# Bifurcation from Periodic and Relative Periodic Solutions in Equivariant Dynamical Systems

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**Abstract**

We review recent developments in the theory for generic bifurcation from periodic and relative periodic solutions in equivariant dynamical systems.

## 1 Introduction

Equivariant bifurcation theory is concerned, to a large extent, with local bifurcation theory for vector fields that are equivariant with respect to the action of a compact Lie group $\Gamma$, see for instance Golubitsky, Stewart and Schaeffer [11]. In particular, a systematic approach to bifurcation from equilibria is laid out in [11]. This approach has been generalized to include bifurcation from relative equilibria (where a single group orbit is flow invariant), see Krupa [14], and also situations where $\Gamma$ is noncompact but acts properly on a finite dimensional manifold, see Fiedler, Sandstede, Scheel and Wulff [7].

Recently, the corresponding theory for bifurcation from periodic solutions has been in development. The simplest case is when the periodic solution has only *spatial* symmetries (symmetries that fix the periodic solution pointwise in phase space), and has been studied by Chossat and Golubitsky [5] (see also Ruelle [20]).

The more complicated case in which a periodic solution has not only spatial, but also *spatiotemporal*, symmetries has been studied by Lamb, Melbourne and Wulff [16, 18]. In these papers, we consider *isolated discrete rotating waves*. These are isolated periodic solutions with a mixture of spatial and spatiotemporal symmetries. We have shown that bifurcations from isolated discrete rotating waves reduce to bifurcations from fixed points of twisted equivariant diffeomorphisms, which in turn reduce (up to flat terms) to bifurcations of equilibria of equivariant vector fields. The novel ingredient in this reduction scheme is the systematic treatment of twisted equivariant diffeomorphisms.

In Wulff, Lamb and Melbourne [23] we consider bifurcation from *relative periodic solutions*. These are flow-invariant sets that reduce to periodic solutions at the $\Gamma$ orbit space level. (Here, $\Gamma$ is possibly noncompact but is assumed to act properly on a finite dimensional manifold.) It is shown in [23] how to reduce the problem to bifurcation from an isolated discrete rotating wave for an equivariant vector field with a compact symmetry group.

Thus, we have the following hierarchy of reductions:

relative periodic solutions
↓
isolated discrete rotating waves
↓
fixed points for twisted equivariant diffeomorphisms
↓
equilibria for equivariant vector fields

In this paper, we review the above results. For transparency, we state our results for dynamical systems in $\mathbb{R}^n$ that are equivariant with respect to a compact Lie group, although our results apply to a more general context (in particular also to noncompact symmetry groups). For more details we refer the reader to [16, 18, 23].

## 2 Periodic and relative periodic solutions in equivariant systems

We consider equivariant dynamical systems in $\mathbb{R}^n$, ie smooth ordinary differential equations of the form

$$\frac{dx}{dt} = f(x), \quad x \in \mathbb{R}^n \tag{2.1}$$

that are equivariant with respect to a linear (orthogonal) representation of a compact Lie group $\Gamma$ on $\mathbb{R}^n$

$$\gamma f(x) = f(\gamma x), \quad \forall\ \gamma \in \Gamma.$$

Consequently, a $\Gamma$-equivariant dynamical system admits a solution $x(t)$ if and only if it also admits $\gamma x(t)$ as a solution, for all $\gamma \in \Gamma$.

Equivariant dynamical systems arise naturally in many problems in physics. The general aim is to study typical (generic) phenomena in such dynamical systems, such as for instance local bifurcations from equilibria and periodic solutions.

A systematic approach towards the bifurcation theory for equilibria and relative equilibria was set out in [11, 14]. In this paper, we review the recently developed corresponding approach towards local bifurcation from periodic and relative periodic solutions.

