Statistical Properties and Decay of Correlations for Interval Maps with Critical Points and Singularities.

Stefano Luzzatto [*]   Ian Melbourne [†]

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Abstract

We consider a class of piecewise smooth one-dimensional maps with critical points and singularities (possibly with infinite derivative). Under mild summability conditions on the growth of the derivative on critical orbits, we prove the central limit theorem and a vector-valued almost sure invariance principle. We also obtain results on decay of correlations and large deviations.

1 Introduction and statement of results

It is well known that deterministic dynamical systems can exhibit "chaotic" and "random-like" behaviour which in many cases gives rise to statistical properties analogous to those which are classical in the probabilistic setting of sequences of random variables. The purpose of this paper is to establish some statistical properties for a large class of deterministic interval maps (studied by Araújo, Luzzatto and Viana [2]) which are allowed to have critical points of arbitrary order, discontinuities and even unbounded derivative.

The method in [2] yields the existence of an absolutely continuous invariant probability measure via the standard approach of *inducing*. However the induced map does not have many desirable properties such as uniformly bounded distortion, long branches, or a Markov partition. This accounts for the generality of the method, but has the drawback (at least temporarily) that standard statistical properties such as the central limit theorem (CLT) are not an immediate consequence. In contrast, Díaz-Ordaz *et al.* [5] establish the existence of a Gibbs-Markov induced map (Markov

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[*]Mathematics Section, Abdus Salam International Centre for Theoretical Physics, Strada Costiera 11, 34151 Trieste, Italy. `luzzatto@ictp.it`

[†]Department of Mathematics, University of Surrey, Guildford, Surrey GU2 7XH, UK. Current Address: Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK. `i.melbourne@warwick.ac.uk`

map with uniformly bounded distortion and long branches) for a more restricted class of maps. The CLT is then guaranteed provided that the associated return time function lies in L^2 (in [5] the return time function has exponential tails and hence lies in L^p for all $p < \infty$). The vector-valued almost sure invariance principle (ASIP) is also immediate in their setting by [9, 14].

The aim of this paper is to establish statistical properties in the more general setting of [2]. Under very mild strengthenings of the hypotheses in [2] (primarily to guarantee sufficient integrability for the return time function), we prove the CLT and vector-valued ASIP. In addition, we obtain results on decay of correlations and large deviations.

In §1.1 we give the definition of the class of maps under consideration. In §§1.2 and 1.3 we state precisely our main results on statistical limit laws and decay of correlations respectively. In §1.4 we give an overview of the strategy for the proofs.

1.1 Interval maps with critical points and singularities

Let $I = [0, 1]$ and suppose $f : I \to I$ is a piecewise smooth map in the sense that I is a union of finitely many intervals such that f is C^2 and monotone on the interior of each interval and extends continuously to the boundary of each interval (Figure 1). To define our specific assumptions on f, we introduce some notation. Let $\mathcal{C}$ denote
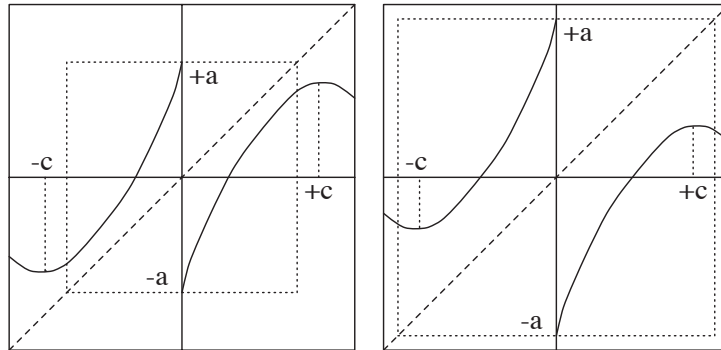

Figure 1: Interval maps with critical points and singularities

the set of all "one-sided" critical/singular points c^+ and c^- ($\mathcal{C}$ is assumed to be finite) and define corresponding one-sided neighbourhoods

$$\Delta(c^+, \delta) = (c^+, c^+ + \delta) \quad \text{and} \quad \Delta(c^-, \delta) = (c^- - \delta, c^-),$$

for each $\delta > 0$. For simplicity, from now on we use c to represent the generic element of $\mathcal{C}$ and write Δ for $\cup_{c \in \mathcal{C}} \Delta(c, \delta)$.

We can now state our assumptions.

(A1) Nondegenerate critical/singular set. *Each $c \in \mathcal{C}$ has a well-defined (one-sided) critical order $\ell = \ell(c) > 0$ in the sense that*

$$|f(x) - f(c)| \approx d(x,c)^{\ell}, \quad |f'(x)| \approx d(x,c)^{\ell-1}, \quad |f''(x)| \approx d(x,c)^{\ell-2} \quad (1.1)$$

for all x in some $\Delta(c,\delta)$. We suppose $\ell(c) \neq 1$ for all $c \in \mathcal{C}$.

Note that we say that $f \approx g$ if the ratio f/g is bounded above and below uniformly in the stated domain. The case $\ell(c) < 1$ corresponds to *singular* points and the case $\ell(c) > 1$ to *critical* points. The case $\ell(c) = 1$ is excluded since this would be a degenerate case which is not hard to deal with but would require having to introduce special notation and special arguments, whereas the other cases can all be dealt with in a unified formalism.

(A2) Uniform expansion away from the critical/singular set. *There exists a constant $\kappa > 0$, independent of δ, such that for every point x and every integer $n \geq 1$ such that $d(f^j x, \mathcal{C}) > \delta$ for all $0 \leq j \leq n-1$ and $d(f^n x, \mathcal{C}) \leq \delta$ we have*

$$|(f^n)'(x)| \geq \kappa.$$

Further, for every $\delta > 0$ there exist constants $c(\delta) > 0$ and $\lambda(\delta) > 0$ such that

$$|(f^n)'(x)| \geq c(\delta)e^{\lambda(\delta)n}$$

for every x and $n \geq 1$ such that $d(f^j x, \mathcal{C}) > \delta$ for all $0 \leq j \leq n-1$.

These conditions are quite natural and are generally satisfied for smooth maps. For example, the first one is satisfied if f is C^3, has negative Schwarzian derivative and satisfies the property that the the derivative along all critical orbits tends to infinity [3, Theorem 1.3]. The second is satisfied if f is C^2 and all periodic points are repelling [10].

The next two conditions generalise condition **(A3) Summability condition along critical orbits** from [2]. Write

$$D_n(c) := |(f^n)'(fc)|, \qquad E_n(c) := D_{n-1}(c)^{1/(2\ell(c)-1)}, \qquad d_n(c) := d(f^n c, \mathcal{C}).$$

These quantities are abbreviated to D_n, E_n, d_n in future. Fix $p \geq 0$. *For every critical point c with $\ell = \ell(c) > 1$ we have*

(A3$_p$) $\sum_{n=1}^{\infty} n^p d_n^{-1} \log d_n^{-1} E_n^{-1} < \infty$.

(A4$_p$) $\sum_{n=1}^{\infty} n^p E_n^{-1} < \infty$.

