_t : X^r \to X^r$ where X is the Poincaré section and r is the first hit time.

Shrinking the cross-section to Y and using the induced roof function R (which need not be the first hit time), we have the alternative model of the geometric Lorenz flow as the suspension flow $S_t : Y^R \to Y^R$ over a uniformly hyperbolic map $F : Y \to Y$ with integrable but unbounded return time function $R : Y \to \mathbb{R}^+$. Again the suspension flow is given by $S_t(y, u) = (y, u + t)$ computed modulo identifications, and the probability measure $\mu = \mu_Y \times \text{Lebesgue}/\int R\, d\mu_Y$ is S_t invariant.

Similarly, we define the quotient suspension semiflow $\bar{S}_t : \bar{Y}^R \to \bar{Y}^R$ with invariant probability measure $\bar{\mu} = \mu_{\bar{Y}} \times \text{Lebesgue}/\int R\, d\mu_{\bar{Y}}$.

The next result enables us to pass from the ambient manifold $\mathbb{R}^3$ to Y^R by means of the projection $p : Y^R \to \mathbb{R}^3$, $p(y, u) = Z_u y$.

Proposition 2.11. *Let (y, u), $(y', u') \in Y^R$. Then*

$$|p(y, u) - p(y', u')| \le C\{d_\theta(y, y') + |u - u'|\}.$$

Proof. Without loss, we can suppose that $u \le u'$. By the mean value theorem, there is a u'' between u and u' such that

$$|p(y', u) - p(y', u')| = |Z_u y' - Z_{u'} y'| \le |\partial_t Z_t(y')|_{t=u''}||u - u'| = |G(Z_{u''} y')||u - u'|,$$

where G is the underlying vector field. Since G is continuous and we are restricting to y lying in the compact attractor Λ, we obtain that there is a constant $C > 0$ such that $|p(y', u) - p(y', u')| \le C|u - u'|$.

Also by Lemma 2.10, $|p(y, u) - p(y', u)| = |Z_u(y) - Z_u(y')| \le C d_\theta(y, y')$.

The result follows by the triangle inequality. □

3. Temporal distortion function

In this section, we introduce the temporal distortion function and prove a result about the dimension of its range.

For all $y, z \in Y$ belonging to the same unstable manifold for $F : Y \to Y$, we define

$$D_0(y, z) = \sum_{j=1}^{\infty} [r(f^{-j} y) - r(f^{-j} z)].$$

We remark that each term in the sum makes sense since f is invertible on the attractor. Moreover we note that property (C) ensures that the roof function can be seen as a $C^{1+\epsilon}$ function on $\bar{X}$ with a logarithmic singularity at 0. We now prove that D_0 is well-defined.

Lemma 3.1. *The function D_0 is measurable and $D_0(y, z)$ is finite for μ_Y-almost every y and every $z \in W^u_F(y)$.*

Moreover, D_0 is continuous in the following sense. Suppose that $D_0(y, z)$ is well-defined and $\epsilon > 0$ is given. Then, there exists $\delta > 0$ such that $|D_0(y', z') - D_0(y, z)| < \epsilon$ for all pairs (y', z') satisfying (i) $D_0(y', z')$ is well-defined, (ii) $|y' - y| < \delta$, $|z' - z| < \delta$, and (iii) $d_\theta(y', y) < \delta$, $d_\theta(z', z) < \delta$.

Proof. Although the iterates $f^{-j} y$, $f^{-j} z$ are close by backward contraction, the values $r(f^{-j} x)$ and $r(f^{-j} y)$ need not be close. Hence, we consider the induced map $F : Y \to Y$ and the induced roof function $R : Y \to \mathbb{R}^+$ given by $R(y) = \sum_{\ell=0}^{\tau(y)-1} r(f^\ell y)$. Note however that F is not invertible (unlike f) so some care is needed in the following argument.

Write $y_0 = y$, $z_0 = z$. By ergodicity of μ_Y under F, we may suppose that there exist $y_i \in Y$, and $a_i \in \alpha$, $i \ge 1$, such that $y_i \in a_i$ and $F y_i = y_{i-1}$ for all $i \ge 1$. Since $\bar{F}$ is full branch, $F(a_1 \cap W^u_F(y_1))$ covers Y and in particular covers $W^u_F(y_0)$. Hence there exists $z_1 \in a_1 \cap W^u_F(y_0)$ such that $F z_1 = z_0$. Inductively, we obtain $z_i \in a_i \cap W^u_F(y_i)$, $i \ge 1$, such that $F z_i = z_{i-1}$.

By construction, $y_i = f^{-\tau(y_i)}y_{i-1}$. Inductively, $y_i = f^{-(\tau(y_1)+\cdots+\tau(y_i))}y$. Hence

$$R(y_i) = \sum_{\ell=0}^{\tau(y_i)-1} r(f^\ell f^{-(\tau(y_1)+\cdots+\tau(y_i))}y) = \sum_{\ell=\tau(y_1)+\cdots+\tau(y_{i-1})+1}^{\tau(y_1)+\cdots+\tau(y_i)} r(f^{-\ell}y).$$

Formally summing up the contributions from y_i and similarly z_i, we obtain the equivalent definition $D_0(y,z) = \sum_{i=1}^\infty [R(y_i) - R(z_i)]$. Moreover, since R is constant along stable leaves, writing $\bar{y}_i = \pi y_i, \bar{z}_i = \pi z_i$,

$$D_0(y,z) = \sum_{i=1}^\infty [R(\bar{y}_i) - R(\bar{z}_i)]. \tag{3.1}$$

To justify these formal manipulations, it suffices to prove that the series in (3.1) converges. In the process, we verify the first statement of the lemma. Recall from Lemma 2.9 that we can choose $\theta \in (0,1)$ so that R is d_θ-Lipschitz with Lipschitz constant $|R|_\theta$. We have $s(y_i, z_i) = i + s(y,z)$ for $i \geq 0$ and so

$$\begin{aligned} |D_0(y,z)| &\leq \sum_{i=1}^\infty |R(\bar{y}_i) - R(\bar{z}_i)| \leq \sum_{i=1}^\infty |R|_\theta d_\theta(y_i, z_i) \\ &= |R|_\theta \sum_{i=1}^\infty \theta^i d_\theta(y_0, z_0) = |R|_\theta \theta (1-\theta)^{-1} d_\theta(y,z) < \infty \end{aligned}$$

as required.

It remains to prove the last statement of the lemma. Let $N \geq 1$. By the above argument,

$$D_0(y,z) - D_0(y',z') = A(y,z) - A(y',z') + B(y,y') - B(z,z'),$$

where

$$A(y,z) = \sum_{i=N}^\infty |R(\bar{y}_i) - R(\bar{z}_i)|, \quad B(y,y') = \sum_{i=0}^{N-1} |R(\bar{y}_i) - R(\bar{y}'_i)|.$$

Moreover, $A(y,z)$, $A(y',z') \leq C\theta^N$.

For (y',z') sufficiently close to (y,z), the sequences of partition elements a_i containing y_i, z_i and a'_i containing y'_i, z'_i coincide for $i = 1, \ldots, N$. Hence $s(y_i, y'_i) = i + s(y,y')$ and $s(z_i, z'_i) = i + s(z,z')$ for $i = 1, \ldots, N$ and so

$$B(y,y') \leq C d_\theta(y,y'), \quad B(z,z') \leq C d_\theta(z,z').$$

Given $\epsilon > 0$, we choose N so that $C\theta^N < \epsilon/4$. Then we choose (y',z') so close to (y,z) that (i) $a'_i = a_i$ for $i = 1, \ldots, N$, (ii) $C d_\theta(y,y') < \epsilon/4$, and (iii) $C d_\theta(z,z') < \epsilon/4$. Then $|D_0(y,z) - D_0(y',z')| < \epsilon$ as required. □

3.1. **A double inducing scheme.** As in [21], the second iterate $y_1 = f^2(0+)$ plays an important role in establishing mixing properties. With that in mind, we consider two inducing schemes $F_i = f^{\tau_i} : Y_i \to Y_i$, $i = 1, 2$, whose quotients $\bar{F}_i : \bar{Y}_i \to \bar{Y}_i$ are $C^{1+\epsilon}$ piecewise expanding maps with full branches. By the l.e.o. condition, we can choose $\bar{Y}_0$ and $\bar{Y}_1$ to be disjoint open intervals containing 0 and y_1 respectively. Setting $Y = Y_0 \cup Y_1$, we obtain a combined (nonergodic) inducing scheme $F = f^\tau : Y \to Y$ where $F|_{Y_i} = F_i$, $\tau|_{Y_i} = \tau_i$. The partition α on Y is the union of the partitions on Y_0 and Y_1.

By Proposition 2.4, for almost every $y \in Y_0$ there is a local unstable manifold $W_F(y)$ that covers Y_0, and then by Proposition 2.5 the product $[y,y']$ is defined for every $y' \in Y_0$. The same statement holds with Y_0 changed to Y_1. In particular, Lemma 3.1 goes through for this inducing scheme by considering points in Y_0 and Y_1 separately. (By convention, $D_0(y,z)$ is never defined for y, z lying in distinct connected components of Y.)

Throughout most of the remainder of this section, until Subsection 3.5, we work with this inducing scheme.

3.2. **Young tower from the inducing scheme.** To the inducing scheme $F = f^\tau : Y \to Y$ constructed in Subsection 3.1, we associate the Young tower [36] $\hat{f} : \Delta \to \Delta$ where $\Delta = \{(y,\ell) : y \in Y,\ \ell = 0,1,\dots,\tau(y)-1\}$ and $\hat{f}(y,\ell) = \begin{cases} (y,\ell+1), & \ell \le \tau(y)-2 \\ (Fy,0), & \ell = \tau(y)-1 \end{cases}$. Note that $F = \hat{f}^\tau$ is a first return map to Y. The projection $\pi : \Delta \to Y$, $\pi(y,\ell) = f^\ell y$, defines a semiconjugacy between $\hat{f} : \Delta \to \Delta$ and $f : X \to X$. Let $\hat{r} = r \circ \pi : \Delta \to \mathbb{R}$ denote the lifted roof function. The partition α of Y extends to a partition $\hat{\alpha} = \{a \times \ell : a \in \alpha,\ 0 \le \ell < \tau|_a\}$ of Δ. Let $\Delta_\ell = \{(y,\ell) : y \in Y,\ 0 \le \ell < \tau(y)\}$ denote the ℓ'th level of the tower. We write $\Delta_\ell = \Delta_{\ell,0} \cup \Delta_{\ell,1}$ where $\Delta_{\ell,i} = \{p = (y,\ell) \in \Delta_\ell : y \in Y_i\}$.

Fix $\ell \ge 0$, $i \in \{0,1\}$. For $p = (y,\ell) \in \Delta_{\ell,i}$ we define the stable and unstable manifolds $W^s(p) = W^s_F(y) \times \ell$, $W^u(p) = W^u_F(y) \times \ell \in \Delta_{\ell,i}$. For $p = (y,\ell)$, $q = (z,\ell) \in \Delta_{\ell,i}$ we define $[p,q] = ([y,z],\ell) \in \Delta_{\ell,i}$. Again if $q \in \hat{a}$ for some $\hat{a} \in \hat{\alpha}$, then $[p,q] \in \hat{a}$.

We say that $p,q \in \Delta$ lie in the same unstable manifold if $p = (y,\ell)$, $q = (z,\ell)$ lie in $\Delta_{\ell,i}$ for some ℓ, i, and y,z lie in the same unstable manifold. In that case we define

$$D_0(p,q) = \sum_{j=1}^{\infty} [\hat{r}(\hat{f}^{-j}p) - \hat{r}(\hat{f}^{-j}q)].$$

Note that

$$D_0(p,q) = D_0(y,z) + \sum_{j=0}^{\ell-1} [r(f^j y) - r(f^j z)], \tag{3.2}$$

so the considerations in Lemma 3.1 for D_0 restricted to points in Y apply also to D_0 on Δ.

