Operator renewal theory for continuous time dynamical systems with finite and infinite measure

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Abstract

We develop operator renewal theory for flows and apply this to obtain results on mixing and rates of mixing for a large class of finite and infinite measure semiflows. Examples of systems covered by our results include suspensions over parabolic rational maps of the complex plane, and nonuniformly expanding semiflows with indifferent periodic orbits.

In the finite measure case, the emphasis is on obtaining sharp rates of decorrelations, extending results of Gouëzel and Sarig from the discrete time setting to continuous time. In the infinite measure case, the primary question is to prove results on mixing itself, extending our results in the discrete time setting. In some cases, we obtain also higher order asymptotics and rates of mixing.

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1 Introduction

This paper is concerned with mixing for continuous time dynamical systems. To set the background for our results, we begin by discussing developments for discrete time.

Much recent research has centered around the statistical properties of smooth dynamical systems with strong hyperbolicity (expansion/contraction) properties. Results such as exponential decay of correlations and statistical limit laws are by now classical for uniformly hyperbolic diffeomorphisms [6, 32, 34]. In particular, if $f : X \to X$ is uniformly hyperbolic and μ is an equilibrium measure corresponding to a Hölder potential, then the correlation function $\int_X v\, w \circ f^n \, d\mu - \int_X v \, d\mu \int_X w \, d\mu$ decays exponentially quickly as $n \to \infty$ for Hölder observables $v, w : X \to \mathbb{R}$.

Young [38] extended this result to a large class of nonuniformly hyperbolic systems, including planar dispersing billiards, and also established polynomial decay of correlations

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for systems that are more slowly mixing [39]. The results were then shown to be optimal by Sarig [33] and Gouëzel [17]. Turning to the infinite measure case, the fundamental difference is that $\lim_{n\to\infty}\int_X v\, w\circ f^n\, d\mu = 0$ for reasonably well-behaved observables v, w. Hence there arises the preliminary problem of showing that $a_n \int_X v\, w\circ f^n\, d\mu \to \int_X v\, d\mu \int_X w\, d\mu$ for a suitable normalising sequence $a_n \to \infty$ and for sufficiently well-behaved v, w. The secondary problem is to estimate the speed of convergence (rate of mixing). Definitive results on the preliminary problem, and first results on the rate of mixing, were obtained recently in [28].

In the continuous time situation, decay of correlations is less well understood. Exponential decay of correlations has been proved only for a very thin set of Anosov flows (those that possess a contact structure or have exceptionally smooth stable and unstable foliations), see [8, 24]. On the other hand, superpolynomial decay of correlations holds for "typical" uniformly hyperbolic flows [9, 12] for observables that are sufficiently regular. The typical set of flows includes those with a pair of periodic points whose ratio of periods is Diophantine [9] and also includes an open and dense set of flows [12]. Results on superpolynomial decay were extended by [26] to nonuniformly hyperbolic flows whose Poincaré map is within the class considered in [38]. For flows whose Poincaré map lies in the class considered in [39], it was shown in [27] that typically polynomial decay holds for sufficiently regular observables.

In the current paper, we develop a continuous time operator renewal theory, and thereby obtain results on sharp lower bounds for finite measure semiflows with polynomial decay of correlations, and mixing (as well as higher order asymptotics) for infinite measure semiflows, extending the discrete time results of [17, 33, 28]. Our results hold typically in the same sense as discussed above (so it suffices that there exists a pair of periods with Diophantine ratio, see hypothesis (A2) and Remark 2.1 below).

1.1 Illustrative examples

To describe the main results, we consider (mainly for convenience) the family of Pomeau-Manneville intermittent maps [31] considered by [25], and their suspensions in the continuous time case. Specifically, define the interval maps $f : X \to X$, $X = [0, 1]$,

$$f(x) = \begin{cases} x(1 + 2^\gamma x^\gamma), & 0 < x < \frac{1}{2} \\ 2x - 1 & \frac{1}{2} < x < 1 \end{cases}, \tag{1.1}$$

where $\gamma > 0$. There is a unique (up to scaling) σ-finite absolutely continuous invariant measure and this measure is finite if and only if $\gamma < 1$. Such maps have a uniformly expanding (or Gibbs-Markov, see Section 2 for precise definitions) first return map to the set $Y = [\frac{1}{2}, 1]$. Set $\beta = 1/\gamma$ and

$$\xi_\beta(t) = \begin{cases} t^{-\beta} & \beta > 2 \\ (\log t)t^{-2} & \beta = 2 \\ t^{-(2\beta-2)} & 1 < \beta < 2 \end{cases}, \quad \xi_{\beta,\epsilon}(t) = \begin{cases} t^{-(\beta-\epsilon)}, & \beta \ge 2 \\ t^{-(2\beta-2)}, & 1 < \beta < 2 \end{cases}. \tag{1.2}$$

Discrete time, finite measure For $\gamma \in (0, 1)$, it follows from [39, 20] that correlations decay like $n^{-(\beta-1)}$:

$$\int_X v\, w \circ f^n - \int_X v \int_X w = O(n^{-(\beta-1)}),$$

for all $v : X \to \mathbb{R}$ Hölder and $w : X \to \mathbb{R}$ lying in L^∞. This decay rate is sharp [17, 33]:

$$\int_X v\, w \circ f^n - \int_X v \int_X w = c \int_X v \int_X w\, n^{-(\beta-1)} + O(\xi_\beta(n)),$$

for all observables v, w supported in a compact subset of $(0, 1]$ with v Hölder and w in L^∞. Here c is a positive constant depending only on γ.