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We do not impose conditions on the orbit of the singular points with $\ell(c) < 1$.

Notice that $(A3_p)$ implies $(A4_{p'})$ if $p \geq p'$ but not if $p < p'$. The two conditions play different roles in the arguments and we will sometimes assume they hold for distinct values p and p'. Condition $(A3_p)$ plays off the derivative against the recurrence in such a way as to optimize to some extent the class of maps to which it applies. As mentioned in [2] we cannot expect to obtain certain dynamical properties in this setting using a condition which only takes into account the growth of the derivative. Note that the conditions are satisfied for any $p \geq 0$ if the derivative is growing exponentially fast and the recurrence is not faster than exponential in the sense that there is a constant $C > 0$ such that $D_n \geq Ce^{\lambda n}$ and $d_n \geq Ce^{-\alpha n}$ with $\alpha < \lambda/(2\ell - 1)$.

Theorem 1.1 (Araújo *et al.* [2]) *Suppose that $f : I \to I$ satisfies assumptions (A1), (A2), $(A3_0)$ and $(A4_1)$. Then it admits an ergodic, absolutely continuous, invariant probability measure μ.*

The proof of Theorem 1.1 relies on the construction of a suitable partition α of the interval I (mod 0) into a finite or countable collection of subintervals and a return time function $\tau : I \to \mathbb{N}$ which is constant on elements of the partition α. Define the *induced map* $F : I \to I$ given by $F|_\alpha = f^{\tau(\alpha)}$. Araújo *et al.* construct the induced map so that (i) $F : I \to I$ is a Rychlik map [19] and (ii) τ is Lebesgue integrable. It follows from Rychlik [19] that there is an absolutely continuous invariant probability measure for the induced map F. Then Theorem 1.1 follows by standard arguments from the integrability of the return time τ.

We note that $(A3_0)$ and $(A4_0)$ are used to show that F is Rychlik, and $(A4_1)$ implies that τ is integrable. Moreover, $(A4_p)$ implies that $\tau \in L^p$ (see Corollary 2.2(a) below). It is natural to explore statistical limit laws such as the central limit theorem when $\tau \in L^2$.

1.2 Statement of results: Statistical limit laws

The first main result of this paper is a vector-valued almost sure invariance principle. For $d \geq 1$, we let $\phi : I \to \mathbb{R}^d$ be an observable lying in the space BV of functions of bounded variation. Suppose that $\int_I \phi \, d\mu = 0$, and define $\phi_n = \sum_{i=0}^{n-1} \phi \circ f^i$.

Definition 1.2 The sequence $\{\phi_n\}$ satisfies the *d-dimensional almost sure invariance principle (ASIP)* if there exists a $\lambda < \frac{1}{2}$ and an abstract probability space supporting a sequence of random variables $\{\phi_n^*\}$ and a d-dimensional Brownian motion $W(t)$ such that:

1. $\{\phi_n\}_{n \geq 1} =_d \{\phi_n^*\}_{n \geq 1}$;

2. $\phi_n^* = W(n) + \mathcal{O}(n^\lambda)$ as $n \to \infty$ a.e.

(The ASIP is *nondegenerate* if the Brownian motion $W(t)$ has a nonsingular covariance matrix.)

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Theorem 1.3 *Suppose that $f : I \to I$ satisfies assumptions (A1), (A2), (A3$_\epsilon$) and (A4$_p$) for some $\epsilon > 0$, $p > 2$. Then the vector-valued ASIP holds for mean zero BV observables $\phi : I \to \mathbb{R}^d$ (for any $\lambda > p/(4p-4)$).*

The ASIP implies a wide range of statistical properties such as the central limit theorem and law of the iterated logarithm and their functional versions (see for example [14]). Many of these results require an L^p condition, $p > 2$, but it is natural to ask whether the central limit theorem (CLT) and its functional version (FCLT) hold if the return time lies in L^2. We prove such a result under an additional assumption on the density of the invariant measure $\hat{\mu}$ for the induced Rychlik map F. Let m denote Lebesgue measure and write $d\hat{\mu} = h \, dm$. (Since the vector-valued (F)CLT is no more difficult than the scalar case, we suppose for simplicity that $d = 1$.)

Theorem 1.4 *Suppose that $f : I \to I$ satisfies assumptions (A1), (A2), (A3$_1$) and (A4$_2$). Suppose moreover that $h^{-1} \in L^1(m)$. Then the CLT holds for mean zero BV observables $\phi : I \to \mathbb{R}$. That is, there exists $\sigma^2 \geq 0$ such that*

$$\lim_{n=\infty} \mu(n^{-\frac{1}{2}}\phi_n < c) = P(G < c), \ \ for \ all \ c \in \mathbb{R},$$

where G is a normal random variable with mean zero and variance σ^2.

Moreover, the FCLT holds: define $W_n(t) = n^{-\frac{1}{2}}\phi_{nt}$ for $t = j/n$, $j = 0, 1, 2, \ldots, n$ and linearly interpolate to form a random element $W_n \in C([0,1])$ (the space of continuous functions on $[0,1]$ with the sup-norm). Let $W(t)$ be a Brownian motion with mean zero and variance parameter σ^2. Then W_n converges weakly to W in $C([0,1])$.

It is immediate from Rychlik [19] that the density h lies in L^∞ and even in BV. Unfortunately, it is not known how to specify a condition directly on the map f to ensure that $h^{-1} \in L^1(m)$ in the general setting of this paper, though it may be possible to verify this assumption in specific situations. In the Markov context, there are classical assumptions on the branches of the induced map such as "finite range structure" which are used to ensure this. However, the flexibility of the approach of Araújo *et al.* [2] and in the current paper is achieved by avoiding such assumptions. Remarkably, the only result that suffers as a consequence is Theorem 1.4.

1.3   Statement of results: Decay of correlations

Our second main set of results concerns decay of correlations. As a consequence we prove a result about large deviations.

In general $f : I \to I$ need not even be mixing. However, there is a spectral decomposition into basic sets that are ergodic and moreover mixing up to a finite cycle. We restrict to such a basic set (which we relabel as I) and assume that it is mixing. Given observables $v, w : I \to \mathbb{R}$, we define the correlation function $\rho_{v,w}(n) = \int_I v \, w \circ f^n \, d\mu - \int_I v \, d\mu \int_I w \, d\mu$.

First we state a result about exponential decay of correlations.

Theorem 1.5 *Suppose that $f : I \to I$ is mixing and satisfies assumptions (A1), (A2), and that there exist constants $c_0, C_0 \geq 1$ such that $d_n E_n \geq C_0 e^{c_0 n}$ for all $n \geq 1$. Then there exists $c > 0$, $C > 0$ such that*

$$|\rho_{v,w}(n)| \leq C\|v\| |w|_\infty e^{-cn},$$

for all $v \in \mathrm{BV}$, $w \in L^\infty$, $n \geq 1$.

Remark 1.6 Note that Theorem 1.5 covers completely the situation discussed before the statement of Theorem 1.1, where the derivative grows exponentially fast, and the recurrence is subexponential or exponential at a sufficiently slow rate.