For $\hat{a},\ \hat{a}' \in \hat{\alpha}$ with $\hat{a}, \hat{a}' \subset \Delta_{\ell,i}$ for some ℓ, i, we define the *temporal distortion function* $D : \hat{a} \times \hat{a}' \to \mathbb{R}$ by setting

$$D(p,q) = \sum_{j=-\infty}^{\infty} [\hat{r}(\hat{f}^j p) - \hat{r}(\hat{f}^j [p,q]) - \hat{r}(\hat{f}^j [q,p]) + \hat{r}(\hat{f}^j q)],$$

for $p \in \hat{a}$, $q \in \hat{a}'$. We note that

$$D(p,q) = \sum_{j=-\infty}^{-1} \big[\hat{r}(\hat{f}^j p) - \hat{r}(\hat{f}^j [p,q]) - \hat{r}(\hat{f}^j [q,p]) + \hat{r}(\hat{f}^j q)\big] = D_0(p,[p,q]) + D_0(q,[q,p]),$$

where the first equality follows since r is constant on stable manifolds and the second is by definition of D_0; see Figure 3. Hence, D is almost everywhere well defined by Proposition 2.5, Lemma 3.1 and (3.2).

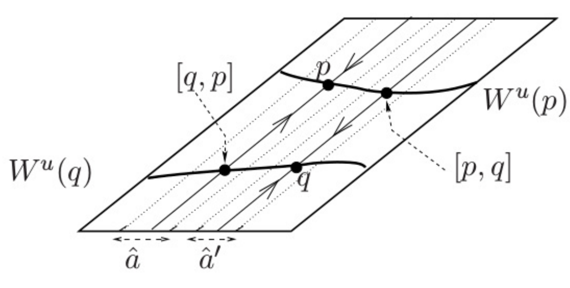

FIGURE 3. The definition of the temporal distortion function.

3.3. **Integrability and locally constant roof functions.** This section follows closely [14, Appendix]. Our purpose is to show that the temporal distortion function is non-zero for geometric Lorenz attractors.

Proposition 3.2. *Let $\hat{a},\ \hat{a}' \in \hat{\alpha}$ with $\hat{a}, \hat{a}' \subset \Delta_{\ell,i}$ for some ℓ, i. Suppose that $D|_{\hat{a}\times\hat{a}'} \equiv 0$. Then for all $p \in \hat{a}$, $q \in \hat{a}'$, the function $D_0(p,[p,q])$ is constant along the stable manifolds of p and q.*

Proof. By Proposition 2.5, $w \mapsto [p,w] = [p,q]$ is constant along stable manifolds of q. Hence, $w \mapsto D_0(p,[p,w])$ is constant along the stable manifold of q. Similarly, $w \mapsto D_0(q,[q,w])$ is constant along the stable manifold of p. But $D|_{\hat{a}\times\hat{a}'} \equiv 0$ implies that these two expressions are equal up to a minus sign and the result follows. ∎

For each $\hat{a} \in \hat{\alpha}$ with $\hat{a} \subset \Delta_{\ell,i}$, we associate a point $q_{\hat{a}} \in \Delta_{\ell,i}$. Then $[p, q_{\hat{a}(p)}]$ is defined for almost every $p \in \Delta$ (here, $\hat{a}(p)$ is the partition element in $\hat{\alpha}$ containing p). Define $\chi, u : \Delta \to \mathbb{R}$ by setting

$$\chi(p) = D_0(p, [p, q_{\hat{a}(p)}]) = \sum_{j=1}^{\infty} \{\hat{r}(\hat{f}^{-j}p) - \hat{r}(\hat{f}^{-j}[p, q_{\hat{a}(p)}])\}, \tag{3.3}$$

$$u(p) = \sum_{j=1}^{\infty} \{\hat{r}(\hat{f}^{-j}[\hat{f}p, q_{\hat{a}(Fp)}]) - \hat{r}(\hat{f}^{-j}[p, q_{\hat{a}(p)}])\}. \tag{3.4}$$

It follows from the definitions that $\hat{r} = \chi \circ \hat{f} - \chi + u$ on Δ.

Proposition 3.3. *If $D \equiv 0$, then $\chi : \Delta \to \mathbb{R}$ is continuous (indeed C^1) on $Y \cong Y \times 0$ and u is constant on partition elements of Δ.*

Proof. By definition, u is constant along local unstable manifolds $W^u(p) \cap a$ for all $p \in \hat{a}$, $\hat{a} \in \hat{\alpha}$. But if $D \equiv 0$, then by Proposition 3.2 we have that χ is constant along stable manifolds. Hence the same holds for $\chi \circ \hat{f}$. But $\hat{r}$ is already constant along stable manifolds, so we deduce that $u = \hat{r} - \chi \circ \hat{f} + \chi$ is constant along stable manifolds.

We have shown that u is constant along stable and unstable manifolds and hence is constant on partition elements.

On Y, we obtain $R = \sum_{\ell=0}^{\tau-1} \hat{r} \circ \hat{f}^{\ell} = \chi \circ F - \chi + \tilde{u}$, where $\tilde{u} = \sum_{\ell=0}^{\tau-1} u \circ \hat{f}^{\ell}$ is constant on partition elements. Since R, χ and u are constant along stable manifolds and hence are well-defined on $\bar{Y}$ we have that $R = \chi \circ \bar{F} - \chi + \tilde{u}$ on $\bar{Y}$.

Restricting to $\bar{Y}_i$ for $i = 0, 1$, and recalling (2.10), we note that the map $\bar{F}_i : \bar{Y}_i \to \bar{Y}_i$ satisfies all the requirements of [9, Proposition 7.4] allowing us to conclude that $\chi_i|_{\bar{Y}_i}$ has a C^1 version. Hence $\chi|_{\bar{Y}}$ has a C^1 version. ∎

However, this property contradicts the structure of geometric Lorenz attractors, as follows.

Theorem 3.4. *For any geometric Lorenz flow, the temporal distortion function D is not identically zero.*

Proof. We adapt the strategy in [21]. Let $y_1 = f^2(0+)$. Recall that 0 and y_1 lie in the interior of $\bar{Y}_0$ and $\bar{Y}_1$ respectively. Choose disjoint subsets $U_0, U_1 \subset Y$ that are the closure of unions of partition elements such that $W^s(y_1) \subset \operatorname{Int} U_1$ while U_0 contains a rectangle of the form $[0,\delta] \times [-1,1]$. Shrinking U_0 if necessary, we can ensure that $\tau|U_0 > 2$ and that $f^2 U_0 \subset U_1$. Fix $z_* \in U_0$ and note that $f^2 z_* \in U_1$.

Let $y \in U_0$. We claim that $f^2[y, z_*] = [f^2 y, f^2 z_*]$. It is clear that $f^2(W^s(z^*)) \subset W^s(f^2 z^*)$, so $f^2[y, z^*] \in W^s(f^2 z^*)$. We need to show that $f^2[y, z^*] \in W^u_F(f^2 y)$.

Now $W^u_F(y) \subset W^u_{loc,f}(y)$ and so $f^2(W^u_F(y)) \subset f^2(W^u_{loc,f}(y))$ and also $W^u_F(f^2 y) \subset W^u_{loc,f}(f^2 y) \subset f^2(W^u_{loc,f}(y))$. Moreover, both $f^2(W^u_F(y))$ and $W^u_F(f^2 y)$ cross $f^2 U_0$, since $W^u_F(y)$ crosses $Y_0 \supset U_0$ and $W^u_F(f^2 y)$ crosses $Y_1 \supset f^2 U_0$. Therefore we conclude that $f^2[y, z^*] \in f^2(W^u_F(y) \cap U_0) = W^u_F(f^2 y) \cap f^2 U_0$. This proves the claim.

Define χ and u as in (3.3) and (3.4) stipulating $q_{(a,2)} = (z_*, 2)$ and $q_{(f^2 a, 0)} = (f^2 z_*, 0)$ for $a \in U_0$.

Suppose for contradiction that $D \equiv 0$. By Proposition 3.3, for $y \in Y$ we have

$$r(y) + r(fy) = \hat{r}(y,0) + \hat{r}(y,1) = \chi(y,2) - \chi(y,0) + \tilde{u}(y,0), \tag{3.5}$$

where χ is continuous on $Y \cong Y \times 0$ and $\tilde{u}(y,0) = u(y,0) + u(y,1)$ is constant on partition elements.

We claim that χ is continuous on $U_0 \times 2$ and that $\lim_{y \to 0^+} \chi(y,2) = \chi(y_1)$. It then follows from (3.5) that $\tilde{\ell}$ is constant on $U_0 \times 0$ and moreover that all terms in (3.5) converge as $y \to 0^+$ with the exception of $r(y)$ which diverges to $+\infty$. This is the desired contradiction.

It remains to verify the claim. For $y \in a$, $a \subset U_0$ we compute that

$$[(y,2), q_{\hat{a}(y,2)}] = [(y,2),(z_*,2)] = ([y,z_*],2)$$

and

$$[(f^2y,0), q_{\hat{a}(f^2y,0)}] = [(f^2y,0),(f^2z_*,0)] = ([f^2y,f^2z_*],0)$$

so that

$$\pi[(y,2), q_{\hat{a}(y,2)}] = f^2[y,z_*] = [f^2y,f^2z_*] = \pi[(f^2y,0), q_{\hat{a}(f^2y,0)}].$$

Hence

$$\begin{aligned}\chi(y,2) &= \sum_{j=1}^{\infty}\big\{\hat{r}(\hat{f}^{-j}(y,2)) - \hat{r}(\hat{f}^{-j}[(y,2),q_{\hat{a}(y,2)}])\big\} \\ &= \sum_{j=1}^{\infty}\big\{r(f^{-j}\pi(y,2)) - r(f^{-j}\pi[(y,2),q_{\hat{a}(y,2)}])\big\} \\ &= \sum_{j=1}^{\infty}\big\{r(f^{-j}f^2y) - r(f^{-j}\pi[(f^2y,0),q_{\hat{a}(f^2y,0)}]\big\} \\ &= \sum_{j=1}^{\infty}\big\{\hat{r}(\hat{f}^{-j}(f^2y,0)) - \hat{r}(\hat{f}^{-j}[(f^2y,0),q_{\hat{a}(f^2y,0)}]\big\} = \chi(f^2y,0).\end{aligned}$$

The claim follows from continuity of χ on Y. □

3.4. **Smoothness of the temporal distortion function.**

Proposition 3.5. *There exists $\hat{a},\hat{a}' \in \hat{\alpha}$ with $\hat{a},\hat{a}' \subset \Delta_{\ell,i}$ for some ℓ, i, and there exists $p=(y,\ell) \in \hat{a}$, $p'=(y',\ell') \in \hat{a}'$, such that*

(a) *y lies in the unstable manifold of a periodic point, and similarly for y'.*
(b) *$D(p,p') \neq 0$.*

Proof. According to Theorem 3.4 there exist $p=(y,\ell)$, $p'=(y',\ell)$ such that $D(p,p') \neq 0$. Let $z_n \to y$ be a sequence of periodic points as in Proposition 2.6 and let $y_n = [z_n,y]$ so $y_n \to y$. Also by Proposition 2.6, $[y_n,y'] = [z_n,y'] \to [y,y']$. We have

$$D(y_n,y') = D_0(y_n,[y_n,y']) + D_0(y',[y',y_n]) = D_0(y_n,[y_n,y']) + D_0(y',[y',y]).$$

By Lemma 3.1, $D_0(y_n,[y_n,y']) \to D_0(y,[y,y'])$.