Discrete time, infinite measure For $\gamma \in (1, 2)$, we showed [28] that there is a constant $c > 0$ (depending only on γ) such that

$$n^{1-\beta} \int_X v\, w \circ f^n \to c \int_X v \int_X w,$$

for all observables v, w supported in a compact subset of $(0, 1]$ with v Hölder and w in L^∞. The same result holds for $\gamma = 1$ with $n^{1-\beta}$ replaced by $\log n$. For $\gamma \geq 2$, such results cannot hold in the generality considered in [28] but, using the extra structure of the maps (1.1), the corresponding results were obtained by [19] for all $\gamma > 1$. In addition, rates of mixing and higher asymptotics for $\gamma \in (1, 2)$ were obtained in [28], improved upon in [35]. and extended to the case $\gamma \geq 2$ in [36].

Continuous time, finite measure Now suppose that $\varphi_X : X \to \mathbb{R}^+$ is a Hölder roof function bounded away from zero. We form the suspension flow $f_t : \Lambda \to \Lambda$ in the usual way (see Section 2 for definitions). This semiflow has an indifferent periodic orbit corresponding to the indifferent fixed point $0 \in X$.

In the finite measure case $\gamma \in (0, 1)$, it follows from [27] that typically $\int_\Lambda v\, w \circ f_t - \int_\Lambda v \int_\Lambda w$ decays at the rate $t^{-(\beta-1)}$ as $t \to \infty$ for observables $v, w : \Lambda \to \mathbb{R}$ where v is Hölder and w is sufficiently smooth in the flow direction.

Here we prove that such results are optimal: for any $\epsilon > 0$

$$\int_\Lambda v\, w \circ f_t - \int_\Lambda v \int_\Lambda w = c \int_\Lambda v \int_\Lambda w\, t^{-(\beta-1)} + O(\xi_{\beta,\epsilon}(t)),$$

for all observables v, w supported away from the indifferent periodic orbit with v Hölder and w sufficiently smooth in the flow direction. If moreover $\int_\Lambda v = 0$, then $\int v\, w \circ f_t = O(t^{-(\beta-\epsilon)})$. This is the direct analogue of the results in [17, 33].

Continuous time, infinite measure Finally, consider the semiflow $f_t : \Lambda \to \Lambda$ for $\gamma \geq 1$. For $\gamma \in (1, 2)$ we prove in this paper that typically

$$t^{1-\beta} \int_\Lambda v\, w \circ f_t \to c \int_\Lambda v \int_\Lambda w,$$

for all observables v, w supported away from the indifferent periodic orbit with v Hölder and w sufficiently smooth in the flow direction. Again the same result holds for $\gamma = 1$ with $t^{1-\beta}$ replaced by $\log t$, and we obtain higher order asymptotics. This is the direct analogue of the results in [28].

1.2 Ingredients of the proofs

The methods in this paper combine

(i) Operator renewal theory developed by [15, 33] for the discrete time finite measure situation.

(ii) The methods we introduced in the infinite ergodic theory setting in [28] which built upon [14, 15, 33].

(iii) The ideas of [9] for uniformly hyperbolic flows and their extension [26, 27] to the nonuniformly hyperbolic setting.

However, there is a fourth and equally important component, namely

(iv) An operator renewal equation for flows (Theorem 3.3).

As far as we can tell, the operator renewal equation for flows introduced in Section 3 below has no counterpart in the existing probability theory literature. Continuous time versions of renewal theory have been developed previously in the probability theory literature. We refer to [11, Ch. XI] for the general framework surrounding Blackwell's renewal theorem [4]. For such a theorem in the infinite mean setting (the continuous time analogue of [14]) we refer to [10, Theorem 1]. We also mention the work of Kingman (see for instance [23]) for the continuous analogue, developed for both finite and infinite mean setting, of Feller's theory on discrete regenerative phenomena. However, it is unclear how to apply these methods here, and our approach seems to have certain advantages as discussed in Remark 3.6.

In Section 2, we state our main results for suspensions of nonuniformly expanding maps, and recover the statements in the introduction (suspensions of intermittent maps) as a special case. The remainder of this paper is then divided into three parts. In Part I, we derive the operator renewal equation for flows. In Part II, we prove our results on infinite measure systems. In Part III, we prove our results on finite measure systems. The paper is written in such a way that Parts II and III can be read independently.

Remark 1.1 For discrete time systems, operator renewal theory was developed first in the finite measure case before being extended to the infinite measure situation. For continuous time systems, we present the material in the reverse order. The reason for this is that having formulated the continuous time operator renewal equation described above in (iv), it is fairly straightforward to deduce our main results for infinite measure semiflows from the existing work described in components (ii) and (iii). (We note however that certain technical estimates in the proof and usage of Lemma 5.2 are considerably more complicated than in the case of discrete time, infinite measure.) In contrast, although our results for finite measure semiflows follow from components (i), (iii) and (iv), it requires significantly more work to glue these methods together.

Notation We use the "big O" and $\ll$ notation interchangeably, writing $a_n = O(b_n)$ or $a_n \ll b_n$ if there is a constant $C > 0$ such that $a_n \le Cb_n$ for all $n \ge 1$. Three positive constants arise frequently throughout the paper: C_1 and C_2 introduced in Section 2, and $c_2 = (C_2 + 1)^{-1}$ which appears for the first time in Section 8.