Theorem 1.5 seems to subsume all previous results in the literature about exponential decay of correlations for interval maps with critical points and singularities. In particular, it extends the afore-mentioned results of Díaz-Ordaz *et al.* [5] and also results of Young [21] who used related methods (inducing to a map satisfying Rychlik's conditions) for certain quadratic maps.

Next, we consider polynomial decay of correlations.

Theorem 1.7 *Suppose that $f : I \to I$ is mixing and satisfies assumptions (A1), (A2) and (A3$_p$) for some $p > 1$. Then for any $q > 0$, there exists $\delta > 0$, $C > 0$ such that*

$$|\rho_{v,w}(n)| \leq C\|v\| |w|_\infty \Big\{ \sum_{j > \delta n} \mu(\tau > j) + n\mu(\tau > \delta n) + n^{-q} \Big\},$$

for all $v \in \mathrm{BV}$, $w \in L^\infty$, $n \geq 1$.

Remark 1.8 (1) Under the hypotheses of Theorem 1.7, if $\mu(\tau > n) = \mathcal{O}(n^{-(\beta+1)})$ for some $\beta > 0$, then taking $q = \beta$ we obtain the optimal decay rate $\rho_{v,w}(n) = \mathcal{O}(1/n^\beta)$. A surprising aspect, inherited from [15], is that we obtain optimal results with minimal assumptions on growth rates of derivatives and recurrence rates (since any $p > 1$ suffices regardless of the size of β).
(2) In particular, if the hypotheses of Theorem 1.7 hold, and $\mu(\tau > n)$ decays superpolynomially (faster than any polynomial rate), then we obtain superpolynomial decay of correlations. Hence, if f is mixing and satisfies (A1), (A2), (A3$_p$) for some $p > 1$, and (A4$_p$) for all p, then we obtain superpolynomial decay of correlations (since (A4$_p$) implies the required assumption on $\mu(\tau > n)$).
(3) In the absence of singularities, i.e. for smooth maps, stronger versions of these results are known. We mention the remarkable result by Rivera-Letelier & Shen [18] who prove superpolynomial decay of correlations for multimodal maps under the assumption that $D_n \to \infty$ for all critical points (no growth rate required). An interesting open problem is whether this result can be generalized to interval maps with critical points and singularities.

Corollary 1.9 *Suppose that $f : I \to I$ is mixing and satisfies assumptions (A1), (A2) and (A3$_p$) for some $p > 1$. Suppose further that $\mu(\tau > n) = O(n^{-(\beta+1)})$ for some $\beta > 0$. Then we obtain the following (optimal) large deviation estimate: For any $v \in \mathrm{BV}$ with $\int_X v \, d\mu = 0$ and any $\epsilon > 0$, there exists a constant $C > 0$ such that $\mu(|\sum_{j=0}^{n-1} v \circ f^j| \geq \epsilon n) \leq C n^{-\beta}$ for all $n \geq 1$.*

Proof This follows immediately from Theorem 1.7 by [11, Theorem 1.2] which slightly improves a result of [13]. ∎

1.4 Strategy of the proofs

We end this introduction with a few remarks about the strategy of the proofs. Given $\phi : [0,1] \to \mathbb{R}^d$, we define the *induced observable* $\Phi : [0,1] \to \mathbb{R}^d$ by setting $\Phi(y) = \sum_{j=0}^{\tau(y)-1} \phi(f^j y)$. If $\int_I \phi \, d\mu = 0$, then $\int_I \Phi \, d\hat{\mu} = 0$. The main part of the proof of Theorems 1.3 and 1.4 is to establish the corresponding statistical limit laws for the induced observable Φ under the induced dynamical system F. The limit laws for ϕ follow in a by now standard way [8, 16, 17].

At the level of the induced map, it would follow easily from [19] that the (F)CLT holds for observables in BV. Extra work is required since the induced observable Φ is only piecewise BV (on elements of the partition α with norm that grows with τ). Our method closely follows the approach of Melbourne & Nicol [12] for piecewise Hölder observables and Gibbs-Markov maps. A scalar ASIP could also be proved along the same lines, and a vector-valued ASIP following [14], but these would again require a condition on h^{-1}. Instead we apply a recent result of Gouëzel [9] which sidesteps this issue.

For decay of correlations, the strategy is somewhat different. The induced map F still plays a major role but we do not consider induced observables. Instead we use the method of operator renewal sequences [20, 7] to relate decay rates of the transfer operator for f to information about the transfer operator for F. However certain key estimates of first return operators in [20, 7] are unavailable since we are not assuming long branches for F. A modification of the method due to Melbourne & Terhesiu [15] avoids sharp estimates for these operators and still yields optimal results.

In Section 2, we recast Theorems 1.3 and 1.4 in terms of conditions on the induced map F. In Section 3, we prove limit theorems for the induced map. In Section 4, we show how the limit laws pass to the original map. In Section 5, we prove Theorems 1.5 and 1.7.

2 Background from [2]

We recall some aspects of the construction from [2] that are needed for our results, in particular the definition of the partition α and the associated return time function

τ. First δ is chosen small enough (prescribing the neighbourhood Δ of the critical/singular set $\mathcal{C}$). To each $x \in \Delta$ is associated an integer $b(x) \geq 1$, called the *binding period* of x, with certain properties listed in Proposition 2.1 below. Next, $q_0 \geq 1$ is fixed sufficiently large. For $x \in I$, define $\ell_0(x) = \min\{j \geq 0 : f^j x \in \Delta\}$. The return time function $\tau : I \to \mathbb{R}$ is then given by $\tau(x) = q_0$ if $\ell_0(x) \geq q_0$ and $\tau(x) = \ell_0(x) + b(f^{\ell_0(x)}(x))$ otherwise. In the latter case, we say that x has binding period $b(x) = b(f^{\ell_0(x)}(x))$. The partition α has the property that the binding period b and return time function τ are constant on partition elements and that the induced map $F = f^\tau$ is smooth and monotone on partition elements. Given $b \geq 1$, we define $\alpha(b)$ to consist of those partition elements a containing points with binding period b and we let $\alpha(0)$ consist of the remaining partition elements.

Proposition 2.1 (Araújo *et al.* [2]) *There exist constants $C, M > 0$ such that*

(a) $\#\alpha(b) \leq M$ *for all* $b \geq 0$.

(b) *If* $a \in \alpha(b)$, *then* $\sup_a |1/F'| \leq C E_b^{-1}$.

(c) *If* $a \in \alpha(b)$, *then* $\operatorname{var}_a |1/F'| \leq C(1 + \log d_{b-1}^{-1}) d_{b-1}^{-1} E_{b-1}^{-1}$.

Proof Part (a) follows from the construction in [2] since f has only finitely many branches. Parts (b) and (c) are [2, Equation (19)] and [2, Equation (20)] respectively. ∎

We use Proposition 2.1 to convert conditions (A1)–(A4) into conditions on the induced map F. For $p \geq 0$, introduce the conditions

(F1$_p$) $\sum_{a \in \alpha} \sup_a(1/|F'|) \tau(a)^p < \infty$.