Now let $q_n = (z_n,\ell)$ and $p_n = [q_n,p] = (y_n,\ell)$. Since $D_0((a,\ell),(b,\ell)) - D_0(a,b)$ is a finite sum (with 2ℓ terms) of continuous functions, it follows that $D_0(p_n,[p_n,p']) \to D_0(p,[p,p'])$ and hence that $D(p_n,p') \to D(p,p')$. Hence there exists n such that $D(p_n,p') \neq 0$ and so we can replace p by the point $p_n = (y_n,\ell)$ where y_n lies in the unstable manifold of the periodic point z_n while maintaining condition (b). Similarly, we can replace p' by a point (y_n',ℓ) where y_n' lies in the unstable manifold of a periodic point. □

Now we fix the points $p=(y,\ell)$, $p'=(y',\ell)$ from Proposition 3.5 and consider the map $g: W^u(p) \to \mathbb{R}$ given by

$$g(q) = D(q,p') = D_0(q,[q,p']) + D_0(p',[p',q]).$$

Since $W^u(p)$ is naturally identified with $f^\ell W^u_F(y)$ it makes sense to speak of smoothness of g.

Proposition 3.6. *The one-dimensional map $g: W^u(p) \to \mathbb{R}$ is C^1.*

Proof. Let $g_1(q) = D_0(q,[q,p']) = \sum_{j=1}^{\infty} r(f^{-j}q) - \sum_{j=1}^{\infty} r(f^{-j}[q,p'])$. Since $[q,p'] = [p,p']$ is independent of $q \in W^u(p)$, the second sum consists of constant functions. For the first sum, note that each $z \in W^u(y)$ converges in backwards time to the periodic orbit y. Since $p = f^\ell y$ and $q = f^\ell z$, the backwards trajectory $\{f^{-j}q,\ j \geq 1\}$ is bounded away from the singularity at 0. It follows that along this trajectory f^{-1} is uniformly contracting and r is uniformly C^1. (The constants are uniform in j but are allowed to depend on q.) Hence the series for $(dg_1)_q : T_qW^u(p) \to \mathbb{R}$ is absolutely convergent and g_1 is C^1.

A similar argument applies to $g_2(q) = D_0(p', [p', q]) = \sum_{j=1}^{\infty} r(f^{-j}p') - \sum_{j=1}^{\infty} r(f^{-j}[p', q])$. This time, it is the first sum that consists of constants. The second sum is like the first sum in g_1 with q replaced by $[p', q]$ which converges in backwards time to the periodic orbit y'. It follows that the dependence of g_2 on $[p', q]$ is C^1. But $z \mapsto [y', z]$ is C^1 by the last statement of Proposition 2.5 and so $q \mapsto [p', q]$ is C^1. Hence g_2 is C^1 and so $g = g_1 + g_2$ is C^1. □

Corollary 3.7. *There exists a nonempty open set $V \subset W^u(p) \cong W^u_F(f^\ell y)$ on which g restricts to a C^1 diffeomorphism.*

Proof. By Proposition 3.6, g is a C^1 map on $W^u(p)$. Since $g([p, p']) = 0$ and $g(p) \neq 0$ by assumption, it follows that g' is not identically zero and the result follows. □

3.5. **Dimension of the range of the temporal distortion function.** If necessary, we now choose a new inducing scheme $F^* : Y^* \to Y^*$ with $Y^* \subset \bigcup_{v \in V} W^s_f(v)$ and such that the properties in Sections 2.3 and 2.4 remain valid. (For this part of the argument it suffices to take Y^* connected and F^* full branch.) The definition of D, and hence g, is unchanged since this is defined in terms of r and f, independent of the choice of inducing scheme. Let α^* denote the associated partition of Y^* and choose two partition elements $a_1, a_2 \in \alpha^*$. Define the finite subsystem $A_0 = \bigcap_{n \geq 0} (F^*)^{-n}(a_1 \cup a_2)$.

Proposition 3.8. *For the finite subsystem A_0 constructed above, the set $D(A_0 \times A_0)$ has positive lower box dimension.*

Proof. At the level of the quotient dynamics, the map $\bar{F}^* : \bar{Y}^* \to \bar{Y}^*$ is uniformly expanding. Moreover, $\bar{F}^* a_i = \bar{Y}^*$ for $i = 1, 2$ and the derivative of $\bar{F}^*$ is bounded on the closure of $a_1 \cup a_2$. It follows (see for example [30, p. 203]) that the Cantor set A_0 has positive Hausdorff dimension. Since $g|_V$ is a C^1 diffeomorphism and $A_0 \subset V$, it follows that $g(A_0)$ has positive Hausdorff dimension. Hence the larger set $D(A_0 \times A_0)$ has positive lower box dimension. □

4. Fast mixing decay of correlations

We are now ready to complete the proof of Theorem A. According to [22, 23], the result is immediate from Proposition 3.8. Unfortunately, the precise formulation of the result we require is not written down there. If the roof function were bounded then we would have all the ingredients required to directly apply [23, Corollary 5.6]. The case of unbounded roof functions is considered in [22, Proposition 3.6] and [23, Section 6.5] for semiflows and flows respectively, but omitting the crucial ingredient provided by the dimension of the range of the temporal distortion function.

Hence, to apply the results in [22, 23] it is necessary to recall several of the definitions and intermediate steps. This is done for semiflows in Subsection 4.1. In Section 4.2, we pass from the semiflow to the flow; here it turns out to be particularly convenient to use a recent approach of [9],

4.1. **Fast mixing for the semiflow.** We assume that $\bar{F} : \bar{Y} \to \bar{Y}$ is a uniformly expanding map with partition α_0 covered by a uniformly hyperbolic map $F : Y \to Y$ with partition α as in Section 2.3. We continue to suppose that $R : Y \to \mathbb{R}^+$ is a possibly unbounded roof function, constant along stable leaves, that is locally Lipschitz in the symbolic metric d_θ on $\bar{Y}$. Moreover, R is bounded below and has exponential tails.

Given $v : \bar{Y}^R \to \mathbb{R}$ continuous, we define $|v|_\theta = \sup |v(y, u) - v(y', u)|/d_\theta(y, y')$ where the supremum is over distinct points (y, u), $(y', u) \in \bar{Y}^R$. (Recall that $\bar{Y}^R$ is an identification space so observables $v : \bar{Y}^R \to \mathbb{R}$ satisfy $v(y, R(y)) = v(\bar{F}y, 0)$.) Define $F_\theta(\bar{Y}^R)$ to be the space of continuous observables $v : \bar{Y}^R \to \mathbb{R}$ such that $\|v\|_\theta = |v|_\infty + |v|_\theta < \infty$.

Let $\partial_t v = \frac{d}{dt} \bar{S}_t v|_{t=0}$ denote the derivative of v in the flow direction. So $\partial_t v(y, u) = \frac{\partial}{\partial u} v(y, u)$ when $0 < u < R(y)$, $\partial_t v(y, 0) = \lim_{t \to 0^+} (v(y, t) - v(y, 0))/t$ and $\partial_t v(y, R(y)) = \lim_{t \to 0^+} (v(y, R(y)) - v(y, R(y) - t))/t$. Provided that $\partial_t v(y, R(y)) = \partial_t v(\bar{F}y, 0)$ for all $y \in \bar{Y}$, this defines a function $\partial_t v : \bar{Y}^R \to \mathbb{R}$. If in addition $v, \partial_t v \in F_\theta(\bar{Y}^R)$ then we write $v \in F_{\theta,1}(\bar{Y}^R)$.

Similarly, define the space $F_{\theta,k}(\bar{Y}^R)$ of observables $v : \bar{Y}^R \to \mathbb{R}$ that are C^k in the semiflow direction with derivatives $\partial_t^j v \in F_\theta(\bar{Y}^R)$ for $j = 0, \dots, k$. Define $\|v\|_{\theta,k} = \sum_{j=0}^k \|\partial_t^j v\|_\theta$.

We require some further definitions from [22, 23] based on [14]. A subset $\bar{A}_0 \subset \bar{Y}$ is a *finite subsystem* of $\bar{Y}$ if $\bar{A}_0 = \bigcap_{n\ge1} \bar{F}^{-n}\bar{A}$ where $\bar{A}$ is a finite union of elements of α_0. Similarly, a subset $A_0 \subset Y$ is a *finite subsystem* of Y if $A_0 = \bigcap_{n\ge1} F^{-n}A$ where A is a finite union of elements of α_0. Such a finite subsystem projects to a finite subsystem $\bar{A}_0 \subset \bar{Y}$.

Definition 4.1. For $b \in \mathbb{R}$ define $M_b : L^\infty(\bar{Y}) \to L^\infty(\bar{Y})$, $M_b v = e^{-ibR} v \circ \bar{F}$. We say that M_b has *approximate eigenfunctions* on a subset $\bar{A}_0 \subset \bar{Y}$ if there exist constants $\alpha, \beta > 0$ arbitrarily large and $C \ge 1$, and sequences $b_k \in \mathbb{R}$ with $|b_k| \to \infty$, $\varphi_k \in [0, 2\pi)$, $u_k : \bar{Y} \to \mathbb{C}$ with $|u_k| \equiv 1$ and $|u_k|_\theta = \sup_{y\ne y'} |u_k(y) - u_k(y')|/d_\theta(y, y') \le C|b_k|$, such that setting $n_k = [\beta \ln |b_k|]$,

$$|(M_{b_k}^{n_k} u_k)(y) - e^{i\varphi_k} u_k(y)| \le C|b_k|^{-\alpha},$$

for all $y \in \bar{A}_0$ and all $k \ge 1$.

Theorem 4.2. *Let $\bar{S}_t : \bar{Y}^R \to \bar{Y}^R$ be a suspension semiflow over a uniformly expanding map $\bar{F} : \bar{Y} \to \bar{Y}$, where the roof function $R : \bar{Y} \to \mathbb{R}^+$ has exponential tails.*

Suppose that there exists a finite subsystem $\bar{A}_0 \subset \bar{Y}$ such that there are no approximate eigenfunctions on $\bar{A}_0$. Then the semiflow has superpolynomial decay for sufficiently smooth observables. That is, for any $\gamma > 0$, there exists $C > 0$ and $k \ge 1$ such that for all observables $v \in F_{\theta,k}(\bar{Y}^R)$, $w \in L^\infty(\bar{Y}^R)$ and all $t > 0$,

$$\Big|\int v\, w \circ \bar{S}_t \, d\bar{\mu} - \int v\, d\bar{\mu} \int w\, d\bar{\mu}\Big| \le C\|v\|_{\theta,k}|w|_\infty t^{-\gamma}. \tag{4.1}$$

Proof. For the quotient suspension $\bar{Y}^R$, we are in the situation of [22, Section 3]. (The induced roof function R is denoted by H in [22].) The exponential tail condition in [22, Definition 3.1] follows from Proposition 2.7 and Lemma 2.9. Hence, in principle, the result follows from [22, Lemma 3.5, Proposition 3.6]. There are two caveats that need to be mentioned.

The first caveat is that the definition of approximate eigenfunctions in [22] is slightly weaker than in Definition 4.1 since the constraint $|u_k|_\theta \le C|b_k|$ is not mentioned. However, as is easily checked, the argument in [22] shows that the failure of fast mixing implies the existence of approximate eigenfunctions that actually satisfy the stronger requirements of Definition 4.1. Only [22, Lemma 3.5] is possibly affected, and it is a consequence of [22, Corollary 3.11 and Lemmas 3.12 and 3.13]. Of these, only [22, Lemma 3.12] is possibly affected by the change in definition. Moreover, this lemma gives a criterion for the existence of approximate eigenfunctions (called w and eventually w_1) and these are shown to satisfy the extra constraint $|w_1|_\theta \le C_{11}|b|$.