2 Statement of the main results

Suspension semiflows Let (Y, μ) be a probability space and $F : Y \to Y$ an ergodic measure-preserving transformation. Let $\varphi : Y \to \mathbb{R}^+$ be a measurable roof function. Form the suspension $Y^\varphi = \{(y, u) \in Y \times \mathbb{R} : 0 \le u \le \varphi(y)\}/\sim$ where $(y, \varphi(y)) \sim (Fy, 0)$. The suspension flow $f_t : Y^\varphi \to Y^\varphi$ is given by $f_t(y, u) = (y, u + t)$ computed modulo identifications and the measure $\mu^\varphi = \mu \times \text{Lebesgue}$ is ergodic and f_t-invariant. In the finite measure case, we normalise by $\bar{\varphi} = \int_Y \varphi \, d\mu$ so that $\mu^\varphi = (\mu \times \text{Lebesgue})/\bar{\varphi}$ is a probability measure.

Gibbs-Markov maps We assume throughout that $F : Y \to Y$ is a full branch Gibbs-Markov map. Roughly speaking F is uniformly expanding with good distortion properties.

We recall the key definitions [1]. Let (Y, μ) be a Lebesgue probability space with countable measurable partition α. Let $F : Y \to Y$ be an ergodic, conservative, measure-preserving, Markov map transforming each partition element bijectively onto Y. For any $\theta \in (0, 1)$, define $d_\theta(y, y') = \theta^{s(y,y')}$ where the *separation time* $s(y, y')$ is the least integer $n \ge 0$ such that $F^n y$ and $F^n y'$ lie in distinct partition elements in α. It is assumed that the partition α separates orbits of F, so $s(y, y')$ is finite for all $y \ne y'$. Then d_θ is a metric. Let $F_\theta(Y)$ be the Banach space of d_θ-Lipschitz functions $v : Y \to \mathbb{R}$ with norm $\|v\| = |v|_\infty + |v|_\theta$ where $|v|_\theta$ is the Lipschitz constant of v.

Define the potential function $g = \log \frac{d\mu}{d\mu \circ F} : Y \to \mathbb{R}$. We require that g is uniformly piecewise Lipschitz: that is, $g|_a$ is d_θ-Lipschitz on each $a \in \alpha$ and the Lipschitz constants can be chosen independent of a.

For $n \ge 1$ we let α_n denote the partition into n-cylinders. Let $g_n = \sum_{j=0}^{n-1} g \circ F^j$. It follows from the Lipschitz property of g together with full branches that there exists a constant $C_1 > 0$ such that

$$e^{g_n(y)} \le C_1 \mu(a), \quad \text{and} \quad |e^{g_n(y)} - e^{g_n(y')}| \le C_1 \mu(a) d_\theta(F^n y, F^n y'), \tag{2.1}$$

for all $y, y' \in a$, $a \in \alpha_n$, $n \ge 1$.

From now on, we adopt a convenient abuse of notation and define $|1_a v|_\theta = \sup_{y,y' \in a : y \ne y'} |v(y) - v(y')|/d_\theta(y, y')$. We write $1_a v \in F_\theta(Y)$ if $1_a v$ is bounded and $|1_a v|_\theta < \infty$.

Roof function The roof function $\varphi : Y \to \mathbb{Z}^+$ is assumed to be piecewise Lipschitz with respect to d_{θ_0} for some $\theta_0 \in (0, 1)$, (ie $1_a \varphi \in F_{\theta_0}(Y)$ for all $a \in \alpha$), and satisfying $\inf \varphi > 0$. For convenience of notation, we suppose that $\inf \varphi > 2$. In particular, the set $\tilde{Y} = Y \times [0, 1]$ lies inside Y^φ.

We make various further assumptions:

(A1) There is a constant $C_2 > 0$ such that $|1_a \varphi|_{\theta_0} \le C_2 \inf_a \varphi$ for all $a \in \alpha$.

(A2) There exist two periodic orbits for f_t with periods τ_1, τ_2 such that τ_1/τ_2 is Diophantine. We require that the periodic orbits intersect Y only in the interior of partition elements.

Remark 2.1 Condition (A1) is automatic for a large class of examples discussed in Subsection 2.1. Condition (A2) is sufficient for a rather technical "approximate eigenfunction" criterion of Dolgopyat [9] to be satisfied. This criterion is stated precisely in Definition 4.2 and holds also for an open dense set of roof functions [12].

Observables We consider observables $v, w : \tilde{Y} \to \mathbb{R}$ of the following form. Writing $v^u(y) = v(y,u)$, define $|v|_\theta = \sup_{u\in[0,1]} |v^u|_\theta$ and $\|v\|_\theta = |v|_\infty + |v|_\theta$. Then $F_\theta(\tilde{Y})$ is the space consisting of those $v \in L^\infty(\tilde{Y})$ with $\|v\|_\theta < \infty$.

For $m \geq 0$, set $|w|_{\infty,m} = \max_{j=0,\dots,m} |\partial_t^j w|_\infty$. We write $w \in L^{\infty,m}(\tilde{Y})$ if w is supported in $Y \times (0,1)$ with $|w|_{\infty,m} < \infty$.