Consider a periodic solution $x(t)$ of (minimal) period $T$, ie there exists a $T > 0$ least such that $x(0) = x(T)$. The symmetries of periodic solutions in equivariant systems fall into two categories: spatial and spatiotemporal symmetries. *Spatial* symmetries are elements $\delta \in \Gamma$ that fix a solution $x(t)$ pointwise, so that $\delta x(t) = x(t)$ for all $t$. *Spatiotemporal* symmetries are elements $\sigma \in \Gamma$ that fix $x(t)$ setwise, but not necessarily pointwise. In particular, they satisfy $\sigma x(t) = x(t + T_\sigma)$ for some $0 \le T_\sigma < T$. Spatiotemporal symmetries that are not spatial (ie with $T_\sigma > 0$) are also called *time-shift* symmetries of $x(t)$.

It is easily verified that the spatiotemporal symmetries of a periodic solution $x(t)$ form a group $\Sigma$ under composition. The spatial symmetries form a subgroup $\Delta$ of $\Sigma$. In fact, $\Delta$ is a normal subgroup of $\Sigma$ and the quotient $\Sigma/\Delta$ is isomorphic to $S^1$ or to a finite cyclic group $\mathbb{Z}_m$. If the quotient is $S^1$, the periodic solution is a *rotating wave*, which is a special case of a relative equilibrium. We speak of a *discrete rotating wave* if

$$\Sigma/\Delta \cong \mathbb{Z}_m.$$

Here $m$ is the largest positive integer such that there exists a time-shift symmetry $\sigma$ in $\Sigma$ for which $\sigma x(t) = x(t + \frac{T}{m})$ (we call such a time-shift symmetry a *minimal* time-shift symmetry of $x(t)$). Consequently, the group $\Sigma$ is generated by $\Delta$ and $\sigma$:

$$\Sigma = \langle \Delta, \sigma \rangle.$$

We say that a discrete rotating wave with spatial symmetry $\Delta$ is *isolated* if $\dim \Delta = \dim \Gamma$. (Note that if $\dim \Gamma > \dim \Delta$, discrete rotating waves come in continuous families due to equivariance.)

**Example 2.1**

(i) Discrete rotating waves are well-known to arise by Hopf bifurcation from a symmetric equilibrium [11]. For example, a fully symmetric equilibrium in a $\mathbb{D}_4$-equivariant dynamical system admits a codimension one Hopf bifurcation to periodic solutions with spatial symmetry $\Delta = \mathbb{D}_2$ and spatiotemporal symmetry $\Sigma = \mathbb{D}_4$. We have $m = 2$, and the time-shift symmetry $\sigma$ may be taken to be any reflection of $\mathbb{D}_4$ that is not in $\mathbb{D}_2$ combined with a time-shift of half the period.

(ii) Discrete rotating waves are observed in many physically relevant situations. For instance, the so-called *Von Kármán vortex street* that arises in a fluid flowing past a cylinder at low Reynolds numbers, is an example of an isolated discrete rotating wave with spatial symmetry $\Delta = SO(2)$ and spatiotemporal symmetry $\Sigma = SO(2) \times \mathbb{Z}_2$ (under the assumption of periodic boundary conditions). Here again, we have $m = 2$, and the spatiotemporal symmetry consists of the generator of $\mathbb{Z}_2$ combined with a time-shift of half the period of the solution. See [16, 17] for more details.

A solution $x(t)$ is called a *relative periodic solution* of (2.1) if there exists a $T > 0$ least such that

$$\sigma x(t) = x(t + T), \text{ for some } \sigma \in \Gamma.$$

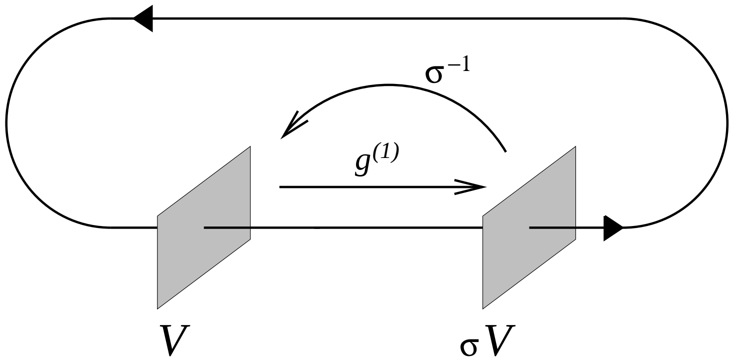


Figure 1: Schematic sketch of the first-hit-pull-back map $f = \sigma^{-1}g^{(1)}$ for an isolated discrete rotating wave.