The second caveat is that [22, Proposition 3.6] has an additional hypothesis, namely that $\bar{S}_t$ is mixing. We claim that if $\bar{S}_t$ is not mixing, then there exist approximate eigenfunctions on the whole of $\bar{Y}$; hence this extra hypothesis is redundant. It remains to verify the claim. A standard characterisation of mixing for suspension semiflows over a mixing transformation $\bar{F}$ is that for each $c \ne 0$ the functional equation $u \circ \bar{F} = e^{icR}u$ has no measurable solutions $u : \bar{Y} \to S^1$ where $S^1 \subset \mathbb{C}$ is the unit circle. Suppose that $c \ne 0$ and $u : \bar{Y} \to S^1$ measurable satisfy such a functional equation. Since $|e^{icR(y)} - e^{icR(y')}| \le |c||R(y) - R(y')|$, the exponential tail condition on R certainly implies that the hypothesis on $f = e^{icR}$ in [15, Theorem 1.1] is satisfied. Hence u has a version with $|u|_\theta < \infty$. For any $\alpha, \beta > 0$, we let $u_k = u^k$, $b_k = kc$, $n_k = [\beta \ln |b_k|]$, $\varphi_k = 0$. In particular, $|u_k|_\theta \le k|u|_\theta \le C|b_k|$ with $C = |u|_\theta/|c|$. Moreover, $M_{b_k}^{n_k} u_k \equiv e^{i\varphi_k} u_k$ so the requirements in Definition 4.1 are satisfied. This completes the proof. □

Proposition 4.3. *Let $A_0 \subset Y$ be a finite subsystem and suppose that $D(A_0 \times A_0)$ has positive lower box dimension. Then there are no approximate eigenfunctions on $\bar{A}_0$.*

Proof. Suppose that there are approximate eigenfunctions on $\bar{A}_0$. The calculation in the proof of [23, Theorem 5.5] (with $\omega_k = 0$) shows that for all $\alpha > 0$, there is a sequence $b_k \in \mathbb{R}$ with $|b_k| \to \infty$ and a constant $C > 0$ such that $|e^{ib_k D(y_1,y_4)} - 1| \le C|b_k|^{-\alpha}$ for all $y_1, y_4 \in A_0$. It then follows from [23, Corollary 5.6] that $D(A_0 \times A_0)$ has lower box dimension zero. □

Corollary 4.4. *Let $G \in \mathcal{U}$ be a vector field defining a strongly dissipative geometric Lorenz flow. Let $\bar{S}_t : \bar{Y}^R \to \bar{Y}^R$ denote the corresponding suspension semiflow. Then for all $\gamma > 0$, there exists $C > 0$ and $k \geq 1$ such that for all observables $v \in F_{\theta,k}(\bar{Y}^R)$, $w \in L^\infty(\bar{Y}^R)$, and all $t > 0$,*

$$\Big| \int v\, w \circ \bar{S}_t \, d\bar{\mu} - \int v \, d\bar{\mu} \int w \, d\bar{\mu} \Big| \leq C \|v\|_{\theta,k} |w|_\infty t^{-\gamma}.$$

Proof. In Proposition 3.8, we constructed a finite subsystem $A_0 \subset Y$ such that $D(A_0 \times A_0)$ has positive lower box dimension. Hence there are no approximate eigenfunctions on $\bar{A}_0$ by Proposition 4.3. The result follows from Theorem 4.2. □

Remark 4.5. As mentioned in Remark 2.8, the full strength of Proposition 2.7 is not required in this paper. A standard and elementary argument using the exponential tails for r and τ implies the stretched exponential estimate $\mu_Y(R > n) = O(e^{-ct^{1/2}})$ which suffices for the methods in [23]. However, the analogue of Theorem 4.2 is not stated explicitly in [23] so for ease of exposition we have made use of Proposition 2.7.

4.2. **Fast mixing for the flow.** We continue to assume the structure from Subsection 4.1 and in addition that there is a $C^{1+\epsilon}$ global exponentially contracting stable foliation.

In Subsection 4.1, we defined the space $F_\theta(\bar{Y}^R)$ of observables on $\bar{Y}$. To define the corresponding space $F_\theta(Y^R)$ it is convenient to choose coordinates (y, z) on Y where $y \in \bar{Y}$ and vertical lines correspond to stable leaves (recall that this can be achieved by a $C^{1+\epsilon}$ change of coordinates). Then we define $F_\theta(Y^R)$ to be the space of continuous observables $v : Y^R \to \mathbb{R}$ respecting the identifications $(y, z, R(y)) \sim (F(y, z), 0)$ and such that $\|v\|_\theta = |v|_\infty + |v|_\theta < \infty$ where

$$|v|_\theta = \sup_{(y,z,u) \neq (y',z',u)} \frac{|v(y, z, u) - v(y', z', u)|}{d_\theta(y, y') + |z - z'|}.$$

Again, we define the space $F_{\theta,k}(Y^R)$ of observables $v : Y^R \to \mathbb{R}$ that are C^k in the flow direction with derivatives $\partial_t^j v \in F_\theta(Y^R)$ for $j = 0, \dots, k$. Define $\|v\|_{\theta,k} = \sum_{j=0}^k \|\partial_t^j v\|_\theta$.

Lemma 4.6. *There is a continuous family of probability measures $\{\eta_y,\, y \in \bar{Y}\}$ on Y with* $\operatorname{supp} \eta_y \subset \pi^{-1}(y)$ *such that $\int_Y v \, d\mu_Y = \int_{\bar{Y}} \int_{\pi^{-1}(y)} v \, d\eta_y \, d\mu_{\bar{Y}}(y)$ for all $v : Y \to \mathbb{R}$ continuous.*

Moreover, there is a constant $C_1 > 0$ such that if $v \in F_{\theta,k}(Y^R)$, and $\bar{v} : \bar{Y}^R \to \mathbb{R}$ is defined to be $\bar{v}(y, u) = \int_{\pi^{-1}(y)} v(y', u) \, d\eta_y(y')$, then $\bar{v} \in F_{\theta,k}(\bar{Y}^R)$ and $\|\bar{v}\|_{\theta,k} \leq C_1 \|v\|_{\theta,k}$.

Proof. A proof of the existence of the continuous disintegration $\mu_Y = \int_{\bar{Y}} \eta_y \, d\bar{\mu}_Y(y)$ in the first statement of the lemma can be found for instance in [10].

Suppose that $v : Y^R \to \mathbb{R}$ is continuous. In particular $v(y, R(y)) = v(Fy, 0)$ for all $y \in Y$. Define $\bar{v}(y, u) = \int_{\pi^{-1}(y)} v(y', u) \, d\eta_y(y')$. As shown below, $\bar{v}(y, R(y)) = \bar{v}(\bar{F}y, 0)$ for all $y \in \bar{Y}$, so that we have a well-defined function $\bar{v} : \bar{Y}^R \to \mathbb{R}$. The estimate $\|\bar{v}\|_\theta \leq C_1 \|v\|_\theta$ follows from [10, Proposition 10]. The case of general k follows since

$$\partial_t^j \bar{v}(y, u) = \int_{\pi^{-1}(y)} \partial_t^j v(y', u) \, d\eta_y(y') = \overline{\partial_t^j v}(y, u), \quad \text{for all } j.$$

It remains to prove that $\bar{v}(y, R(y)) = \bar{v}(\bar{F}y, 0)$ for $y \in \bar{Y}$. Throughout, we regard $\{\eta_y,\, y \in \bar{Y}\}$ as a family of probability measures on Y. In particular, $F_*\eta_y$ denotes the pushforward of η_y (so $(F_*\eta_y)(E) = \eta_y(F^{-1}E)$).

Define $v_0 : Y \to \mathbb{R}$ by setting $v_0(y) = v(y, 0)$. Then $\bar{v}(y, 0) = \eta_y(v_0)$, and so $\bar{v}(\bar{F}y, 0) = \eta_{\bar{F}y}(v_0)$. Also, using that R is constant along the stable foliation, we obtain

$$\begin{aligned} \bar{v}(y, R(y)) &= \int_Y v(y', R(y)) \, d\eta_y(y') = \int_Y v(y', R(y')) \, d\eta_y(y') \\ &= \int_Y v(Fy', 0) \, d\eta_y(y') = \eta_y(v_0 \circ F) = (F_*\eta_y)(v_0). \end{aligned}$$

By continuity of $\bar{v}$, it remains to show that $F_*\eta_y = \eta_{\bar{F}y}$ for $\mu_{\bar{Y}}$-a.e. $y \in \bar{Y}$.

We claim that

$$\int_{\bar{Y}} F_*\eta_y \, d\mu_{\bar{Y}}(y) = \int_{\bar{Y}} \eta_{\bar{F}y} \, d\mu_{\bar{Y}}(y).$$

By uniqueness of a family of conditional measures with respect to a given measure and measurable partition, we deduce from the claim that $F_*\eta_y = \eta_{\bar{F}y}$ for $\mu_{\bar{Y}}$-a.e. $y \in \bar{Y}$ as required.

It remains to prove the claim. Since $\mu_{\bar{Y}}$ is $\bar{F}$-invariant, we have on the one hand

$$\int_{\bar{Y}} \eta_{\bar{F}y} \, d\mu_{\bar{Y}}(y) = \int_{\bar{Y}} \eta_y \, d(\bar{F}_*\mu_{\bar{Y}})(y) = \int_{\bar{Y}} \eta_y \, d\mu_{\bar{Y}}(y).$$

On the other hand, by F-invariance of μ_Y,

$$\int_{\bar{Y}} \eta_y \, d\mu_{\bar{Y}}(y) = \mu_Y = F_*\mu_Y = \int_{\bar{Y}} F_*\eta_y \, d\mu_{\bar{Y}}(y),$$

completing the proof of the claim. □

Proof of Theorem A We follow the argument in [9, Section 8].

Given $\gamma > 0$, choose $C > 0$ and $k \geq 1$ as in Corollary 4.4. We suppose that $v \in C^k(\mathbb{R}^3)$ and that $w \in C^\alpha(\mathbb{R}^3)$ for some $\alpha > 0$.

Recall that $p : Y^R \to \mathbb{R}^3$ is the semiconjugacy $p(y, u) = Z_u y$. It suffices to prove the result for observables $v \circ p$ and $w \circ p$ at the level of the suspension flow on Y^R. By Proposition 2.11, $v \circ p \in F_{\theta,k}(Y^R)$.

Without loss, we may suppose that $\int_{Y^R} v \circ p \, d\mu = 0$. Define the semiconjugacy $\pi^R : Y^R \to \bar{Y}^R$, $\pi^R(y, u) = (\pi y, u)$, so $\bar{S}_t \circ \pi^R = \pi^R \circ S_t$.