Define

$$\rho_{v,w}(t) = \int_{Y^\varphi} v\, w \circ f_t \, d\mu^\varphi,$$

and write $\bar{v} = \int_{Y^\varphi} v\, d\mu^\varphi$, $\bar{w} = \int_{Y^\varphi} w\, d\mu^\varphi$. In the finite measure case, the correlation function of v and w is given by $\rho_{v,w}(t) - \bar{v}\bar{w}$. We can now state our main theorems.

Theorem 2.2 (Infinite measure) *Assume that $F : Y \to Y$ is a full branch Gibbs-Markov map with roof function $\varphi : Y \to \mathbb{R}^+$ satisfying conditions (A1) and (A2).*

(a) Suppose that φ is nonintegrable and $\mu(\varphi > t) = \ell(t)t^{-\beta}$ where $\beta \in (\frac{1}{2}, 1]$ and ℓ is a measurable slowly varying function (so $\lim_{x\to\infty} \ell(\lambda x)/\ell(x) = 1$ for all $\lambda > 0$).

Let $d_\beta = \frac{1}{\pi}\sin\beta\pi$ for $\beta < 1$ and $d_\beta = 1$ for $\beta = 1$. Define $\tilde{\ell}(t) = \ell(t)$ for $\beta < 1$ and $\tilde{\ell}(t) = \int_1^t \ell(s)s^{-1}\, ds$ for $\beta = 1$.

Then there exist $\theta \in (0,1)$, $m \geq 1$, and a function $a : (0,\infty) \to (0,\infty)$ with $\lim_{t\to\infty} a(t) = 0$ such that

$$|\tilde{\ell}(t)t^{1-\beta}\rho_{v,w}(t) - d_\beta \bar{v}\bar{w}| \leq \|v\|_\theta |w|_{\infty,m}\, a(t),$$

for all $v \in F_\theta(\tilde{Y})$, $w \in L^{\infty,m}(\tilde{Y})$.

(b) Suppose moreover that $\mu(\varphi > t) = ct^{-\beta} + O(t^{-q})$ where $\beta \in (\frac{1}{2}, 1)$, $q \in (1, 2\beta)$ and $c > 0$. There exist constants $d_1 = c^{-1}d_\beta, d_2, d_3, \ldots \in \mathbb{R}$, and for any $\epsilon > 0$, there exist $\theta \in (0,1)$, $m \geq 1$, such that

$$\rho_{v,w}(t) = \sum_j d_j t^{-j(1-\beta)}\bar{v}\bar{w} + O(\|v\|_\theta |w|_{\infty,m}\, t^{-\beta(1-q^{-1}(2\beta-1)-\epsilon)}),$$

for all $v \in F_\theta(\tilde{Y})$, $w \in L^{\infty,m}(\tilde{Y})$, $t > 0$. Here, the sum is over those $j \geq 1$ with $j(1-\beta) \leq \beta(1 - q^{-1}(2\beta - 1) - \epsilon)$.

In particular, if $\mu(\varphi > t) = ct^{-\beta} + O(t^{-2\beta})$, then the error term is of the form $O(\|v\|_\theta |w|_{\infty,m}\, t^{-(\frac{1}{2}-\epsilon)})$. If in addition $\beta > \frac{3}{4}$ and $d_2 \neq 0$, then we obtain second order asymptotics.

Remark 2.3 Explicit formulas for the constants d_j, $j \ge 2$, can be found in [28, Section 9]. Write $\mu(\varphi > t) = ct^{-\beta}(1 + H(t))$. Then $d_j = e_j \int_0^\infty H(t)\, dt$ where e_j is a nonzero constant depending only on j and β. In particular, either all the constants d_j are nonzero (the typical case) or $d_j = 0$ for all $j \ge 2$. (The functions H and H_1 in [28, Lemma 3.2] coincide in the continuous time context.)

In the finite case, define

$$\zeta(t) = \int_t^\infty \mu(\varphi > \tau)\, d\tau, \quad \xi_{\beta,\epsilon}(t) = \begin{cases} t^{-(\beta-\epsilon)}, & \beta \ge 2 \\ t^{-(2\beta-2)}, & 1 < \beta < 2 \end{cases}. \tag{2.2}$$

Theorem 2.4 (Finite measure) *Assume that $F : Y \to Y$ is a full branch Gibbs-Markov map with roof function $\varphi : Y \to \mathbb{R}^+$ satisfying conditions (A1) and (A2).*

(a) Suppose that $\mu(\varphi > t) = O(t^{-\beta})$ where $\beta > 1$. Then for any $\epsilon > 0$, there exists $\theta \in (0, 1)$, $m \ge 1$, such that

$$\rho_{v,w}(t) - \bar{v}\bar{w} = (1/\bar{\varphi})\bar{v}\bar{w}\, \zeta(t) + O(\|v\|_\theta |w|_{\infty,m}\, \xi_{\beta,\epsilon}(t)),$$

for all $v \in F_\theta(\tilde{Y})$, $w \in L^{\infty,m}(\tilde{Y})$, $t > 0$.

(b) Suppose further that $\bar{v} = 0$. Then for any $\epsilon > 0$, there exists $\theta \in (0, 1)$, $m \ge 1$, such that

$$\rho_{v,w}(t) = O\big(\|v\|_\theta |w|_{\infty,m}\, t^{-(\beta-\epsilon)}\big),$$

for all $v \in F_\theta(\tilde{Y})$, $w \in L^{\infty,m}(\tilde{Y})$, $t > 0$.