We call $T$ the *relative period* of $x(t)$, and $\sigma$ a minimal time-shift symmetry of $x(t)$. From this definition it is evident that periodic solutions are special examples of relative periodic solutions. However, relative periodic solutions need not be periodic. They are called *relative periodic* because they correspond to periodic solutions in the orbit space $\mathbb{R}^n/\Gamma$.

As in the case of periodic solutions, the spatial symmetry of a relative periodic solution is defined as those elements of $\Gamma$ that fix $x(t)$ pointwise. The spatiotemporal symmetry group $\Sigma$ of a relative periodic solution is the closure of the group generated by $\Delta$ and a minimal time-shift symmetry $\sigma$.

**Example 2.2** An elementary example of a relative periodic solution is a quasiperiodic solution in an $SO(2)$-equivariant dynamical system with spatial symmetry $\Delta = 1$, and minimal time-shift symmetry $\sigma$ satisfying $x(t+T) = \sigma x(t)$, where $\sigma$ is a typical element in $SO(2)$ so that the closure of the group generated by $\sigma$ is equal to $\Sigma = SO(2)$. Notice that in this situation the solution is not periodic. Examples of this kind of relative periodic solution (often referred to as modulated rotating waves) have been observed in Taylor-Couette experiments (wavy vortices) [13] and in flame patterns inside circular burners [12].

# 3 Bifurcation from isolated discrete rotating waves

We consider local bifurcations from an isolated periodic solution with a given spatiotemporal symmetry group.

To study bifurcation from an isolated discrete rotating wave $x(t)$, it turns out to be useful to construct a *first-hit-pull-back* map $f$ on a $\Delta$-invariant local section $V$ transverse to $x(t)$. The map $f$ consists of two parts: a first-hit map $g^{(1)}$ between the section $V$ and another section $\sigma V$, followed by a pull-back to $V$ by $\sigma^{-1}$ (see Figure 1). Consequently we have:

$$f = \sigma^{-1}g^{(1)}.$$

There is a simple relationship between the Poincaré return map $G : V \mapsto V$ and the diffeomorphism $f$. Namely, $G = \sigma^m f^m$. The periodic solution corresponds to a fixed point of both $f$ and $G$.

The strategy is to study bifurcations of $f$ and relate them to bifurcations of the underlying flow. One complication arising from this approach is that the structure of the map $f$ is in general not (standard) equivariance but *twisted equivariance*. In particular, by $\Delta$-equivariance of $g^{(1)}$ we have:

$$f\delta = \phi(\delta)f, \quad \text{where} \quad \phi(\delta) = \sigma^{-1}\delta\sigma.$$

Note that the definition of twisted equivariance depends on the group $\Delta$ and the group automorphism $\phi \in \mathrm{Aut}(\Delta)$.

It turns out that we can always chose $\sigma$ such that there exists a finite positive integer $k$ for which $\phi^k$ is equal to the identity automorphism on $\Delta$. This value of $k$ plays an important role in the analysis. (Note that in previous work, twisted equivariance has also been called $k$-symmetry, see for example [15]).

When $k = 1$, the bifurcation problem is that of a general $\Delta$-equivariant diffeomorphism. This situation is well known to arise when one considers local bifurcations from periodic solutions with only spatial symmetry [5], or periodic solutions with an Abelian spatiotemporal symmetry group [6, 4]. In general this situation occurs whenever periodic solutions have a minimal time-shift symmetry that commutes with all the spatial symmetries.

When $k \geq 2$, we require the methods in [16] (see also [17] for a less technical discussion). We use the integer $k$ to define the following extension of $\Delta$: $\Delta \rtimes \mathbb{Z}_{2k}$ is the (abstract) group generated by $\Delta$ and an element $\tau$ satisfying $\tau^{2k} = 1$ (identity) and $\tau^{-1}\delta\tau = \phi(\delta)$ for all $\delta \in \Delta$.