Define $w_t : Y^R \to \mathbb{R}$ by setting

$$w_t(y, u) = \int_{\pi^{-1}(y)} w \circ p \circ S_t(y', u) \, d\eta_y(y').$$

Then $\int_{Y^R} v \circ p \; w \circ p \circ S_{2t} \, d\mu = I_1(t) + I_2(t)$, where

$$I_1(t) = \int_{Y^R} v \circ p \; w \circ p \circ S_{2t} \, d\mu - \int_{Y^R} v \circ p \; w_t \circ \bar{S}_t \circ \pi^R \, d\mu$$
$$I_2(t) = \int_{Y^R} v \circ p \; w_t \circ \bar{S}_t \circ \pi^R \, d\mu.$$

Now $I_1(t) = \int_{Y^R} v \circ p \, \{(w \circ p \circ S_t - w_t \circ \pi^R) \circ S_t\} \, d\mu$, so $|I_1(t)| \leq |v|_1 |w \circ p \circ S_t - w_t \circ \pi^R|_\infty$. Using the definitions of π^R and w_t,

$$\begin{aligned} w \circ p \circ S_t(y, u) - w_t \circ \pi^R(y, u) &= w \circ p \circ S_t(y, u) - w_t(\pi y, u) \\ &= \int_{\pi^{-1}(y)} (w \circ p \circ S_t(y, u) - w \circ p \circ S_t(y', u)) \, d\eta_{\pi(y)}(y'). \end{aligned}$$

Since S_t contracts exponentially along stable manifolds, there are constants $C_2, a > 0$ (dependent on α) such that $|w \circ p \circ S_t(y, u) - w \circ p \circ S_t(y', u)| \leq C_2 |w|_\alpha e^{-at}$ for all $y' \in \pi^{-1}(y)$, $(y, u) \in Y^R$, and so $|w \circ p \circ S_t - w_t \circ \pi^R|_\infty \leq C_2 |w|_\alpha e^{-at}$. Hence

$$|I_1(t)| \leq C_2 |v|_1 |w|_\alpha e^{-at}.$$

Next, define $\bar{v} : \bar{Y}^R \to \mathbb{R}$ by setting $\bar{v}(y, u) = \int_{\pi^{-1}(y)} v \circ p(y', u) \, d\eta_y(y')$. Since $\int_{Y^R} v \circ p \, d\mu = 0$, it follows from Lemma 4.6 that $\int_{\bar{Y}^R} \bar{v} \, d\bar{\mu} = 0$. Moreover, $I_2(t) = \int_{\bar{Y}^R} \bar{v} \; w_t \circ \bar{S}_t \, d\bar{\mu}$. By Lemma 4.6, $\bar{v} \in F_{\theta,k}(\bar{Y}^R)$ and $\|\bar{v}\|_{\theta,k} \leq C_1 \|v \circ p\|_{\theta,k}$. Clearly, $|w_t|_\infty \leq |w|_\infty$. Hence it follows from Corollary 4.4 that $|I_2(t)| \leq C \|\bar{v}\|_{\theta,k} |w_t|_\infty \, t^{-\gamma} \leq C C_1 \|v\|_{\theta,k} |w|_\infty \, t^{-\gamma}$ completing the proof. □

5. ASIP for time-1 map of a nonuniformly expanding semiflow

In this section, we prove the ASIP for time-1 maps of a general class of sufficiently mixing nonuniformly expanding semiflows.

Suppose that $\bar F:\bar Y\to\bar Y$ is a Gibbs-Markov map (uniformly expanding with bounded distortion and big images – for notational convenience we assume full branches) with ergodic invariant measure $\mu_{\bar Y}$. Let d_θ denote a symbolic metric on $\bar Y$ for some $\theta\in(0,1)$. Let $R:\bar Y\to\mathbb{R}^+$ be a possibly unbounded roof function satisfying

(i) R is bounded below,
(ii) $\mu_{\bar Y}(R>t)=O(t^{-\beta})$ for some $\beta>1$,
(iii) $|R|_\theta=\sup_{y\neq y'}|R(y)-R(y')|/d_\theta(y,y')<\infty$.

(This includes the case of uniformly expanding semiflows where R is bounded.) Define the suspension semiflow $\bar S_t:\bar Y^R\to\bar Y^R$ with ergodic invariant measure $\bar\mu=\mu_{\bar Y}\times\mathrm{Leb}/\int_{\bar Y}R\,d\mu_{\bar Y}$. Let $F_{\theta,k}(\bar Y^R)$ be the space of observables introduced in Section 4.1.

Remark 5.1. Condition (iii), although satisfied for geometric Lorenz flows, is unnecessarily restrictive. At the end of the section, we show how this condition can be relaxed.

Theorem 5.2. *Suppose that* (4.1) *holds with* $\beta>2\sqrt{2}+1$. *Let* $v\in F_{\theta,k+1}(\bar Y^R)$ *be an observable with mean zero. Then the ASIP holds for the time-*1 *map* $\bar S=\bar S_1$*: passing to an enriched probability space, there exists a sequence* $X_0,X_1,\dots$ *of iid normal random variables with mean zero and variance* σ^2*, such that*

$$\sum_{j=0}^{n-1}v\circ\bar S^j=\sum_{j=0}^{n-1}X_j+O(n^{1/4}(\log n)^{1/2}(\log\log n)^{1/4}),\quad a.e.$$

The variance is given by

$$\sigma^2=\lim_{n\to\infty}\frac1n\int(\sum_{j=0}^{n-1}v\circ\bar S^j)^2\,d\bar\mu=\sum_{n=-\infty}^{\infty}\int v\cdot\big(v\circ\bar S^n\big)\,d\bar\mu.$$

The degenerate case $\sigma^2=0$ *occurs if and only if* $v=\chi\circ\bar S-\chi$ *for some* $\chi\in L^2$. *Moreover, for any finite* p *we have* $\chi\in L^p$ *for* k *sufficiently large.*

Remark 5.3. (a) The CLT and functional CLT can be proved more directly under weaker assumptions: it suffices that $\beta>1$, see [32, Theorem 1]. Also, the statements about the variance in Theorem 5.2 are a standard consequence of the methods there.

(b) An ASIP with weaker error term can be proved for smaller values of β. However, in our application to the geometric Lorenz attractor, we can obtain any desired value of β by increasing the smoothness of the observable, and there is no easy relationship between the degree of smoothness and the size of β, so there seems little point in pursuing this here.

First, we relate the transfer operators for the semiflow and the induced map $\bar F$. Let L_t be the transfer operator corresponding to the semiflow $\bar S_t$, so $\int_{\bar Y^R}L_tv\,w\,d\bar\mu=\int_{\bar Y^R}v\,w\circ\bar S_t\,d\bar\mu$ for all $v\in L^1(\bar Y^R)$, $w\in L^\infty(\bar Y^R)$. In particular, L_n is the transfer operator for $\bar S^n$. Let P denote the transfer operator for $\bar F:\bar Y\to\bar Y$, so $\int_{\bar Y}Pv\,w\,d\mu_{\bar Y}=\int_{\bar Y}v\,w\circ\bar F\,d\mu_{\bar Y}$ for all $v\in L^1(\bar Y)$, $w\in L^\infty(\bar Y)$.

Proposition 5.4. *Let* $t>0$. *The transfer operator* L_t *is given by a finite sum of the form*

$$(L_tv)(y,u)=\sum_{j=0}^{\infty}(P^j\tilde v_{t,u,j})(y),\quad \textit{with}\quad \tilde v_{t,u,j}(y)=v(y,u-t+R_j(y)).$$

The number of nonzero terms in the sum is bounded by $t/\inf R+1$.

Proof. Let $\bar{R} = \int_Y R \, d\mu_{\bar{Y}}$ and write

$$\begin{aligned}\int_{\bar{Y}^R} v\, w \circ \bar{S}_t \, d\bar{\mu} &= (1/\bar{R}) \int_Y \int_0^{R(y)} v(y,u)\, w \circ \bar{S}_t(y,u)\, du\, d\mu_{\bar{Y}} \\ &= (1/\bar{R}) \sum_{j=0}^{\infty} \int_Y \int_{R_j(y)-t}^{R_{j+1}(y)-t} 1_{[0,R(y)]}(u) v(y,u)\, w(\bar{S}_t(y,u))\, du\, d\mu_{\bar{Y}}.\end{aligned}$$

For the j'th term to give a nonzero contribution to the sum, it is necessary that $R_{j+1}(y) - t < R(y)$ for some y, equivalently $R_j(\bar{F}y) < t$, leading to the condition that $0 \le j \le t/\inf R$. Now

$$\begin{aligned}&\int_{\bar{Y}} \int_{R_j(y)-t}^{R_{j+1}(y)-t} 1_{[0,R(y)]}(u) v(y,u)\, w(\bar{S}_t(y,u))\, du\, d\mu_{\bar{Y}} \\ &\quad = \int_{\bar{Y}} \int_{R_j(y)-t}^{R_{j+1}(y)-t} 1_{[0,R(y)]}(u) v(y,u)\, w(\bar{F}^j y, u + t - R_j(y))\, du\, d\mu_{\bar{Y}} \\ &\quad = \int_{\bar{Y}} \int_0^{R(\bar{F}^k y)} 1_{[0,R(y)]}(u - t + R_j(y)) v(y, u - t + R_j(y))\, w(\bar{F}^k y, u)\, du\, d\mu_{\bar{Y}} \\ &\quad = \int_0^{\infty} \int_{\bar{Y}} 1_{[0,R(\bar{F}^k(y))]}(u) 1_{[0,R(y)]}(u - t + R_j(y)) v(y, u - t + R_j(y))\, w(\bar{F}^k y, u)\, d\mu_{\bar{Y}}\, du \\ &\quad = \int_0^{\infty} \int_{\bar{Y}} \tilde{v}_{t,u,j}(y)\, 1_{[0,R(\bar{F}^k(y))]}(u) w(\bar{F}^k y, u)\, d\mu_{\bar{Y}}\, du \\ &\quad = \int_0^{\infty} \int_{\bar{Y}} (P^j \tilde{v}_{t,u,j})(y)\, 1_{[0,R(y)]}(u) w(y,u)\, d\mu_{\bar{Y}}\, du \\ &\quad = \int_{\bar{Y}} \int_0^{R(y)} (P^j \tilde{v}_{t,u,j})(y)\, w(y,u)\, du\, d\mu_{\bar{Y}},\end{aligned}$$

as required. □

Theorem 5.2 is a consequence of Cuny & Merlevède [12, Theorem 3.2]. To apply [12], we are required to check that the following three conditions hold:

$$\sum_{n=1}^{\infty} (\log n)^3 n^{5/2} |L_n v|_4^4 < \infty, \tag{5.1}$$

$$\sum_{n=1}^{\infty} (\log n)^3 n |L_n v|_2^2 < \infty, \tag{5.2}$$

$$\sum_{n=1}^{\infty} (\log n)^3 n^{-2} \Big(\sum_{i=1}^{n} \sum_{j=0}^{n-i} |L_i(vL_j v) - \textstyle\int_{\bar{Y}^R} v L_j v\, d\bar{\mu}|_2 \Big)^2 < \infty. \tag{5.3}$$

Proposition 5.5. *Let $p \in [1,\infty)$. Then there is a constant $C > 0$ such that $|L_t v - \int v\, d\bar{\mu}|_p \le C\|v\|_{\theta,k} t^{-\beta/p}$ for all $v \in F_{\theta,k}(\bar{Y}^R)$.*

Proof. Following [24], we set $w = \operatorname{sgn} L_t v$ in (4.1) to obtain $|L_t v|_1 \le C\|v\|_{\theta,k} t^{-\beta}$. Since $v \in L^\infty$ and $|L_t v|_\infty \le |v|_\infty$, we obtain $|L_t v|_p^p \le |v|_\infty^{p-1} |L_t v|_1 \le C\|v\|_{\theta,k}^p t^{-\beta}$. □

Lemma 5.6. *There exists a constant C_1 (depending on k and θ) such that*

$$\|vL_t v\|_{\theta,k} \le C_1(t+1)(|R|_\theta + 1)\|v\|_{\theta,k+1}^2, \tag{5.4}$$

for all $t \ge 0$.

Proof. It is clear that $\|vL_t v\|_{\theta,k} \le C\|v\|_{\theta,k}\|L_t v\|_{\theta,k}$ where C depends only on k. It remains to estimate $\|L_t v\|_{\theta,k}$. By Proposition 5.4, it suffices to estimate $\|P^j \tilde{v}_{t,u,j}\|_{\theta,k}$ uniformly in j and t, since there are at most $t/\inf R + 1$ elements in the sum.