Remark 2.5 It is well-known that the regular variation assumption is necessary for Theorem 2.2. The assumption of polynomial tails in Theorem 2.4 can be relaxed as in [16] or [30] but we do not pursue that here.

2.1 Examples with full branch Gibbs-Markov first return maps

In formulating Theorems 2.2 and 2.4, we considered suspensions where the map $F : Y \to Y$ is uniformly expanding and the roof function $\varphi : Y \to \mathbb{R}^+$ is unbounded. Often it is convenient to reverse the roles and to start with a map $f : X \to X$ that is less well-behaved (nonuniformly expanding instead of uniformly expanding) together with a bounded roof function $\varphi_X : X \to \mathbb{R}$.

In particular a large class of examples covered by our methods are those where the map $f : X \to X$ has a first return map $F : Y \to Y$ that is full branch and Gibbs-Markov and where φ_X is globally Lipschitz. This includes suspensions of parabolic rational maps of the complex plane (Aaronson *et al.* [3]) and Thaler's class of interval maps with indifferent fixed points [37] (in particular the family (1.1) defined above).

The separation time s, and hence the metric d_θ, extends from Y to X: define $s(f^\ell y, f^\ell y') = s(y, y')$ for all $y, y' \in a$, $a \in \alpha$, $0 \le \ell < \tau(y)$. Suppose that the roof function $\varphi_X : X \to \mathbb{R}^+$ is locally Lipschitz with respect to this metric and define the induced roof function $\varphi : Y \to \mathbb{Z}^+$, $\varphi(y) = \sum_{j=0}^{\tau(y)-1} \varphi_X(y)$. Thus we obtain equivalent semiflows using either (f, φ_X) or (F, φ). Furthermore, condition (A1) is automatic if φ_X is globally Lipschitz, and the statement of condition (A2) is unchanged in the new setting (the set of periods for (f, φ_X) or (F, φ) are identical).

Example 2.6 Consider the family of intermittent maps (1.1) discussed in the introduction. Such maps $f : X \to X$ have a full branch Gibbs-Markov first return map to the set $Y = [\frac{1}{2}, 1]$. Recall that $v : X \to \mathbb{R}$ is C^η for $\eta \in (0, 1]$ if v is continuous and $\sup_{x \neq x'} |v(x) - v(x')|/|x - x'|^\eta < \infty$.

Proposition 2.7 *Suppose that $\varphi_X : X \to \mathbb{R}^+$ is C^η, $\eta \in (0, 1]$. Then*

$$\mu(y \in Y : \varphi(y) > t) = c_0 t^{-\beta}(1 + O(t^{-\beta\eta})),$$

where $\beta = 1/\gamma$, $c_0 = \frac{1}{4}\beta^\beta \varphi_X(0)^\beta h(\frac{1}{2})$, and $h : Y \to \mathbb{R}^+$ is the density for μ.

Proof See [15, Theorem 1.3] for a similar calculation in the case $\gamma < 1$. Recall (see for example [28, Proposition 11.12]) that the first return time $\tau : Y \to \mathbb{Z}^+$ satisfies

$$\mu(\tau > n) = c_1 n^{-\beta}(1 + O(n^{-\beta})), \tag{2.3}$$

where $c_1 = \frac{1}{4}\beta^\beta h(\frac{1}{2})$.

If $\tau(y) = k$, then write

$$\varphi(y) = \sum_{j=0}^{k-1} \varphi_X(f^j y) = \varphi_X(y) + (k-1)\varphi_X(0) + \sum_{j=1}^{k-1} (\varphi_X(f^j y) - \varphi_X(0)),$$

where $f^j y = O((k - j)^{-\beta})$ for $j = 1, \ldots, k - 1$. It follows that

$$\Big|\sum_{j=1}^{k-1} (\varphi_X(f^j y) - \varphi_X(0))\Big| \le |\varphi_X|_\eta \sum_{j=1}^{k-1} |f^j y)|^\eta \ll k^{1-\eta\beta}.$$

Hence $\varphi(y) = k(\varphi_X(0) + O(k^{-\beta\eta})) = \varphi_X(0)\tau(y)(1 + O(\tau(y)^{-\beta\eta})$. This combined with (2.3) yields the result. ∎

Corollary 2.8 *Suppose that $X = [0, 1]$ and that $f : X \to X$ is an intermittent map of the form (1.1), with $\gamma \in (0, 2)$. Let $\varphi_X : X \to \mathbb{R}^+$ be a C^η-roof function, $\eta \in (0, 1]$. Suppose further that the suspension semiflow possesses a pair of periodic orbits with Diophantine ratio of periods. Let $\beta = 1/\gamma$ and $c_0 = \frac{1}{4}\beta^\beta \varphi_X(0)^\beta h(\frac{1}{2})$. Then*

(a) For $\gamma \in [1, 2)$, the conclusion of Theorem 2.2(a) holds with $\tilde{\ell}(t) \sim c_0$ for $\gamma \in (1, 2)$ and $\tilde{\ell}(t) \sim c_0 \log t$ for $\gamma = 1$.

Moreover if $\eta \in (0, 1]$ is sufficiently large ($\eta \in (\frac{1-\beta}{\beta}, 1]$ suffices), then the conclusion of Theorem 2.2(b) holds.