(F2$_p$) $\sum_{a \in \alpha} \operatorname{var}_a(1/|F'|) \tau(a)^p < \infty$.

Here $\sup_a(1/|F'|) = |1_a(1/F')|_\infty$, and $\operatorname{var}_a(1/F')$ denotes the variation of the function $1/F'$ on the interval $a \in \alpha$.

Corollary 2.2 (a) *If* $\sum_{n=1}^{\infty} n^p E_n^{-1} < \infty$, *then* (F1$_p$) *holds and* $\tau \in L^p(m)$.

(b) *If* $\sum_{n=1}^{\infty} n^p d_n^{-1} \log d_n^{-1} E_n^{-1} < \infty$, *then* (F2$_p$) *holds.*

Proof It follows from the definition of τ and Proposition 2.1(a,b) that

$$\sum_{a \in \alpha} \sup_a(1/|F'|) \tau(a)^p = \sum_{b=0}^{\infty} \sum_{a \in \alpha(b)} \sup_a(1/|F'|) \tau(a)^p \leq CM \sum_{b=0}^{\infty} E_b^{-1}(b + q_0)^p < \infty.$$

Also, by the mean value theorem $m(a) \leq \sup_a(1/|F'|) m(Fa) \leq \sup_a(1/|F'|)$, so it follows similarly that

$$\int |\tau|^p \, dm = \sum_{a \in \alpha} m(a) \tau(a)^p < \infty,$$

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proving part (a). In the same manner, part (b) follows from Proposition 2.1(a,c). ∎

Conditions (F1$_0$) and (F2$_0$) imply that F satisfies the *Rychlik conditions* [19] and thus admits an F-invariant absolutely continuous probability measure $\hat{\mu}$ with a bounded density $h = d\hat{\mu}/dm \in \mathrm{BV}$. The condition $\tau \in L^1(m)$ then implies, by standard arguments, that there exists a corresponding f-invariant absolutely continuous probability measure μ.

The crucial ingredient in the proof of Theorems 1.3 and 1.4 turns out to be existence of an induced map satisfying (F1$_1$) and (F2$_\epsilon$) and a condition on the integrability of the return time function τ. Assumptions (A1) and (A2) are important in the construction of such an induced map, but become redundant once such a map is assumed. By Corollary 2.2, Conditions (A3$_p$) and (A4$_p$) imply (F1$_1$) and (F2$_p$) and thus we reduce Theorems 1.3 and 1.4 to the following.

Theorem 2.3 *Suppose that $f : I \to I$ admits an induced map satisfying (F1$_1$) and (F2$_\epsilon$) for some $\epsilon > 0$ and that $\tau \in L^p$ for some $p > 2$. Let $\phi : I \to \mathbb{R}^d$ be an observable in* BV *with $\int_I \phi \, d\mu = 0$. Then ϕ satisfies the vector-valued ASIP for any $\lambda > p/(4p - 4)$.*

Theorem 2.4 *Suppose that $f : I \to I$ admits an induced map satisfying (F1$_1$) and (F2$_\epsilon$) for some $\epsilon > 0$ and that $\tau \in L^2$. Suppose further that $h^{-1} \in L^1(m)$. Let $\phi : I \to \mathbb{R}^d$ be an observable in* BV *with $\int_I \phi \, d\mu = 0$. Then ϕ satisfies the CLT and functional CLT.*

3 Limit theorems for the induced map

For generality and coherency with notation to be used below, we let $Y = I$ and let $F : Y \to Y$ be a piecewise uniformly expanding map with respect to a partition α of intervals and μ_Y an ergodic, F-invariant, absolutely continuous probability measure with density $h = d\mu_Y/dm$. Let $\tau : \alpha \to \mathbb{Z}^+$ be a "weight function" (which will of course be the return time function in the application of these results to be given below, but for the moment we are not assuming that F is an induced map) and define

$$\|\Phi\|_{\mathrm{BV}_\tau} := \sup_{a \in \alpha} \left\{ \frac{\sup_a(\Phi) + \mathrm{var}_a(\Phi)}{\tau(a)} \right\}.$$

Then we define the space of functions $\Phi : Y \to \mathbb{R}^d$ of *bounded weighted variation* as

$$\mathrm{BV}_\tau := \{\Phi : \|\Phi\|_{\mathrm{BV}_\tau} < \infty\}.$$

In particular, if $\tau \equiv 1$, then $\| \ \|_{\mathrm{BV}_\tau} = \| \ \|_{\mathrm{BV}}$ and $\mathrm{BV}_\tau = \mathrm{BV}$.

Proposition 3.1 *Suppose that (F1$_1$) and (F2$_\epsilon$) hold for some $\epsilon > 0$. Let $p > 2$. Then the vector-valued ASIP holds for the induced map $F : Y \to Y$ for mean zero observables $\Phi \in \mathrm{BV}_\tau \cap L^p(Y, \mu_Y)$.*

Proposition 3.2 *Suppose (F1$_1$) and (F2$_1$) hold and that $h \in \mathrm{BV}$, $h^{-1} \in L^1(m)$. Then the CLT and functional CLT hold for the induced map $F : Y \to Y$ for mean zero observables $\Phi \in \mathrm{BV}_\tau \cap L^2(Y, \mu_Y)$.*

We prove these two Propositions in the following two subsections, then in the next section we show how they imply the Theorems above. We let L and P denote the transfer operators for F with respect to Lebesgue measure m and the invariant measure μ_Y respectively:

$$\int_Y Lv \cdot w \, dm = \int_Y v \cdot w \circ F \, dm \quad \text{for all } v \in L^1(m), w \in L^\infty(m),$$

and

$$\int_Y Pv \cdot w \, d\mu_Y = \int_Y v \cdot w \circ F \, d\mu_Y \quad \text{for all } v \in L^1(\mu_Y), w \in L^\infty(\mu_Y).$$

We have $Lh = h$, $P1 = 1$ and $P = h^{-1}Lh$. Also, $(Lv)(x) = \sum_{Fy=x} v(y)/|F'(y)|$.

3.1 ASIP for the induced map

In this subsection, we prove Proposition 3.1. We assume throughout assumptions (F1$_1$) and (F2$_\epsilon$) of the Proposition for some $\epsilon > 0$. Our goal is to apply a general result of Gouëzel [9, Theorem 2.1]. Given $\Phi \in \mathrm{BV}_\tau$, $t \in \mathbb{R}^d$ and $v \in \mathrm{BV}$, let

$$L_t v := L(e^{it\Phi} v) \quad \text{and} \quad P_t v := P(e^{it\Phi} v).$$

Here $t\Phi$ is shorthand for $t \cdot \Phi$. Note that $P_t = h^{-1} L_t h$. We also define the Banach space $\mathcal{B} = h^{-1} \mathrm{BV}$ with norm $\|v\|_{\mathcal{B}} = \|hv\|_{\mathrm{BV}}$.