Note also that $\partial_t P^j\tilde{v} = P^j\widetilde{(\partial_t v)}$. Hence it suffices to prove that $\|P^j\tilde{v}_{t,u,j}\|_{\theta,0} \le C(|R|_\theta+1)\|v\|_{\theta,1}$ uniformly in j and t.

For general reasons, $|P^j\tilde{v}_{t,u,j}|_\infty \le |\tilde{v}_{t,u,j}|_\infty = |v|_\infty$. Next we recall that $(Pv)(y) = \sum_{a\in\alpha} e^{p(y_a)}v(y_a)$ where α is the underlying partition, y_a is the unique preimage $\bar{F}^{-1}y$ lying in a (this is where we assume full branches; otherwise there may be no preimage and the term is simply omitted) and p is the potential. Iterating, we obtain

$$(P^j\tilde{v}_{t,u,j})(y) = \sum_{a\in\alpha_j} e^{p_j(y_a)}v(y_a, u-t+R_j(y_a))$$

where α_j is the partition of j-cylinders and $p_j = \sum_{i=0}^{j-1} p\circ\bar{F}^i$, $R_j = \sum_{i=0}^{j-1} R\circ\bar{F}^i$. (Again y_a denotes the unique preimage $\bar{F}^{-j}y$ lying in a.) We recall the standard estimate for Gibbs-Markov expanding maps: there is a constant $C>0$ such that

$$|e^{p_n(y)} - e^{p_n(y')}| \le Ce^{p_n(y)}d_\theta(\bar{F}^n y, \bar{F}_n y'), \quad \text{for all } y,y'\in a,\ a\in\alpha_n,\ n\ge 1. \tag{5.5}$$

Also, an easy calculation shows that (see the proof of Lemma 2.9)

$$|R_n(y) - R_n(y')| \le (1-\theta)^{-1}|R|_\theta d_\theta(\bar{F}^n y, \bar{F}^n y'), \qquad \text{for all } y,y'\in a,\ a\in\alpha_n,\ n\ge 1. \tag{5.6}$$

Let $(y,u),(y',u')\in\bar{Y}^R$. We suppose without loss that $u'\le u$. Then

$$(P^j\tilde{v}_{t,u,j})(y) - (P^j\tilde{v}_{t,u',j})(y') = I + II + III + IV, \tag{5.7}$$

where

$$\begin{aligned}
I &= \sum_{a\in\alpha_j}(e^{p_j(y_a)} - e^{p_j(y'_a)})v(y_a, u-t+R_j(y_a)),\\
II &= \sum_{a\in\alpha_j} e^{p_j(y'_a)}\cdot[v(y_a,u-t+R_j(y_a)) - v(y'_a,u-t+R_j(y_a))],\\
III &= \sum_{a\in\alpha_j} e^{p_j(y'_a)}\cdot[v(y'_a,u-t+R_j(y_a)) - v(y'_a,u-t+R_j(y'_a))],\\
IV &= \sum_{a\in\alpha_j} e^{p_j(y'_a)}\cdot[v(y'_a,u-t+R_j(y'_a)) - v(y'_a,u'-t+R_j(y'_a))].
\end{aligned}$$

Using (5.5) and (5.6), $|I| \le C\sum_{a\in\alpha_j} e^{p_j(y_a)}|v|_\infty = C|v|_\infty$, and

$$|III| \le \sum_{a\in\alpha_j} e^{p_j(y'_a)}|\partial_t v|_\infty|R_j(y_a) - R_j(y'_a)| \le (1-\theta)^{-1}|\partial_t v|_\infty|R|_\theta d_\theta(y,y').$$

Also $|II| \le \sum_{a\in\alpha_j} e^{p_j(y'_a)}|v|_\theta d_\theta(y_a,y'_a) = \theta^j|v|_\theta d_\theta(y_a,y'_a)$ and $|IV| \le |\partial_t v|_\infty|u-u'|$. Hence $\|P^j\tilde{v}_{t,u,j}\|_{\theta,0} \le C(|R|_\theta+1)\|v\|_{\theta,1}$ as required. □

Corollary 5.7. *There exists a constant C (depending on k, θ and $|R|_\theta$) such that*

$$\sum_{i=1}^{n}\sum_{j=0}^{n-i}|L_i(vL_jv) - \textstyle\int_{\bar{Y}^R} vL_jv\,d\bar{\mu}|_2 \le C(n^{\sqrt{2}+1-\beta/2}+1)\|v\|^2_{\theta,k+1},$$

for all mean zero $v\in F_{\theta,k}(\bar{Y}^R)$.

Proof. On one hand, by Proposition 5.5 and Lemma 5.6,

$$|L_i(vL_jv) - \textstyle\int_{\bar{Y}^R} vL_jv\,d\bar{\mu}|_2 \le Ci^{-\beta/2}\|vL_jv\|_{\theta,k} \le C'i^{-\beta/2}(j+1)\|v\|^2_{\theta,k+1}.$$

On the other hand,

$$\begin{aligned}
|L_i(vL_jv) - \textstyle\int_{\bar{Y}^R} vL_jv\,d\bar{\mu}|_2 &\le |vL_jv - \textstyle\int_{\bar{Y}^R} vL_jv\,d\bar{\mu}|_2\\
&\le 2|vL_jv|_2 \le 2|v|_\infty|L_jv|_2 \le C(j+1)^{-\beta/2}\|v\|^2_{\theta,k}.
\end{aligned}$$

Using the first estimate,

$$\sum_{i=1}^{n}\sum_{j=0}^{i^{1/\sqrt{2}}}|L_i(vL_jv)-\textstyle\int_{\bar{Y}^R}vL_jv\,d\bar{\mu}|_2\le C\|v\|_{\theta,k+1}^2\sum_{i=1}^{n}\sum_{j=0}^{i^{1/\sqrt{2}}}(j+1)i^{-\beta/2}\le C'\|v\|_{\theta,k+1}^2(n^{\sqrt{2}-\beta/2+1}+1).$$

Using the second estimate,

$$\sum_{j=0}^{n}\sum_{i=1}^{j^{\sqrt{2}}}|L_i(vL_jv)-\textstyle\int_{\bar{Y}^R}vL_jv\,d\bar{\mu}|_2\le C\|v\|_{\theta,k+1}^2\sum_{j=0}^{n}\sum_{i=1}^{j^{\sqrt{2}}}(j+1)^{-\beta/2}\le C'\|v\|_{\theta,k+1}^2(n^{\sqrt{2}-\beta/2+1}+1).$$

Combining these gives the required estimate. □

Now we can complete the proof of Theorem 5.2.

Proof of Theorem 5.2. By Proposition 5.5, $|L_nv|_p^p\le C^p\|v\|_{\theta,k}^p n^{-\beta}$. Hence conditions (5.1) and (5.2) are satisfied for $\beta>\frac{7}{2}$. By Corollary 5.7, condition (5.3) is satisfied for $\beta>2\sqrt{2}+1$. □

Finally, as promised in Remark 5.1, we show how condition (iii) can be relaxed. Indeed it suffices that

(iii$_1$) $\sum_{a\in\alpha}\mu_{\bar{Y}}(a)\operatorname{Lip}_a R<\infty$,

where $\operatorname{Lip}_A g=\sup_{x,y\in A,\,x\ne y}|g(x)-g(y)|/d_\theta(x,y)$ for $g:\bar{Y}\to\mathbb{R}$, $A\subset\bar{Y}$.

We begin by recalling a standard estimate for Gibbs-Markov maps which we did not explicitly make use of earlier: there is a constant $C>0$ such that $|1_ae^{p_j}|_\infty\le C\mu_{\bar{Y}}(a)$ for all $a\in\alpha_j$, $j\ge1$.

Note that condition (iii) was only used in the estimate of term III in the proof of Lemma 5.6. But alternatively, we compute that $|III|\le|\partial_tv|_\infty\sum_{i=0}^{j-1}A_{i,j}$ where

$$\begin{aligned}A_{i,j}=\sum_{a\in\alpha_j}e^{p_j(y_a')}|R\circ\bar{F}^i(y_a)-R\circ\bar{F}^i(y_a')|&\le C\sum_{a\in\alpha_j}\mu_{\bar{Y}}(a)\operatorname{Lip}_{\bar{F}^ia}R d_\theta(\bar{F}^iy_a,\bar{F}^iy_a')\\&=C\theta^{j-i}d_\theta(y,y')\sum_{a\in\alpha_j}\mu_{\bar{Y}}(a)\operatorname{Lip}_{\bar{F}^ia}R.\end{aligned}$$

Now

$$\begin{aligned}\sum_{a\in\alpha_j}\mu_{\bar{Y}}(a)\operatorname{Lip}_{\bar{F}^ia}R&=\sum_{b\in\alpha_{j-i}}\sum_{a\in\alpha_j:\bar{F}^ia=b}\mu_{\bar{Y}}(a)\operatorname{Lip}_bR=\sum_{b\in\alpha_{j-i}}\mu_{\bar{Y}}(\bar{F}^{-i}b)\operatorname{Lip}_bR=\sum_{b\in\alpha_{j-i}}\mu_{\bar{Y}}(b)\operatorname{Lip}_bR\\&=\sum_{c\in\alpha}\sum_{b\in\alpha_{j-i}:b\subset c}\mu_{\bar{Y}}(b)\operatorname{Lip}_bR\le\sum_{c\in\alpha}\sum_{b\in\alpha_{j-i}:b\subset c}\mu_{\bar{Y}}(b)\operatorname{Lip}_cR=\sum_{c\in\alpha}\mu_{\bar{Y}}(c)\operatorname{Lip}_cR.\end{aligned}$$

Hence $A_{i,j}\le C\theta^{j-i}d_\theta(y,y')\sum_{a\in\alpha}\mu_{\bar{Y}}(a)\operatorname{Lip}_aR$ and

$$|III|\le C\theta(1-\theta)^{-1}|\partial_tv|_\infty\sum_{a\in\alpha}\mu_{\bar{Y}}(a)\operatorname{Lip}_aR\,d_\theta(y,y').$$

Hence, assuming condition (iii$_1$) instead of (iii), we obtain an estimate analogous to the one in Lemma 5.6 with the conclusion (5.4) replaced by the estimate

$$\|vL_tv\|_{\theta,k}\le C_1(t+1)\Big(\sum_{a\in\alpha}\mu_{\bar{Y}}(a)\operatorname{Lip}_aR+1\Big)\|v\|_{\theta,k+1}^2,$$

for all $t\ge0$.

6. Nondegeneracy in the CLT and ASIP

In this section, we prove that the degenerate case $\sigma^2 = 0$ in Theorem 5.2 is of infinite codimension. Suppose as in Theorem 5.2 that $v = \chi \circ \bar{S} - \chi$ for some $\chi \in L^2$. Following [24, Proposition 2], we define $\psi_t = \int_0^t v \circ \bar{S}_s \, ds$ and $h = \int_0^1 \chi \circ \bar{S}_s \, ds$. Then ψ_t is a continuous cocycle for the semiflow $\bar{S}_t$; that is $\psi_{t+s} = \psi_s \circ \bar{S}_t + \psi_t$ for all $s, t \geq 0$. Moreover, $h \in L^2$ and $\psi_t = h \circ \bar{S}_t - h$, so ψ_t is an L^2 coboundary.

In the Axiom A setting of [24] it now follows from a Livšic regularity theorem of [34] that h has a Hölder version. Hence if q is a periodic point of period T for $\bar{S}_t$, then $\int_0^T v(\bar{S}_t q) \, dt = \psi_T(q) = h(\bar{S}_T q) - h(q) = 0$. Since $\bar{S}_t$ has infinitely many periodic orbits, this places infinitely many restrictions on v.