Codimension one bifurcations are governed by the following theorem.

**Theorem 3.1 ([16] Codimension one bifurcation)** *Codimension one bifurcations from a fixed point of a twisted equivariant diffeomorphism are in one-to-one "correspondence" with codimension one bifurcations from an equilibrium of a $\Delta \rtimes \mathbb{Z}_{2k}$-equivariant vector field.*

**Remark 3.2**

(a) We write "correspondence" in quotes since the proof of the theorem relies on the symmetry of formal normal forms for twisted equivariant maps [15], so aspects of the bifurcations that are beyond all orders are not necessarily preserved by the correspondence. However, a finite determinacy result of Field [9] ensures that many branches of solutions (and their stability) are preserved by the correspondence.

(b) Theorem 3.1 is a reformulation of results in [16]. In particular, nonHopf and Hopf bifurcation in [16] correspond to respectively steady state and Hopf bifurcation of $\Delta \rtimes \mathbb{Z}_{2k}$-equivariant vector field.

(c) The analysis of codimension one steady state and Hopf bifurcation for $\Delta \rtimes \mathbb{Z}_{2k}$-equivariant vector fields involves the computation of respectively absolutely irreducible representations and irreducible representations of complex type for $\Delta \rtimes \mathbb{Z}_{2k}$. In [16], it is shown how to obtain these representations from the representations of $\Delta$ using induced representation theory.

To generalize Theorem 3.1 to higher codimension bifurcation, we need to introduce a notion of resonance. Let $x_0$ denote the fixed point of $f$ corresponding to the underlying isolated discrete rotating wave. We consider the linearization $Df$ of $f$ at $x_0$, restricted to the *center subspace* $E_c$ (ie the subspace on which all eigenvalues of $Df$ have absolute value one.)

**Proposition 3.3 ([18])** *There exists a decomposition $Df = L_0 A = A L_0$, where $A$ is an equivariant linear map, $L_0$ is a twisted equivariant linear map and $L_0^{2k} = I$.*

**Definition 3.4 (Resonance)** An eigenvalue $\mu$ of an equivariant matrix $A$ is called *resonant* if $\mu = \exp(2\pi i p/q)$ where $p, q \in \mathbb{N}$ are in their lowest terms and $q \geq 3$. A bifurcation is called *nonresonant* if the decomposition $Df = L_0 A$ in Proposition 3.3 can be chosen such that $A$ has no resonant eigenvalues. Otherwise the bifurcation is called *resonant.*

**Theorem 3.5 ([18] Nonresonant bifurcation)** *In the absence of resonances, bifurcations of arbitrary codimension from a fixed point for a twisted equivariant diffeomorphism are in one-to-one "correspondence" with bifurcations of the same codimension from an equilibrium for a $\Delta \rtimes \mathbb{Z}_{2k}$-equivariant vector field.*

Let $\mu_i = \exp(2\pi i p_i/q_i)$ be the resonant eigenvalues of $A$, and define $\ell = \mathrm{lcm}(\{q_i\}) \geq 3$. In the nonresonant case we define $\ell = 1$. It turns out that the decomposition $Df = L_0 A$ can always be chosen such that $\gcd(2k, \ell) = 1$. We define the group $\Delta \rtimes \mathbb{Z}_{2k\ell}$ as above, but now with $\tau$ having order $2k\ell$ instead of $2k$.

**Theorem 3.6 ([18] Resonant bifurcation)** *Resonant bifurcations from a fixed point for a twisted equivariant diffeomorphism "correspond" to bifurcations from an equilibrium point for a $\Delta \rtimes \mathbb{Z}_{2k\ell}$-equivariant vector field.*

**Remark 3.7** In the resonant case ($\ell > 1$), bifurcation for twisted equivariant diffeomorphisms reduces to bifurcation for an equivariant vector field, but (unlike the nonresonant case) in the process of reduction one loses the correspondence of genericity between families of diffeomorphisms and families of vector fields, since the equivariant vector fields satisfy some special conditions on the linear part. Note that this phenomenon already occurs in the context of resonant bifurcations in nonsymmetric systems, see eg [1, 2].