(b) *For $\gamma \in (0,1)$, the conclusion of Theorem 2.4(a) holds in the form*

$$\rho_{v,w}(t) - \bar{v}\bar{w} = (1/\bar{\varphi})c_0(\beta-1)^{-1}\bar{v}\bar{w}\, t^{-(\beta-1)} + O(t^{-p}),$$

where $p = \min\{\beta - 1 + \beta\eta, 2\beta - 2\}$ for $\beta < 2$ and $p = \min\{\beta - 1 + \beta\eta, \beta - \epsilon\}$ for $\beta \geq 2$.

Moreover, if $\bar{v} = 0$, then we obtain $\rho_{v,w}(t) = O(t^{-(\beta-\epsilon)})$ as in Theorem 2.4(b).

Proof Condition (A1) is automatic, and we have explicitly assumed condition (A2). Proposition 2.7 gives the required estimates on $\mu(\varphi > t)$. Hence the results follow from Theorems 2.2 and 2.4. ∎

Part I

Continuous time operator renewal theory

In this part of the paper, we formulate an operator renewal equation for flows.

3 The operator renewal equation

Transfer operators Let $R : L^1(Y) \to L^1(Y)$ denote the transfer operator for $F : Y \to Y$ and let $L_t : L^1(Y^\varphi) \to L^1(Y^\varphi)$ denote the family of transfer operators for f_t. (So $\int_Y Rv\, w\, d\mu = \int_Y v\, w \circ F\, d\mu$, $\int_{Y^\varphi} L_t v\, w\, d\mu^\varphi = \int_{Y^\varphi} v\, w \circ f_t\, d\mu^\varphi$ for suitable test functions v, w.)

Recall that $\tilde{Y} = Y \times [0,1]$. We define the probability measure $\tilde{\mu} = \mu \times$ Lebesgue on $\tilde{Y}$. Note that in the infinite measure case $\mu^\varphi|_{\tilde{Y}} = \tilde{\mu}$, whereas in the finite measure case $\mu^\varphi|_{\tilde{Y}} = (1/\bar{\varphi})\tilde{\mu}$.

Define $\tilde{F} : \tilde{Y} \to \tilde{Y}$ by setting $\tilde{F}(y,u) = (Fy,u)$. Note that $\tilde{F}(y,u) = f_{\varphi(y)}(y,u)$. Define $\tilde{\varphi} : \tilde{Y} \to \mathbb{R}^+$, $\tilde{\varphi}(y,u) = \varphi(y)$. Then $\tilde{F} = f_{\tilde{\varphi}}$. Let $\tilde{R}$ denote the transfer operator corresponding to the map $\tilde{F} : \tilde{Y} \to \tilde{Y}$ ($\int_{\tilde{Y}} \tilde{R}v\, w\, d\tilde{\mu} = \int_{\tilde{Y}} v\, w \circ \tilde{F}\, d\tilde{\mu}$). Given $v \in L^1(\tilde{Y})$ and $u \in [0,1]$ we define $v^u \in L^1(Y)$, $v^u(y) = v(y,u)$. It is easily verified that

$$(\tilde{R}v)(y,u) = (Rv^u)(y). \tag{3.1}$$

Renewal operators For $t > 0$, define $T_t, U_t : L^1(\tilde{Y}) \to L^1(\tilde{Y})$ by setting

$$T_t v = 1_{\tilde{Y}} L_t(1_{\tilde{Y}} v), \quad U_t v = 1_{\tilde{Y}} L_t(1_{\{\tilde{\varphi} > t\}} v).$$

For $s \in \mathbb{C}$, define the families of operators on $L^1(\tilde{Y})$,

$$\hat{R}(s)v = \tilde{R}(e^{-s\tilde{\varphi}} v), \quad \hat{T}(s)v = \int_0^\infty e^{-st} T_t v\, dt, \quad \hat{U}(s)v = \int_0^\infty e^{-st} U_t v\, dt.$$

Note that $\hat{R}$, $\hat{T}$, $\hat{U}$ are analytic on $\mathbb{H} = \{\operatorname{Re} s > 0\}$ and that $\hat{R}$ is well-defined on $\overline{\mathbb{H}} = \{\operatorname{Re} s \geq 0\}$. We also define $\hat{R}_0(s) : L^1(Y) \to L^1(Y)$ for $s \in \overline{\mathbb{H}}$: $\hat{R}_0(s)v = R(e^{-s\varphi} v)$.

Remark 3.1 Throughout the paper, we write $\hat{a}(s)$ to denote a function that is analytic on $\mathbb{H}$, with inverse Laplace transform $a(t)$.

We note that $\hat{R}_0(s)$ has the formal inverse Laplace transform $R_0(t)v = R(\delta_\varphi(t)v)$, where δ_x is the δ-measure at x, but this is not used explicitly in the paper.