Lemma 3.3 *There exist constants $C > 0$, $\gamma \in (0, 1)$ such that*

(a) $\|L^n v - h \int v \, dm\|_{\mathrm{BV}} \leq C\gamma^n \|v\|_{\mathrm{BV}}$ *for all $n \geq 1$, $v \in \mathrm{BV}$.*

(b) $\|L_t - L_0\|_{\mathrm{BV}} \to 0$ *as $t \to 0$.*

Proof Part (a) is in [19] (using (F1$_0$) and (F2$_0$)). Next, write

$$|(L_t - L_0)v(x)| = |L(\{e^{it\Phi} - 1\}v)(x)| \leq \sum_{Fy=x} (1/|F'(y)|)|e^{it\Phi(y)} - 1||v(y)|$$

$$\leq \sum_{a \in \alpha} \sup_a |(1/F')(e^{it\Phi} - 1)v|.$$

Since $|e^{ix} - 1| = \mathcal{O}(x^\epsilon)$ for any $\epsilon \in [0, 1]$,

$$|(L_t - L_0)v|_\infty \leq \sum_a \sup_a(1/|F'|)|t|^\epsilon \sup_a |\Phi|^\epsilon \sup_a(v)$$

$$\leq |t|^\epsilon(\|\Phi\|_{\mathrm{BV}_\tau})^\epsilon |v|_\infty \sum_{a \in \alpha} \sup_a(1/|F'|)\tau(a)^\epsilon.$$

Further,

$$\operatorname{var}((L_t - L_0)v) \le \sum_{a \in \alpha} \operatorname{var}_a((1/F')(e^{it\Phi} - 1)v) + 2 \sum_{a \in \alpha} \sup_a |(1/F')(e^{it\Phi} - 1)v|.$$

The second term is estimated as above. Since $\operatorname{var}_a(e^{i\Phi} - 1) \le \operatorname{var}_a(\Phi)$,

$$\begin{aligned}
\operatorname{var}_a((1/F')(e^{it\Phi} - 1)v) &\le \operatorname{var}_a(1/F')\sup_a|(e^{it\Phi} - 1)v| \\
&\quad + \operatorname{var}_a(v)\sup_a|(1/F')(e^{it\Phi} - 1)| + \operatorname{var}_a(e^{it\Phi} - 1)\sup_a|(1/F')v| \\
&\le |t|^\epsilon \operatorname{var}_a(1/F')\sup_a|\Phi|^\epsilon|v|_\infty + |t|^\epsilon\sup_a(1/|F'|)\sup_a|\Phi|^\epsilon \operatorname{var}(v) \\
&\quad + |t| \operatorname{var}_a(\Phi)\sup_a(1/|F'|)|v|_\infty \\
&\le |t|^\epsilon\|\Phi\|_{\mathrm{BV}_\tau}\|v\|_{\mathrm{BV}}\{\operatorname{var}_a(1/F')\tau(a)^\epsilon + \sup_a(1/|F'|)\tau(a)\}.
\end{aligned}$$

By conditions (F1) and (F2$_\epsilon$), $\|L_t - L_0\|_{\mathrm{BV}} = \mathcal{O}(|t|^\epsilon)$ proving part (b). ∎

Corollary 3.4 *There exist constants $C > 0$, $\gamma \in (0, 1)$ such that*

(a) $\|P^n v - \int v \, d\mu_Y\|_\mathcal{B} \le C\gamma^n\|v\|_\mathcal{B}$ for all $n \ge 1$, $v \in \mathcal{B}$.

(b) $\|P_t - P_0\|_\mathcal{B} \to 0$ as $t \to 0$.

Proof From Lemma 3.3 we have

$$\begin{aligned}
\|P^n v - \int v \, d\mu_Y\|_\mathcal{B} &= \|h(P^n v - \int hv \, dm)\|_{\mathrm{BV}} = \|L^n(hv) - h\int(hv) \, dm)\|_{\mathrm{BV}} \\
&\le C\gamma^n\|hv\|_{\mathrm{BV}} = C\gamma^n\|v\|_\mathcal{B},
\end{aligned}$$

and

$$\begin{aligned}
\|(P_t - P_0)v\|_\mathcal{B} &= \|h(P_t - P_0)v\|_{\mathrm{BV}} = \|(L_t - L_0)(hv)\|_{\mathrm{BV}} \\
&\le \|(L_t - L_0)\|_{\mathrm{BV}}\|hv\|_{\mathrm{BV}} = \|(L_t - L_0)\|_{\mathrm{BV}}\|v\|_\mathcal{B}
\end{aligned}$$

so that $\|P_t - P_0\|_\mathcal{B} \le \|L_t - L_0\|_{\mathrm{BV}} \to 0$ as $t \to 0$. ∎

Proof of Proposition 3.1 We have verified the hypotheses (I) of Gouëzel [9, Theorem 2.1], so the result follows. ∎

Remark 3.5 From the proof it follows that condition (F1$_1$) can be replaced by the assumptions that (F1$_\epsilon$) holds for some $\epsilon > 0$ and $\sum_{a \in \alpha} \sup_a(1/|F'|)\tau(a) \operatorname{var}_a(\phi) < \infty$.

3.2 CLT for the induced map

In this subsection, we prove Proposition 3.2 following the approach in [12]. We assume throughout the assumptions of the Proposition.

Lemma 3.6 *The operator* $L : \mathrm{BV}_\tau \to \mathrm{BV}$ *is bounded.*

Proof Let $\Phi \in \mathrm{BV}_\tau$. Then, using the definition of $\| \cdot \|_{\mathrm{BV}_\tau}$, we have

$$|L\Phi(x)| \le \sum_{Fy=x} |\Phi(y)|/|F'(y)| \le \|\Phi\|_{\mathrm{BV}_\tau} \sum_{a\in\alpha} \sup_a (1/|F'|)\tau(a),$$

so $|L\Phi|_\infty \le \{\sum_{a\in\alpha} \sup_a(1/|F'|)\tau(a)\}\|\Phi\|_{\mathrm{BV}_\tau}$ and $\sum_{a\in\alpha} \sup_a(1/|F'|)\tau(a) < \infty$ by assumption (F1$_1$). Next, we note that

$$\mathrm{var}(L\Phi) \le \sum_{a\in\alpha} \mathrm{var}_a(\Phi/F') + 2\sum_{a\in\alpha} \sup_a |\Phi/F'|.$$

The second sum is estimated as above using (F1$_1$). For the first sum, we have

$$\mathrm{var}_a(\Phi/F') \le \mathrm{var}_a(\Phi)\sup_a(1/|F'|) + \sup_a|\Phi| \, \mathrm{var}_a(1/F')$$
$$\le \|\Phi\|_{\mathrm{BV}_\tau}\{\sup_a(1/|F'|) + \mathrm{var}_a(1/F')\}\tau(a).$$

Hence $\mathrm{var}(L\Phi) \le \{\sum_{a\in\alpha} \mathrm{var}_a(1/|F'|)\tau(a) + 4\sum_{a\in\alpha}\sup_a(1/|F'|)\tau(a)\}\|\Phi\|_{\mathrm{BV}_\tau}$ and $\sum_{a\in\alpha} \mathrm{var}_a(1/|F'|)\tau(a) + 4\sum_{a\in\alpha}\sup_a(1/|F'|)\tau(a) < \infty$ by (F1$_1$) and (F2$_1$). ∎

Corollary 3.7 *There exist* $C > 0$, $\gamma \in (0,1)$ *such that* $|P^n\Phi|_{L^2(Y,\mu_Y)} \le C\gamma^n|h^{-1}|_2\|h\|_{\mathrm{BV}}\|\Phi\|_{\mathrm{BV}_\tau}$ *for all* $n \ge 1$. *In particular* $\chi := \sum_{n=1}^\infty P^n\Phi \in L^2(Y,\mu_Y)$.