There is the usual distinction between *strong* and *weak* resonances [1, 2]. Roughly speaking, the resonance corresponding to an eigenvalue $\mu = e^{2\pi i p/q}$ is strong if $q$ is small and weak if $q$ is large. In general, this distinction depends on the number of nonresonant and resonant eigenvalues (taking into account algebraic and geometric multiplicities) and also depends on the desired completeness of the analysis of the dynamics.

We note that the simplest solutions that occur in Hopf bifurcation are branches of invariant two-tori. It follows from Field [9] (building upon work of Ruelle [20]) that in most cases the branching and stability of two-tori are the same for bifurcations with resonance as for nonresonant bifurcations — only resonances of a specified low order are "strong" in this context. In particular, if one focuses on the simplest aspects of the theory, it usually suffices to consider bifurcation of $\Delta \rtimes \mathbb{Z}_{2k}$-equivariant vector fields.

More delicate dynamics such as phaselocking on the invariant tori is influenced by resonances of all orders. In this context, a weak resonance is one for which the analysis is comparatively straightforward; see [1, 2] for Hopf bifurcation from periodic solutions in systems without symmetry.

In order to be able to apply Theorems 3.5–3.6 to concrete bifurcation problems, one needs to know how to interpret the results of the reduced equivariant bifurcation problem in terms of solutions for the underlying flow.

The simplest interpretation is for equilibria. Suppose the underlying discrete rotating wave has period one. Let $x^{\rm bif}$ be an equilibrium arising in a local bifurcation of the $\Delta \rtimes \mathbb{Z}_{2k\ell}$-equivariant vector field, corresponding to a periodic solution $P^{\rm bif}$. The spatiotemporal symmetry group $\Sigma^{\rm bif}$ of $P^{\rm bif}$ is intimately related to the symmetry properties of $x^{\rm bif}$ in the reduced problem. We define

$$\Delta^{\rm bif} = \{\delta \in \Delta \mid \delta x^{\rm bif} = x^{\rm bif}\}.$$

**Proposition 3.8 (Symmetry and period of bifurcating solutions)** *Let $p \geq 1$ be least such that $x^{\rm bif} = \tau^p \delta_0 x^{\rm bif}$ for some $\delta_0 \in \Delta$. Then the spatial symmetry of $P^{\rm bif}$ is equal to $\Delta^{\rm bif}$ and $\Sigma^{\rm bif}$ is generated by $\Delta^{\rm bif}$ and the minimal time-shift symmetry $\sigma^{\rm bif} = \sigma^p \delta_0$. The time-shift associated with $\sigma^{\rm bif}$ is $\frac{p}{m}$.*

*Let $m^{\rm bif} \geq 1$ be least such that $(\sigma^{\rm bif})^{m^{\rm bif}} \in \Delta^{\rm bif}$, and hence $\Sigma^{\rm bif}/\Delta^{\rm bif} \cong \mathbb{Z}_{m^{\rm bif}}$. Then the period of $P^{\rm bif}$ is close to the integer $\frac{p}{m} \cdot m^{\rm bif}$.*

For similar interpretations of periodic solutions for the $\Delta \rtimes \mathbb{Z}_{2k\ell}$-equivariant vector field, see [16, 18].

**Remark 3.9** Suppose that we have a codimension one nonHopf bifurcation (cf Theorem 3.1 and Remark 3.2(b), and [16]). Then $\ell = 1$ and we reduce to a codimension one steady state bifurcation for a $\Delta \rtimes \mathbb{Z}_{2k}$-equivariant vector field. By [16, Proposition 4.5], it is generically the case that we have either a *period preserving bifurcation* where $\frac{p}{m} \cdot m^{\rm bif} = 1$ for each periodic solution $P^{\rm bif}$ corresponding to an equilibrium $x^{\rm bif}$, or a *period doubling bifurcation* where $\frac{p}{m} \cdot m^{\rm bif} = 2$ for each of these periodic solutions.