In the nonuniformly expanding case, the situation is similar once we have a Livšic regularity theorem for nonuniformly expanding semiflows. As we now show, this is a straightforward combination of results of [15] for Gibbs-Markov maps and [34] for uniformly expanding semiflows.

In the remainder of this section, we suppose as in Section 5 that $\bar{F} : \bar{Y} \to \bar{Y}$ is a full-branch Gibbs-Markov map and that $R : \bar{Y} \to \mathbb{R}^+$ is a roof function satisfying conditions (i), (ii) and (iii$_1$). (The full-branch condition is relaxed in Remark 6.3.)

Given a continuous cocycle ψ_t on $\bar{Y}^R$, we define $I_\psi : \bar{Y} \to \mathbb{R}$

$$I_\psi(y) = \psi_{R(y)}(y, 0).$$

Lemma 6.1. *Let ψ be a cocycle for $\bar{S}_t$ such that $I_\psi : X \to G$ satisfies*

$$\sum_{a \in \alpha} \mu_{\bar{Y}}(a) \operatorname{Lip}_a I_\psi < \infty.$$

Suppose that there exists $h : \bar{Y}^R \to \mathbb{R}$ measurable such that for all $t \geq 0$: $\psi_t = h \circ \bar{S}_t - h$ a.e. Then h has a version that is continuous.

Proof. We begin by following the proof of [34, Theorem 3.3]. Note that the set of zero measure where $\psi_t = h \circ \bar{S}_t - h$ fails for each $t \geq 0$ can be made independent of t; see e.g. [11, p. 13]. Hence for almost every $(y, u) \in \bar{Y}^R$,

$$\psi_{R(y)}(y, u) = h(\bar{S}_{R(y)}(y, u)) - h(y, u). \tag{6.1}$$

Then $\Psi(y, u) = \psi_{R(y)}(y, u) - h(\bar{S}_{R(y)}(y, u)) + h(y, u) = 0$, $\bar{\mu}$-a.e. and so by Fubini's Theorem there exists $0 < u_0 < \inf R$ such that $\Psi(y, u_0) = 0$ for $\mu_{\bar{Y}}$-a.e $y \in \bar{Y}$. Since ψ is a cocycle,

$$\begin{aligned} \psi_{R(y)}(y, u_0) &= \psi_{R(y)}(\bar{S}_{u_0}(y, 0)) = \psi_{R(y)+u_0}(y, 0) - \psi_{u_0}(y, 0) \\ &= \psi_{u_0}(\bar{S}_{R(y)}(y, 0)) + \psi_{R(y)}(y, 0) - \psi_{u_0}(y, 0) \\ &= \psi_{u_0}(\bar{S}_{R(y)}(y, 0)) + I_\psi(y) - \psi_{u_0}(y, 0). \end{aligned}$$

Substituting in (6.1) with $u = u_0$ and using $\bar{S}_{R(y)}(y, u_0) = (\bar{F}y, u_0)$ we obtain

$$I_\psi(y) = h(\bar{F}y, u_0) - \psi_{u_0}(\bar{F}y, 0) - (h(y, u_0) - \psi_{u_0}(y, 0)) = g(\bar{F}y) - g(y),$$

where $g(y) = h(y, u_0) - \psi_{u_0}(y, 0)$ is measurable.

We have shown that $I_\psi = g \circ \bar{F} - g$ satisfies the hypotheses of [15, Theorem 1.1]. It follows that g has a version that is continuous (even Lipschitz) on $\bar{Y}$.

Let us now define

$$\tilde{h}(y, u) = \psi_u(y, 0) + g(y).$$

As in the proof of [34, Theorem 3.3], it follows from the definitions that $\tilde{h}$ is a well-defined function on $\bar{Y}^R$ (ie $\tilde{h}(y, R(y)) = \tilde{h}(\bar{F}, 0)$) and that $\tilde{h}$ is a version of h. Since ψ_u and g are continuous, it follows that $\tilde{h}$ is continuous as required. ☐

Corollary 6.2. *Suppose that $v \in F_{\theta,0}(\bar{Y}^R)$ satisfies $v = \chi \circ \bar{S} - \chi$ for some $\chi \in L^2$. Then $\int_0^T v(\bar{S}_t q) \, dt = 0$ for all periodic points $q \in \bar{Y}^R$ of period T.*

Proof. Define $\psi_t = \int_0^t v \circ \bar{S}_s \, ds$ and $h = \int_0^1 \chi \circ \bar{S}_s \, ds$, so $\psi_t = h \circ \bar{S}_t - h$. We claim that I_ψ satisfies the assumption in Lemma 6.1. Then h has a continuous version. Hence it follows as in the Axiom A setting that $\int_0^T v(\bar{S}_t q)\, dt = 0$ for all periodic points $q \in \bar{Y}^R$ of period T.

It remains to verify the claim. For $y \in \bar{Y}$, we compute that $I_\psi(y) = \psi_{R(y)}(y, 0) = \int_0^{R(y)} v \circ \bar{S}_s(y, 0) ds = \int_0^{R(y)} v(y, u)\, du$. For $a \in \alpha$, $x, y \in a$, and taking $R(y) \geq R(x)$,

$$\begin{aligned}|I_\psi(x) - I_\psi(y)| &= \Big| \int_0^{R(x)} \big(v(x, u) - v(y, u)\big)\, du + \int_{R(x)}^{R(y)} v(y, u)\, du \Big| \\ &\leq R(x)|v|_\theta d_\theta(x, y) + |R(y) - R(x)||v|_\infty \leq (\sup_a R + \operatorname{Lip}_a R)\|v\|_\theta d_\theta(x, y).\end{aligned}$$

Hence $\operatorname{Lip}_a I_\psi \leq (\sup_a R + \operatorname{Lip}_a R)\|v\|_\theta$ and so

$$\sum_{a \in \alpha} \mu_{\bar{Y}}(a) \operatorname{Lip}_a I_\psi \leq \|v\|_\theta \sum_{a \in \alpha} \mu_{\bar{Y}}(a)(\sup_a R + \operatorname{Lip}_a R) < \infty,$$

as required. □

Remark 6.3. The condition that $\bar{F} : \bar{Y} \to \bar{Y}$ is full-branch can be relaxed as in [15]. The function $g : \bar{Y} \to \mathbb{R}$ constructed in the proof of Lemma 6.1 will no longer be continuous in general, but it is continuous on each partition element of the partition α_* generated by the images $\bar{F}a$ of the elements of α. We conclude that h has a version that is continuous on $\{(y, u) \in a_* \times [0, \infty) : u \leq R(y)\}$ for each $a_* \in \alpha_*$. Hence in Corollary 6.2 we obtain that $\int_0^T v(\bar{S}_t q)\, dt = 0$ for periodic points $q \in \bar{Y}^R$ of period T such that the orbit of q intersects one of the partition elements a_*.

7. CLT and ASIP for the time-1 map of geometric Lorenz flows

In this section we prove Theorem C (and as a consequence Theorem B), by reducing from the geometric Lorenz flow to the quotient flow, enabling the application of Theorem 5.2.

To achieve this reduction, we modify the argument in [24, Appendix A] which deals with the Axiom A case and bounded roof function. We note that the argument in [24] is unnecessarily complicated, since having reduced without loss to the situation where r depends only on future coordinates, the quantity Δ in [24, Proposition 5] is identically zero.

On the other hand, the situation for geometric Lorenz attractors is made complicated since (a) there is no convenient metric on the symbolic flow Y^R, and (b) the roof function is unbounded. To deal with (a), we reduce directly to $\bar{Y}^R$. For (b), we make crucial use of Lemma 2.9 and Proposition 2.11.

Theorem 7.1. *Let $v : \mathbb{R}^3 \to \mathbb{R}$ be a C^{k+1} observable of mean zero. There exists $\hat{v}, \hat{\chi} : Y^R \to \mathbb{R}$ continuous and bounded such that*

- (i) $v \circ p = \hat{v} + \hat{\chi} - \hat{\chi} \circ S$,
- (ii) *$\hat{v}$ depends only on future coordinates and hence projects to a mean zero observable $\bar{v} : \bar{Y}^R \to \mathbb{R}$,*
- (iii) *$\bar{v} \in F_{\theta', k}(\bar{Y}^R)$ for some $\theta' \in (0, 1)$.*

We recall that $p : Y^R \to \mathbb{R}^3$ denotes the measure-preserving semiconjugacy between the flows $S_t : Y^R \to Y^R$ and $Z_t : \mathbb{R}^3 \to \mathbb{R}^3$. Recall also that the projection $\pi : X \to \bar{X}$ along stable manifolds restricts to a projection $\pi : Y \to \bar{Y}$. Also, we defined $\pi(y, u) = (\pi y, u)$ provided that $u \in [0, R(y))$. This induces a measure-preserving semiconjugacy $\pi : Y^R \to \bar{Y}^R$ between the flow $S_t : Y^R \to Y^R$ and the semiflow $\bar{S}_t : \bar{Y}^R \to \bar{Y}^R$.

We now show how Theorem C follows from Theorem 7.1.

Proof of Theorem C. Let $v : \mathbb{R}^3 \to \mathbb{R}$ be a C^{k+2} mean zero observable. By Theorem 7.1, $\sum_{j=0}^{n-1} v \circ Z^j \circ p = \sum_{j=0}^{n-1} \hat{v} \circ S^j + \hat{\chi} - \hat{\chi} \circ S^n = \sum_{j=0}^{n-1} \bar{v} \circ S^j \circ \pi + O(1)$ uniformly on Y^R, where $\bar{v} \in F_{\theta', k+1}(\bar{Y}^R)$. Hence the ASIP for v is equivalent to the ASIP for $\bar{v}$ and follows from Theorem 5.2.

It remains to verify the statement about the degenerate case $\sigma^2 = 0$ in Theorem B. By Theorem 5.2, $\bar{v} = \chi \circ S - \chi$ for some $\chi \in L^p(\bar{Y}^R)$. Working still on $\bar{Y}^R$, define $\psi_t = \int_0^t \bar{v} \circ S_s \, ds$, $h = \int_0^1 \chi \circ S_s \, ds$, so $\psi_t = h \circ \bar{S}_t - h$. By Lemma 6.1, h has a continuous version.

Lifting to Y^R, we have that $\int_0^t \hat{v} \circ S_s \, ds = \hat{h} \circ S_t - \hat{h}$ where $\hat{h} = h \circ \pi$ is continuous. Hence using Theorem 7.1(i),

$$\int_0^t v \circ Z_s \circ p \, ds = \hat{h} \circ p \circ S_t - \hat{h} \circ p + \int_0^t \tilde{\chi} \circ S_s \, ds - \int_0^t \tilde{\chi} \circ S_{s+1} \, ds = \tilde{h} \circ S_t - \tilde{h}, \tag{7.1}$$

where $\tilde{h} = \hat{h} \circ p - \int_0^1 \tilde{\chi} \circ S_s \, ds$ is continuous.

Now suppose that q is a periodic point of period T_1 for the geometric Lorenz flow Z_t. Then $q = Z_{u_0}(y_0) = p(y_0, u_0)$ for some $y_0 \in Y$, $u_0 \in [0, R(y_0)]$. Since R is not necessarily the first return time to Y it need not be the case that (y_0, u_0) has period T_1 under S_t. However, certainly there exists $T > 0$, an integer multiple of T_1, such that $S_T(y_0, u_0) = (y_0, u_0)$. By (7.1), $\int_0^T v(Z_t q) \, dt = \tilde{h}(S_T(y_0, u_0)) - \tilde{h}(y_0, u_0) = 0$ as required. □

In the remainder of this section, we prove Theorem 7.1.