Proposition 3.2 *$\hat{\rho}(s) = \int_{\tilde{Y}} \hat{T}(s)v\, w\, d\mu^\varphi$ for $v \in L^1(\tilde{Y})$, $w \in L^\infty(\tilde{Y})$, $s \in \mathbb{H}$.*

Proof We have $\rho(t) = \int_{\tilde{Y}} v\, w \circ f_t\, d\mu^\varphi = \int_{\tilde{Y}} L_t v\, w\, d\mu^\varphi = \int_{\tilde{Y}} T_t v\, w\, d\mu^\varphi$, so that

$$\hat{\rho}(s) = \int_0^\infty e^{-st}\rho(t)\, dt = \int_0^\infty e^{-st} \int_{\tilde{Y}} T_t v\, w\, d\mu^\varphi\, dt = \int_{\tilde{Y}} \hat{T}(s) v\, w\, d\mu^\varphi,$$

as required. ∎

Theorem 3.3 *$\hat{T}(s)\hat{R}(s) = \hat{T}(s) - \hat{U}(s)$ for $s \in \mathbb{H}$.*

Proof Let $w \in L^\infty(\tilde{Y})$. We compute that

$$\begin{aligned}
\int_{\tilde{Y}} \hat{T}(s)\hat{R}(s)v\, w\, d\tilde{\mu} &= \int_{\tilde{Y}} \int_0^\infty e^{-st} L_t \tilde{R}(e^{-s\tilde{\varphi}}v)\, w\, dt\, d\tilde{\mu} = \int_0^\infty e^{-st} \int_{\tilde{Y}} e^{-s\tilde{\varphi}} v\, w \circ f_t \circ \tilde{F}\, d\tilde{\mu}\, dt \\
&= \int_{\tilde{Y}} \int_0^\infty e^{-s(t+\tilde{\varphi})} v\, w \circ f_{t+\tilde{\varphi}}\, dt\, d\tilde{\mu} = \int_{\tilde{Y}} \int_{\tilde{\varphi}}^\infty e^{-st} v\, w \circ f_t\, dt\, d\tilde{\mu} \\
&= \int_{\tilde{Y}} \int_0^\infty e^{-st} v\, w \circ f_t\, dt\, d\tilde{\mu} - \int_{\tilde{Y}} \int_0^{\tilde{\varphi}} e^{-st} v\, w \circ f_t\, dt\, d\tilde{\mu} \\
&= \int_{\tilde{Y}} \hat{T}(s)v\, w\, d\tilde{\mu} - \int_{\tilde{Y}} \int_0^\infty 1_{\{\tilde{\varphi}>t\}} e^{-st} v\, w \circ f_t\, dt\, d\tilde{\mu} = \int_{\tilde{Y}} \hat{T}(s) v\, w\, d\tilde{\mu} - \int_{\tilde{Y}} \hat{U}(s) v\, w\, d\tilde{\mu}
\end{aligned}$$

so $\hat{T}\hat{R} = \hat{T} - \hat{U}$ as required. ∎

For future reference, we record the following formula for U_t.

Proposition 3.4 *Suppose that $v \in L^1(\tilde{Y})$. Then*

$$(U_t v)(y,u) = \begin{cases} v(y, u-t)1_{[t,1]}(u), & 0 \le t \le 1 \\ (\tilde{R} v_t)(y,u), & t > 1 \end{cases},$$

where $v_t(y,u) = 1_{\{t<\varphi(y)<t+1-u\}} v(y, u - t + \varphi(y))$.

Proof For $t \le 1$, we have $U_t v = T_t v$ and so

$$\begin{aligned}
\int_{\tilde{Y}} (U_t v)\, w\, d\tilde{\mu} &= \int_{\tilde{Y}} v\, w \circ f_t\, d\tilde{\mu} = \int_{\tilde{Y}} v(y,u)\, w(y, u+t)\, d\tilde{\mu} \\
&= \int_Y \int_0^{1-t} v(y,u) w(y,u+t)\, du\, d\mu = \int_Y \int_t^1 v(y, u-t) w(y,u)\, du\, d\mu \\
&= \int_{\tilde{Y}} v(y, u-t) 1_{[t,1]}(u) w(y,u)\, d\tilde{\mu}.
\end{aligned}$$

For $t > 1$,

$$\begin{aligned}
\int_{\tilde Y}(U_t v)\, w\, d\tilde\mu &= \int_{\tilde Y} T_t(1_{\{\tilde\varphi>t\}}v)\, w\, d\tilde\mu = \int_{\tilde Y} 1_{\{\tilde\varphi>t\}}v\, w\circ f_t\, d\tilde\mu \\
&= \int_{\tilde Y}(1_{\{\tilde\varphi>t\}}v)(y,u)w(Fy,u+t-\varphi(y))\, d\tilde\mu \\
&= \int_Y\int_{\varphi(y)-t}^1 1_{\{\varphi(y)>t\}}v(y,u)w(Fy,u+t-\varphi(y)))\, du\, d\mu \\
&= \int_Y\int_0^{1+t-\varphi(y)} 1_{\{\varphi(y)>t\}}v(y,u-t+\varphi(y))w(Fy,u))\, du\, d\mu \\
&= \int_{\tilde Y} 1_{\{t<\varphi(y)<t+1-u\}}v(y,u-t+\varphi(y))w\circ\tilde F(y,u)\, d\tilde\mu = \int_{\tilde Y}(\tilde R v_t)(y,u)w(y,u)\, d\tilde\mu,
\end{aligned}$$

as required. ∎

In Section 2.1, we defined the symbolic metric d_θ on Y and the Banach space $F_\theta(Y)$ of d_θ-Lipschitz functions $v : Y \to \mathbb{R}$. The next result makes use of the Gibbs-Markov structure and a weakened version of condition (A2).

Proposition 3.5 *Let $\theta \in (0,1)$. Viewing the family of twisted transfer operators $\hat R_0(s)$ as operators on $F_\theta(Y)$,*

(a) The spectral radius of $\hat R_0(s)$ is less than 1 for $s \in \overline{\mathbb{H}} - \{0\}$ and is equal to 1 for $s = 0$.