**Example 3.10**

(i) In Example 2.1(i) we presented a discrete rotating wave with spatial symmetry $\Delta = \mathbb{D}_2$ and spatiotemporal symmetry $\Sigma = \mathbb{D}_4$. Taking any reflection in $\Sigma \setminus \Delta$ as our minimal time-shift symmetry, we find that in this case $k = 2$. Hence, the bifurcation problem reduces to a $\mathbb{D}_2 \rtimes \mathbb{Z}_{4\ell}$-equivariant bifurcation problem. For a detailed discussion of codimension one bifurcations for this example, see [16].

In Example 2.1(ii), we have $k = 1$, and the bifurcation problem reduces to a $SO(2) \times \mathbb{Z}_{2\ell}$-equivariant bifurcation problem. Again, see [16] and also [17].

(ii) (*Suppression of period doubling.*) Swift and Wiesenfeld [22] observed that periodic solutions with (only) a time-shift symmetry of order two do not have generic (codimension one) bifurcations of period doubling type. To illustrate our approach, we will discuss this result from our viewpoint.

The bifurcation problem for a discrete rotating wave with period one and a single time-shift symmetry of order two reduces to a $\mathbb{Z}_2$-equivariant bifurcation problem (because $k = 1$, and not because $\Sigma \cong \mathbb{Z}_2$). Here the generator $\tau$ of $\mathbb{Z}_2$ corresponds to the minimal time-shift symmetry $\sigma$ combined with a half period time-shift. For any bifurcating fixed point $x^{\mathrm{bif}}$, we now observe that it is fixed with respect to $\tau^2$ (because $\tau^2 = 1$). This implies that the solution has a pure time-shift symmetry that takes the form of a unit time-shift (two times a half). Hence the corresponding solutions must have period (close to) one, and so period doubling cannot occur. In fact, generically we have a $\mathbb{Z}_2$ symmetry breaking pitchfork bifurcation producing fixed points with no symmetry. The bifurcating points correspond to periodic solutions with approximately the same period as the original solution, but without the $\mathbb{Z}_2$ time-shift symmetry.

Note that when a periodic solution has no symmetry, the bifurcation problem generically (codimension one) also reduces to a $\mathbb{Z}_2$-equivariant bifurcation problem. However then $\tau$ represents a pure time-shift symmetry consisting of a unit time-shift. The generic $\mathbb{Z}_2$ symmetry breaking pitchfork bifurcation produces new fixed points with no symmetry, that are thus not invariant with respect to $\tau$ but only with respect to $\tau^2$. Hence, they represent periodic solutions with period (close to) two.

# 4 Local bifurcations from relative periodic solutions

In the case of bifurcation from relative periodic solutions, the strategy is to reduce to a problem of bifurcation from isolated discrete rotating waves.

For relative periodic solutions, it is more complicated than in the discrete rotating wave case to set up a surface of section and define a return map. Following Sandstede, Scheel and Wulff [21], however, in analogy to the approach towards bifurcation from relative equilibria [14, 7], it is possible to find convenient coordinates in which to describe the differential equation in an open neighborhood of a relative periodic solution $\mathcal{P}$. These equations take the form of a *skew-product* between dynamics in a neighborhood of an isolated discrete rotating wave and dynamics on the group $\Gamma$. In particular, the dynamics near the discrete rotating wave is independent of the dynamics on $\Gamma$, but the dynamics on $\Gamma$ is driven by the dynamics near the discrete rotating wave. The dynamics on $\Gamma$ is usually referred to as *drift* (along the group orbit). We will not give full details on these equations, but refer to [23] instead. Certain aspects of our results are presented in the following theorem.

Define $\Delta \rtimes \mathbb{Z}_k$ in the usual way, now with $\tau^k = 1$.