Recall that the projection $p : Y^R \to \mathbb{R}^3$ is given by $p(y, u) = Z_u(y)$. Note that if $(y, u) \in Y^R$ then $\pi Z_u y = Z_u \pi y$. However, we caution that for general $t > 0$, $x \in \mathbb{R}^3$ it is *not* the case that $\pi Z_t x$ and $Z_t \pi x$ coincide.

Proof of Theorem 7.1. Define $\hat{\chi} : Y^R \to \mathbb{R}$ by setting

$$\hat{\chi}(y, u) = \sum_{n=0}^{\infty} \{v(Z_n Z_u y) - v(Z_n \pi Z_u y)\}.$$

It follows from exponential contraction along stable manifolds that there are constants $C, a > 0$ such that

$$\begin{aligned} |v(Z_n(Z_u y) - v(Z_n(\pi Z_u y))| &\le |Dv|_\infty |Z_n(Z_u y) - Z_n(\pi Z_u y)| \\ &\le C|Dv|_\infty e^{-an} |Z_u y - \pi Z_u y| \le C'|Dv|_\infty e^{-an}, \end{aligned} \tag{7.2}$$

so that $\hat{\chi}$ is continuous and bounded.

Define $\hat{v} : Y^R \to \mathbb{R}$ by setting

$$\hat{v} = v \circ p + \hat{\chi} \circ S - \hat{\chi},$$

so that (i) is satisfied by definition. Also

$$\hat{v}(y, u) = v(\pi Z_u y) + \sum_{n=0}^{\infty} \{v(Z_n(Z \pi Z_u y)) - v(Z_n(\pi Z_{u+1} y))\},$$

and so (ii) is satisfied.

When proving (iii), we note that the formula for $\partial_t^j \hat{v}$ is identical to that for $\hat{v}$ with v replaced by $\partial_t^j v$ throughout. Hence it suffices to consider the case $k = 0$ and to prove that $\bar{v} \in F_{\theta', 0}(\bar{Y}^R)$ for $v \in C^1(\mathbb{R}^3)$.

By the triangle inequality it suffices to show that

$$|\bar{v}(y, u) - \bar{v}(y, u')| \le C|Dv|_\infty |u - u'|, \tag{7.3}$$

$$|\bar{v}(y, u) - \bar{v}(y', u)| \le C|Dv|_\infty (1 + |R|_\theta) d_{\theta'}(y, y'). \tag{7.4}$$

First we prove (7.3). The n'th term of $\tilde{v}$ is given by

$$w_n(u) = v(Z_n(Z \pi Z_u y)) - v(Z_n(\pi Z_{u+1} y)).$$

We have $|w_n(u) - w_n(u')| \le |Dw_n|_\infty |u - u'|$. But

$$\begin{aligned} |Dw_n(u)| &\le |Dv|_\infty |Z_n(Z \pi Z_u y) - Z_n(\pi Z_{u+1} y)| \\ &\le C|Dv|_\infty e^{-an} |Z \pi Z_u y - \pi Z_{u+1} y| \le C'|Dv|_\infty e^{-an}. \end{aligned}$$

Hence

$$|w_n(u) - w_n(u')| \le C''|Dv|_\infty e^{-an}|u-u'|,$$

and (7.3) follows.

It remains to prove (7.4). For the initial term in the formula for $\hat v$, we note that

$$\begin{aligned}|v(\pi Z_u y) - v(\pi Z_u y')| &\le |Dv|_\infty|\pi Z_u y - \pi Z_u y'| = |Dv|_\infty|p(\pi y,u) - p(\pi y',u)| \\ &\le C|Dv|_\infty d_\theta(y,y'),\end{aligned} \tag{7.5}$$

by Proposition 2.11.

Let $N \ge 1$. We write the remainder of $\hat v(y,u) - \hat v(y',u)$ as

$$A(Z\pi Z_u y, Z\pi Z_u y') + A(\pi Z_{u+1}y, \pi Z_{u+1}y') + B(y) + B(y'),$$

where

$$A(x,x') = \sum_{n=0}^{N-1}\{v(Z_n x) - v(Z_n x')\},$$

$$B(y) = \sum_{n=N}^{\infty}\{v(Z_n(Z\pi Z_u y)) - v(Z_n(\pi Z_{u+1}y))\}.$$

We again use exponential contraction along stable directions as in (7.2) to show that

$$|B(y)|,\ |B(y')| \le C|Dv|_\infty e^{-aN}. \tag{7.6}$$

Let $j = j(y,t)$ be the lap number for $y \in Y$ under Z_t, so $t \in [R_j(y), R_{j+1}(y))$ and $Z_t(y,0) = p(F^j y, t - R_j(y))$. Then the n'th term of $A(Z\pi Z_u y, Z\pi Z_u y') = A(Z_{u+1}\pi y, Z_{u+1}\pi y')$ has the form

$$a_n = v\circ p(F^j\pi y, n+u+1-R_j(y)) - v\circ p(F^{j'}\pi y', n+u+1-R_{j'}(y')),$$

where

$$j = j(y,n+u+1), \quad j' = j(y',n+u+1). \tag{7.7}$$

Note that $j,\ j' \le (n+1)/\inf R \le N/\inf R$.

Let $q = [1/\bar r]+2$. Suppose that $s(y,y') = qN$. Choose N so large that $(1-\theta)^{-1}|R|_\theta\theta^N < \inf R$. Then

$$\begin{aligned}|R_j(y) - R_j(y')| &\le (1-\theta)^{-1}|R|_\theta\theta^{-j+1}\theta^{Nq} \le (1-\theta)^{-1}|R|_\theta\theta^{N(q-1/\inf R)} \\ &\le (1-\theta)^{-1}|R|_\theta\theta^N < \inf R,\end{aligned}$$

for all $0 \le j \le [N/\inf R]$. Hence for this range of j, the intervals $[R_j(y), R_{j+1}(y)]$ and $[R_j(y'), R_{j+1}(y')]$ almost coincide (the initial points are within distance $\inf R$, as are the final points). It follows that the lap numbers j and j' in (7.7) satisfy $|j-j'| \le 1$ for all $0 \le n \le N$. The estimation of the terms in $A(Z\pi Z_u y, Z\pi Z_u y')$ now splits into three cases.

When $j = j'$, we obtain the term

$$a_n = v\circ p(F^j\pi y, n+u+1-R_j(y)) - v\circ p(F^j\pi y', n+u+1-R_j(y')).$$

Hence by Proposition 2.11,

$$\begin{aligned}|a_n| &\le C|Dv|_\infty\{\theta^{s(F^j\pi y, F^j\pi y')}| + |R_j(y) - R_j(y')|\} \\ &\le C'|Dv|_\infty(1+|R|_\theta)\theta^{s(y,y')-j} \le C'|Dv|_\infty(1+|R|_\theta)\theta^{qN-n/\inf R}.\end{aligned} \tag{7.8}$$

If $j' = j+1$, then

$$\begin{aligned}a_n =& v\circ p(F^j\pi y, n+u+1-R_j(y)) - v\circ p(F^j\pi y, R(F^j y)) \\ &+ v\circ p(F^{j+1}\pi y, 0) - v\circ p(F^{j+1}\pi y', n+u+1-R_{j+1}(y')),\end{aligned}$$

so that

$$\begin{aligned}|a_n| &\le C|Dv|_\infty\{R_{j+1}(y)-n-u-1\} \\ &\quad + C|Dv|_\infty\{\theta^{s(F^{j+1}y,F^{j+1}y')}+n+u+1-R_{j+1}(y')\} \\ &= C|Dv|_\infty\{\theta^{s(F^{j+1}y,F^{j+1}y')}+R_{j+1}(y)-R_{j+1}(y')\},\end{aligned}$$

yielding the same estimate as in (7.8). Similarly for the case $j'=j-1$. Hence in all three cases, we obtain the estimate (7.8). Summing over n, we obtain that

$$|A(Z\pi Z_u y, Z\pi Z_u y')| \le C|Dv|_\infty(1+|R|_\theta)\theta^{(q-1/\inf R)N} \le C|Dv|_\infty(1+|R|_\theta)\theta^N. \tag{7.9}$$

To deal with the n'th term $A(\pi Z_{u+1}y, \pi Z_{u+1}y')$ we need to introduce four lap numbers. First let $j_1 \le 1/\inf R$ be the lap number corresponding to $Z_{u+1}y$, so

$$\pi Z_{u+1}y = \pi Z_{u+1-R_{j_1}(y)}(F^{j_1}y) = Z_{u+1-R_{j_1}(y)}(\pi F^{j_1}y).$$

Then let $j=j_1+j_2$ where $j_2 \le n/\inf R$ is the lap number corresponding to $Z_{u+1-R_{j_1}(y)}(F^{j_1}y)$ under Z_n. Altogether, we obtain

$$Z_n\pi Z_{u+1}y = Z_{n+u+1-R_{j_1}(y)-R_{j_2}(F^{j_1}y)}(F^{j_2}\pi F^{j_1}y) = p(F^{j_2}\pi F^{j_1}y, n+u+1-R_j(y)).$$

Similarly, we write

$$Z_n\pi Z_{u+1}y' = p(F^{j_2'}\pi F^{j_1'}y', n+u+1-R_{j'}(y')).$$

Again, we consider the three cases $j=j'$, $j=j'+1$, $j=j'-1$ separately. For example, if $j'=j+1$, then

$$\begin{aligned}&|Z_n\pi Z_{u+1}y - Z_n\pi Z_{u+1}y'| \\ &\quad = |p(F^{j_2}\pi F^{j_1}y, n+u+1-R_j(y)) - p(F^{j_2'}\pi F^{j_1'}y', n+u+1-R_{j'}(y'))| \\ &\quad \le |p(F^{j_2}\pi F^{j_1}y, n+u+1-R_j(y)) - p(F^{j_2}\pi F^{j_1}y, R(F^jy)| \\ &\qquad\quad + |p(F^{j_2+1}\pi F^{j_1}y, 0) - p(F^{j_2'}\pi F^{j_1'}y', n+u+1-R_{j+1}(y'))| \\ &\quad \le C\{R_{j+1}(y)-n-u-1\} + C\{\theta^{s(F^{j_2+1}\pi F^{j_1}y, F^{j_2'}\pi F^{j_1'}y')}+n+u+1-R_{j+1}(y')\} \\ &\quad = C\{\theta^{s(F^{j+1}y,F^{j+1}y')}+R_{j+1}(y)-R_{j+1}(y')\},\end{aligned}$$

and the calculation proceeds as for (7.9). Hence we obtain

$$|A(\pi Z_{u+1}y, \pi Z_{u+1}y')| \le C|Dv|_\infty(1+|R|_\theta)\theta^N. \tag{7.10}$$

Combining (7.5), (7.6), (7.9), (7.10), we obtain that $|\bar v(y,u)-\bar v(y',u)| \le C|Dv|_\infty\{e^{-aN}+(1+|R|_\theta)\theta^{N/q}\}$. Hence (7.4) holds with $\theta'=\max\{e^{-a},\theta^{1/q}\}$ completing the proof. □

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Vitor Araújo and Paulo Varandas, Departamento de Matemática, Universidade Federal da Bahia, Av. Ademar de Barros s/n, 40170-110 Salvador, Brazil.

E-mail address: `vitor.d.araujo@ufba.br, www.sd.mat.ufba.br/∼vitor.d.araujo`

E-mail address: `paulo.varandas@ufba.br, www.pgmat.ufba.br/varandas/`

Ian Melbourne, Institute of Mathematics, University of Warwick, Coventry CV4 7AL, UK

E-mail address: `i.melbourne@warwick.ac.uk`