Proof We use the fact that $h^{-1} \in L^1(m)$, or equivalently that $h^{-1} \in L^2(Y,\mu_Y)$ (since $d\mu_Y = h\,dm$ we have $\int(h^{-1})^2 d\mu = \int(h^{-1})^2 h\,dm = \int h^{-1}\,dm$). Thus, writing $|\ |_2 = |\ |_{L^2(Y,\mu_Y)}$ and $P^n\Phi = h^{-1}hP^n\Phi$ we have

$$|P^n\Phi|_2 \le |h^{-1}|_2|hP^n\Phi|_\infty \le |h^{-1}|_2\|hP^n\Phi\|_{\mathrm{BV}} = |h^{-1}|_2\|L^{n-1}L(h\Phi)\|_{\mathrm{BV}}.$$

Since $h \in \mathrm{BV}$, it follows that $h\Phi \in \mathrm{BV}_\tau$ and, by Lemma 3.6, that $L(h\Phi) \in \mathrm{BV}$. Hence by Lemma 3.3(a),

$$|P^n\Phi|_2 \le C\gamma^{n-1}|h^{-1}|_2\|L(h\Phi)\|_{\mathrm{BV}}.$$

Finally, by Lemma 3.6, $\|L(h\Phi)\|_{\mathrm{BV}} \le C\|h\Phi\|_{\mathrm{BV}_\tau} \le C\|h\|_{\mathrm{BV}}\|\Phi\|_{\mathrm{BV}_\tau}$ which gives the result. ∎

Proof of Proposition 3.2 The proof follows from Corollary 3.7 by a standard Gordin-type argument [6]. We sketch the steps of the argument.

Write $\Phi = \hat{\Phi} + \chi \circ F - \chi$. Then it suffices to prove the CLT for $\hat{\Phi}$. Now $\hat{\Phi} \in L^2(Y,\mu_Y)$ (since $\Phi, \chi \in L^2(Y,\mu_Y)$) and it follows from the definitions that $\hat{\Phi} \in \ker P$. This means that $\{\hat{\Phi} \circ F^n\}$ defines a sequence of reverse martingale increments. Passing to the natural extension and applying the (F)CLT for ergodic L^2 martingales we obtain the required limit laws as $n \to -\infty$. Since the limit laws are distributional, this is equivalent to the result as $n \to \infty$. ∎

4 Limit theorems for the original map

We are now ready to complete the proof of Theorems 1.3 and 1.4. Starting with $F : Y \to Y$ and $\tau : Y \to \mathbb{N}$, we build a tower map $\tilde{f} : X \to X$ for which $F : Y \to Y$ is a *first* return map. Define

$$X = \{(y, \ell) \in Y \times \mathbb{Z} : 0 \le \ell \le \tau(y) - 1\}$$

and

$$\tilde{f}(y, \ell) = \begin{cases} (y, \ell + 1), & \ell \le \tau(y) - 2 \\ (Fy, 0), & \ell = \tau(y) - 1 \end{cases}.$$

Then Y is identified with the base $\{(y, 0) : y \in Y\}$ of the tower and $F = \tilde{f}^\tau$ is the first return map to Y. Let $\mu_Y = \hat{\mu}$ denote the absolutely continuous F-invariant probability measure on Y. Since τ is integrable, we can define a $\tilde{f}$-invariant probability measure $\mu_X = \mu_Y \times \nu / \int \tau \, d\mu_Y$ on X where ν denotes counting measure.

Also, we have a natural projection $\pi : X \to I$ given by $\pi(y, \ell) = f^\ell y$. This is a semiconjugacy: $\pi \circ \tilde{f} = f \circ \pi$. The pushforward measure $\mu = \pi_* \mu$ is the absolutely continuous f-invariant probability measure on I constructed in [2].

Now suppose that $\phi : X \to \mathbb{R}^d$ is a mean zero BV observable as in Theorem 2.3 or Theorem 2.4. This lifts to an observable $\tilde{\phi} = \phi \circ \pi : X \to \mathbb{R}^d$ on the tower. We can then define the induced observable $\Phi : Y \to \mathbb{R}^d$ given by

$$\Phi(y) = \sum_{\ell=0}^{\tau(y)-1} \tilde{\phi}(y, \ell) = \sum_{\ell=0}^{\tau(y)-1} \phi(f^\ell y).$$

The following abstract result shows that limit theorems for the induced map are inherited by the original map.

Proposition 4.1 *Suppose that $f : I \to I$ is ergodic and that $\phi \in L^\infty(I)$ with $\int_I \phi \, d\mu = 0$.*

(a) If $\tau \in L^p(Y, \mu_Y)$ for some $p > 2$ and Φ and $\tau - \int_Y \tau \, d\mu_Y$ satisfy the ASIP for (Y, μ_Y, F), then ϕ satisfies the ASIP for (I, μ, f).

(b) If $\tau \in L^2(Y, \mu_Y)$ and Φ and $\tau - \int_Y \tau \, d\mu_Y$ satisfy the (F)CLT for (Y, μ_Y, F), then ϕ satisfies the (F)CLT for (I, μ, f).

Proof The ASIP for (Y, μ_Y, F, Φ) lifts by [4, 16] to an ASIP for $(X, \mu_X, \tilde{f}, \tilde{\phi})$. The CLT lifts by [8, Theorem A.1] (cf. [16, Theorem 1.1]). The functional CLT lifts by [17]. It follows from the definition of π, $\tilde{\phi}$ and μ that the limit laws for $(X, \mu_X, \tilde{f}, \tilde{\phi})$ push down to the required limit laws for (I, μ, f, ϕ). ∎

Proof of Theorems 2.3 and 2.4 The assumptions of Theorem 2.3 guarantee that $\tau \in L^p(Y, m)$ for some $p > 2$. Since $h \in$ BV, it follows that $\tau \in L^p(Y, \mu_Y)$.

Since $\phi \in$ BV, it follows from the definition of Φ that $\Phi \in$ BV$_\tau \cap L^2(Y, \mu_Y)$. By Proposition 3.1, Φ and τ satisfy the ASIP for (Y, μ_Y, F). Apply Proposition 4.1(a) to obtain the ASIP for ϕ.

The argument for Theorem 2.4 is identical. ∎

5   Decay of correlations

In this section, we prove Theorems 1.5 and 1.7. By assumption $f : I \to I$ is mixing. To begin with we make the simplifying assumption that the tower map $\tilde{f} : X \to X$ constructed in Section 4 is mixing. This simplifying assumption is relaxed at the end of the section.