**Theorem 4.1** *The dynamics in a neighborhood of $\mathcal{P}$ is equivalent, modulo drifts along group orbits, to the dynamics in the neighborhood of an isolated discrete rotating wave $y(t)$ with spatial symmetry $\Delta$ and spatiotemporal symmetry $\Delta \rtimes \mathbb{Z}_k$.*

*More precisely, there is a $\Gamma$-invariant neighborhood $U$ of $\mathcal{P}$ and a $\Delta \rtimes \mathbb{Z}_k$-invariant neighborhood $W$ of $y(t)$ such that the dynamics on the orbit spaces $U/\Gamma$ and $W/(\Delta \rtimes \mathbb{Z}_k)$ are topologically conjugate.*

*Moreover, each symmetry $\tau^j\delta \in \Delta \rtimes \mathbb{Z}_k$, $\delta \in \Delta$, acting on $W$ corresponds to a symmetry of the form $\gamma\sigma^j\delta \in \Gamma$ acting on $U$, where $\gamma \in \Gamma$ is near identity.*

Theorem 4.1 reduces bifurcation from a relative periodic solution to bifurcation from an isolated discrete rotating wave. In particular, there is a one-to-one correspondence between (group orbits of) periodic solutions lying close to $y(t)$, and relative periodic solutions lying close to $\mathcal{P}$.

Combining the results of Theorems 3.5–3.6 with Theorem 4.1, we conclude that bifurcation from a relative periodic solution with spatial symmetry $\Delta$ and spatiotemporal symmetry generator $\sigma$ reduces to bifurcation from an equilibrium of a $\Delta \rtimes \mathbb{Z}_{2k\ell}$-equivariant vector field.

**Remark 4.2** Note that in the general problem of bifurcation from discrete rotating waves (as treated in [16, 18]) we have either an orientable or nonorientable local flow around the periodic solution. However, in [23] we always obtain a trivial (orientable) bundle structure. Alternatively, following Theorem 4.1, we may directly reduce to a general discrete rotating wave without orientability conditions on the local bundle structure.

**Remark 4.3** Theorem 4.1 represents only part of the theory given in Wulff *et al.* [23]. For example, Theorem 4.1 does not provide a means of computing the element $\gamma$ in the spatiotemporal symmetry $\gamma\sigma^p\delta$. We note that for a bifurcating relative periodic solution with spatial symmetry $\Delta^{\mathrm{bif}}$, $\gamma$ is a general near identity element in $Z(\Delta^{\mathrm{bif}})$, and hence the results of Krupa [14] and Field [8] can be used to determine the expected drift on the bifurcating relative periodic solutions.

In addition, the implications of such drifts in phase space for phenomena viewed in physical space have been studied recently [10] in the context of Hopf bifurcation from relative equilibria. The corresponding analysis for Hopf bifurcation from relative periodic solutions requires the full strength of the results in [23].

More generally, the results in [23] hold for many noncompact Lie groups $\Gamma$, including the Euclidean group. We note that the computation of the slow drift $\gamma$ is particularly important in this context, since the value of $\gamma$ determines whether the drift is compact or unbounded. (The genericity results of [14, 8] generalize to the noncompact group setting, see [3] and [23, Section 5(b)], but do not predict the *actual* value of $\gamma$.) Again, $\gamma$ may be determined using the results in [23].

# 5 Extensions in the presence of additional structure

The theory developed in [16, 23, 18] provides a systematic approach towards the study of bifurcation of (relative) periodic solutions in equivariant dynamical systems, including mode interactions.

However, dynamical systems may have additional structure, apart from equivariance. If so, this structure needs to be taken into account.

We here mention two additional structures which recently have attracted much attention: time-reversal symmetry (in general, reversible equivariant systems), and symplectic structure (reversible equivariant Hamiltonian systems). Lamb and Wulff [19] show that bifurcation from reversible relative periodic solutions reduces to bifurcation from isolated reversible discrete rotating waves. This in turn reduces [18] to bifurcation from equilibria of reversible equivariant vector fields. The analogous theory for bifurcation from relative periodic solutions in reversible equivariant Hamiltonian systems is treated in Wulff, Lamb and Roberts [24].

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