Let $\mathcal{B}_1(Y)$ denote the space of uniformly piecewise bounded variation observables, namely those $v : Y \to \mathbb{R}$ such that $\|v\|_{\mathcal{B}_1(Y)} = \sup_{a \in \alpha} \|1_a v\|_{\mathrm{BV}} < \infty$. Take $\mathcal{B}(Y) = h^{-1} \mathcal{B}_1(Y)$ (with norm $\|v\|_{\mathcal{B}(Y)} = \|hv\|_{\mathcal{B}_1(Y)}$).

Proposition 5.1   *(a) BV$(Y) \subset \mathcal{B}_1(Y) \subset \mathcal{B}(Y) \subset L^1(Y, \mu_Y)$.*

(b) The unit ball in $\mathcal{B}_1(Y)$ is compact in $L^1(Y, \mu_Y)$.

Proof   The first inclusion in (a) is obvious, and the second inclusion holds since $h \in$ BV(Y). Also $\mathcal{B}_1(Y) \subset L^\infty(Y)$ and $h^{-1} \in L^1(Y, \mu_Y)$ (since $\int_Y h^{-1} d\mu_Y = \int_Y 1 \, dm = 1$) so $\mathcal{B}(Y) \subset L^1(Y, \mu_Y)$ completing the proof of (a).

It is well-known that the unit ball in BV(Y) is compact in $L^1(Y, m)$, and hence in $L^1(Y, \mu_Y)$ since the density h is bounded. Let $v_n \in \mathcal{B}_1(Y)$, $n \geq 1$, with $\|v_n\|_{\mathcal{B}_1(Y)} \leq 1$. Then for each $a \in \alpha$, $1_a v_n$ lies in the unit ball in BV(Y). Passing to a subsequence, we obtain a function $v : a \to \mathbb{R}$ with $\|1_a v\|_{\mathrm{BV}(Y)} \leq 1$ such that $|1_a(v_n - v)|_1 \to 0$ for all a. Putting these together, we obtain a limit function $v \in \mathcal{B}_1(Y)$ with $\|v\|_{\mathcal{B}_1(Y)} \leq 1$ and a common subsequence n_k such that $\int_a |v_{n_k} - v| \, d\mu_Y \to 0$ as $k \to \infty$ for all $a \in \alpha$. Since $|v_{n_k} - v|_\infty \leq 2$, it follows that $\int_Y |v_{n_k} - v| \, d\mu_Y \to 0$ as $k \to \infty$. This concludes the proof of (b). ∎

Recall that P denotes the transfer operator for the induced map $F : Y \to Y$ with respect to the invariant measure μ_Y. Define operators $P_n : \mathcal{B}(Y) \to \mathcal{B}(Y)$ for $n \geq 1$, and $P(z) : \mathcal{B}(Y) \to \mathcal{B}(Y)$ for $z \in \mathbb{C}$, $|z| \leq 1$:

$$P_n v = P(1_{\{\tau = n\}} v), \qquad P(z) = \sum_{n=1}^\infty P_n z^n.$$

Proposition 5.2   *(i) The eigenvalue 1 is simple and isolated in the spectrum of $P = P(1)$.*

(ii) For $z \neq 1$, the spectrum of $P(z)$ does not contain 1.

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Proof Let L denote the transfer operator for the induced map $F : Y \to Y$ with respect to Lebesgue measure m. Define $L_n v = L(1_{\{\tau=n\}}v)$ and $L(z) = \sum_{n=1}^{\infty} L_n z^n$. We show that properties (i) and (ii) are valid for $L(z) : \mathcal{B}_1(Y) \to \mathcal{B}_1(Y)$. Since $P(z)v = h^{-1}L(z)(hv)$ and $\mathcal{B}(Y) = h^{-1}\mathcal{B}_1(Y)$, the properties are inherited for $P(z)$.

First, let $z = 1$. Rychlik [19] establishes the "basic inequality" $\|L^n v\|_{\mathrm{BV}(Y)} \leq C(|v|_1 + \gamma^n \|v\|_{\mathrm{BV}(Y)})$ for all $v \in \mathrm{BV}(Y)$, $n \geq 1$. Lemma 3.6 (with $\tau = 1$) guarantees that $L : \mathcal{B}_1(Y) \to \mathrm{BV}(Y)$ is a bounded operator, so that $\|L^n v\|_{\mathcal{B}_1(Y)} \leq C'(|v|_1 + \gamma^n \|v\|_{\mathcal{B}_1(Y)})$ for all $v \in \mathcal{B}_1(Y)$, $n \geq 1$. Combined with Proposition 5.1(b), we deduce as in [19] that $L : \mathcal{B}_1(Y) \to \mathcal{B}_1(Y)$ is quasicompact, and so has essential spectral radius strictly less than 1. By ergodicity of F, 1 is a simple eigenvalue, and property (i) follows.

Aaronson *et al.* [1, Proposition 4], extend the argument of [19] to cover the case of $z \in \mathbb{C}$, $|z| = 1$, and the case $|z| < 1$ is simpler. Hence for all $z \in \mathbb{C}$, $|z| \leq 1$, we have the basic inequality: there exist constants $C > 0$, $\gamma \in (0,1)$ such that $\|L(z)^n v\|_{\mathrm{BV}(Y)} \leq C(|v|_1 + \gamma^n \|v\|_{\mathrm{BV}})$ for all $v \in \mathrm{BV}(Y)$, $n \geq 1$. Again, the operator $L(z) : \mathcal{B}_1(Y) \to \mathrm{BV}(Y)$ is bounded (this is immediate from the proof of Lemma 3.6 since $L(z)v = L(z^\tau v)$ and τ is constant on partition elements so z^τ does not contribute to the estimates). Hence $L(z)$ is quasicompact with essential spectral radius strictly less than 1 for all z, $|z| \leq 1$.

Next, we note that $L^n(z)v = L^n(z^{\tau_n}v)$ where $\tau_n = \sum_{j=0}^{n-1} \tau \circ F^j \geq n$. By the basic inequality, it is certainly the case that $|L^n|_\infty \leq C$ for some constant C. It follows that $|L^n(z)v|_\infty = |L^n(z^{\tau_n}v)|_\infty \leq C|z^{\tau_n}|_\infty |v|_\infty \leq C|z^n||v|_\infty$. In particular, if λ is an eigenvalue then $|\lambda| \leq |z|$. Hence, condition (ii) is satisfied for all z with $|z| < 1$.

Finally, let $z = e^{i\theta}$, $0 < \theta < 1$, and suppose that $L(e^{i\theta})v = v$ for some nonzero $v \in \mathcal{B}_1(Y)$. The L^2 adjoint of $v \mapsto L(e^{i\theta})v = L(e^{i\theta\tau}v)$ is $v \mapsto e^{-i\theta}v \circ F$, and it follows easily that $e^{-i\theta}v \circ F = v$. Form the tower map $\tilde{f} : X \to X$ as in Section 4 and define $w : X \to \mathbb{R}$, $w(y,\ell) = e^{-i\ell\theta}v(y)$. Then trivially $w \circ \tilde{f} = e^{-i\theta}w$ and the constraint $e^{-i\theta}v \circ F = v$ means that $w(y,\tau(y)) = w(Fy,0)$ so that w is well-defined. Certainly w is measurable. We have constructed a nontrivial eigenfunction for $\tilde{f}$ which contradicts the assumption that $\tilde{f}$ is mixing. Hence, condition (ii) is satisfied for all $z = e^{i\theta}$, $z \neq 1$, completing the proof. ∎

Lemma 5.3 (a) *Suppose that $f : I \to I$ satisfies assumptions (A1), (A2) and (A3$_p$) for some $p \geq 1$. Then $\sum_{n=1}^{\infty} n^{p-1} \sum_{j>n} \|P_j\|_{\mathcal{B}(Y)} < \infty$.*

(b) *Suppose that $f : I \to I$ satisfies assumptions (A1) and (A2), and that there exist constants $c_0, C_0 \geq 1$ such that $d_n E_n \geq C_0 e^{c_0 n}$ for all $n \geq 1$. Then $\|P_n\|_{\mathcal{B}(Y)} = O(e^{-c_1 n})$ for some $c_1 > 0$.*

Proof We prove part (a). Part (b) is similar.

Define $L_n v = L(1_{\{\tau=n\}}v)$ regarded as an operator on $\mathcal{B}_1(Y)$ as in the proof of Proposition 5.2. We prove that $\sum_{n=1}^{\infty} n^{p-1} \sum_{j>n} \|L_j\|_{\mathcal{B}_1(Y)} < \infty$. The required result is then immediate.

As in the proof of Lemma 3.6,

$$|L_n v|_\infty \leq \sum_{a \in \alpha : \tau(a) = n} \sup_a (1/F') |v|_\infty,$$

$$\text{var}(L_n v) \leq \sum_{a \in \alpha : \tau(a) = n} \text{var}_a(v/F') + 2 \sum_{a \in \alpha : \tau(a) = n} \sup_a(v/F'),$$

so it suffices to show that

$$\sum_{n=1}^\infty n^{p-1} \sum_{a \in \alpha : \tau(a) > n} \sup_a(1/F') < \infty, \quad \sum_{n=1}^\infty n^{p-1} \sum_{a \in \alpha : \tau(a) > n} \text{var}_a(1/F') < \infty. \quad (5.1)$$

Recall that if $a \in \alpha(b)$, then $\tau(a) \in [b, q_0 + b]$. Hence if $\tau(a) > n$ then $a \in \alpha(b)$ with $b > n - q_0$, and so

$$\sum_{a \in \alpha : \tau(a) > n} \sup_a(1/F') \leq \sum_{b > n - q_0} \sum_{a \in \alpha(b)} \sup_a(1/F') \leq CM \sum_{b > n - q_0} E_b^{-1},$$

by Proposition 2.1(a,b). Note that $(A3_p)$ implies $(A4_p)$ which is equivalent to the condition that $\sum_{n=1}^\infty n^{p-1} \sum_{b > n} E_b^{-1} < \infty$ verifying the first condition in (5.1). The second condition is verified in an identical manner using Proposition 2.1(a,c) and $(A3_p)$. ∎

Proposition 5.4 *Let* $v : I \to \mathbb{R}$ *be* BV *and fix a function* $j : Y \to \mathbb{Z}^+$ *that is constant on partition elements and satisfies* $0 \leq j(y) < \tau(y)$. *Define* $\hat{v} : Y \to \mathbb{R}$, $\hat{v}(y) = v(f^{j(y)}y)$. *Then* $\|\hat{v}\|_{\mathcal{B}(Y)} \leq \|h\|_{\text{BV}} \|v\|_{\text{BV}}$.

Proof Compute that

$$\|\hat{v}\|_{\mathcal{B}(Y)} = \|h\hat{v}\|_{\mathcal{B}_1(Y)} = \sup_{a \in \alpha} \|1_a h \hat{v}\|_{\text{BV}} = \sup_{a \in \alpha} \|1_a h\, v \circ f^{j(a)}\|_{\text{BV}}$$

$$\leq \|h\|_{\text{BV}} \sup_{a \in \alpha} \|1_a\, v \circ f^{j(a)}\|_{\text{BV}} = \|h\|_{\text{BV}} \sup_{a \in \alpha} \|1_a v\|_{\text{BV}} \leq \|h\|_{\text{BV}} \|v\|_{\text{BV}},$$

as required. ∎

Proof of Theorems 1.5 and 1.7 The Banach space $\mathcal{B}(Y)$ lies in $L^1(Y, \mu_Y)$ by Proposition 5.1(a) and contains constant functions. The conclusion of Lemma 5.3(a) with $p = 1$ corresponds to [15, Hypothesis (†)], and Proposition 5.2 corresponds to [15, Hypothesis (H2)]. Hence $F : Y \to Y$ is a *good* inducing scheme in the sense of [15]. Moreover, Proposition 5.4 means that BV observables on I are *exchangeable* in the sense of [15].

Lemma 5.3(a) with $p > 1$ is the remaining hypothesis in [15, Theorem 1.11] yielding Theorem 1.7. Lemma 5.3(b) puts us in the position of [15, Example 5.3] yielding Theorem 1.5. ∎

Remark 5.5 Theorem 1.7 extends naturally to a much larger class of exchangeable observables on I, namely those that are uniformly piecewise BV. More precisely, we define $\mathcal{B}_1(I)$ to consist of those $v : I \to \mathbb{R}$ such that $\|v\|_{\mathcal{B}_1(I)} = \sup_{a \in \alpha} \sup_{0 \leq j < \tau(a)} \|1_{f^j a} v\|_{\mathrm{BV}} < \infty$. To verify that $\mathcal{B}_1(I)$ is exchangeable in the sense of [15] it suffices to show that Proposition 5.4 holds for $v \in \mathcal{B}_1(I)$, and the proof of this is the same as before.

It remains to relax the assumption that the tower map $\tilde{f} : X \to X$ is mixing. In general there exists an integer $k \geq 1$ such that $\tilde{f}$ is mixing up to a k-cycle. That is, X is the disjoint union of k subsets that are cyclically permuted by $\tilde{f}$ and are mixing under $\tilde{f}^k$. Label one of these subsets X_k. Then we obtain a semiconjugacy $\pi_k : X_k \to I$ between $\tilde{f}^k : X_k \to X_k$ and $f^k : I \to I$. By the proof in this section,

$$|\rho_{v,w}(kn)| \leq C\|v\|\|w\|_\infty \Big\{ \sum_{j > \delta n} \mu(\tau > j) + n\mu(\tau > \delta n) + n^{-q} \Big\},$$

for all $v \in \mathrm{BV}$, $w \in L^\infty$, $n \geq 1$. Redefining C and δ, and replacing w by $w \circ f^j$, $0 \leq j \leq k - 1$, we obtain the required result.

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