(b) 1 is a simple eigenvalue for $\hat R_0(0)$ and is isolated in the spectrum of $\hat R_0(0)$.

Proof This is standard. By for example the proof of [28, Proposition 11.4], $\hat R_0(s)$ has spectral radius at most 1 and essential spectral radius at most θ for all $s \in \overline{\mathbb{H}}$. Also the spectral radius is less than 1 for all $s \in \mathbb{H}$. Hence it suffices to consider eigenvalues at 1 for $s = ib$. It follows from ergodicity of F that 1 is a simple eigenvalue for its transfer operator $\hat R_0(0)$, so it remains to rule out 1 as an eigenvalue for $\hat R_0(ib)$, $b \neq 0$.

Consider the family of operators $M_b : F_\theta(X) \to F_\theta(X)$ given by $M_b v = e^{ib\varphi}v \circ F$. The operators $\hat R_0(ib)$ and M_b are L^2 adjoints so it is equivalent to show that 1 is not an eigenvalue for M_b. We claim that if 1 is an eigenvalue, then every period τ corresponding to a periodic orbit for the semiflow lies in $(2\pi/b)\mathbb{Z}$ which violates condition (A2). (For this proposition it suffices to have two irrationally related periods.)

To prove the claim, suppose that $M_b v = v$ for some $v \in F_\theta(X)$, $v \not\equiv 0$. In other words, $e^{ib\varphi}v \circ F = v$. In particular, $|v| \circ F = |v|$ and it follows by ergodicity that $|v|$ is constant. Hence v is nonvanishing. Iterating, we have $e^{ib\sum_{j=0}^{k-1}\varphi\circ F^j}v \circ F^k = v$. Now suppose that y is a periodic point for F of period k. The period τ of the corresponding periodic orbit for f_t is given by $\tau = \sum_{j=0}^{k-1}\varphi(F^j y)$ and so $v(y) = e^{ib\tau}v(y)$. Dividing by $v(y)$, we obtain $e^{ib\tau} = 1$ verifying the claim. ∎

Let $\hat T_0(s) = (I - \hat R_0(s))^{-1}$; this is well-defined for $s \in \overline{\mathbb{H}} - \{0\}$. Write $v^u(y) = v(y,u)$. Then

$$((I - \hat R(s))^{-1}v)(y,u) = (\hat T_0(s)v^u)(y), \quad s \in \overline{\mathbb{H}} - \{0\}.$$

Also we obtain the renewal equation

$$\hat{T}(s) = \hat{U}(s)(I - \hat{R}(s))^{-1}, \quad s \in \overline{\mathbb{H}} - \{0\}.$$

By Proposition 3.2, we obtain an analytic extension

$$\hat{\rho}(s) = \int_{\tilde{Y}} \hat{U}(s)(I - \hat{R}(s))^{-1} v\, w\, d\mu^{\varphi}, \tag{3.2}$$

defined on a neighbourhood of $\overline{\mathbb{H}} - \{0\}$.

Remark 3.6 The operator renewal equation $\hat{T}(s) = \hat{U}(s)(I - \hat{R}(s))^{-1}$ has the desired effect of relating the Laplace transform of the transfer operators T_t for the flow with the perturbed transfer operator $\hat{R}_0(s)v = R(e^{-s\varphi}v)$ where R is the transfer operator for the Poincaré map F.

In dynamical systems theory, there are two standard types of discrete time system that can be obtained from a continuous time system: Poincaré maps such as F and the time-h map f_h for fixed $h > 0$. In the probability theory literature, a standard technique after Kingman [22] is to consider discrete time "skeletons" f_h and to pass to the continuous time limit as $h \to 0$. However, for the properties studied in the current paper, the partially hyperbolic time-h map is as difficult to study as the underlying continuous time system. In contrast, the uniformly expanding Poincaré map F is much more tractable.

Hence, at least for certain situations in dynamical systems theory, and perhaps in probability theory too, the renewal equation presented here seems a more useful approach than passing to discrete time skeletons.

The following elementary result is required in both Part II and Part III.

Proposition 3.7 *Let $m \ge 1$. Suppose that $v \in L^1(\tilde{Y})$ and $w \in L^{\infty,m}(\tilde{Y})$. Then*

$$\hat{\rho}_{v,w}(s) = \sum_{j=1}^{m} \rho_{v,\partial_t^{j-1}w}(0)s^{-j} + s^{-m}\hat{\rho}_{v,\partial_t^m w}(s).$$

Proof First note that $\rho_{v,w}$ is m-times differentiable and ${\rho_{v,w}}^{(j)} = \rho_{v,\partial_t^j w}$ for $j = 0, \dots, m$. By Taylor's Theorem, $\rho_{v,w}(t) = P_m(t) + H_m(t)$, where

$$P_m(t) = \sum_{j=0}^{m-1} \frac{1}{j!}{\rho_{v,w}}^{(j)}(0)t^j, \quad H_m(t) = \int_0^t g(t-\tau){\rho_{v,w}}^{(m)}(\tau)\, d\tau, \quad g(t) = \frac{t^{m-1}}{(m-1)!}.$$

Hence $\hat{\rho}_{v,w}(s) = \sum_{j=0}^{m-1} \rho_{v,\partial_t^j w}(0)s^{-(j+1)} + \hat{H}_m(s)$, where $\hat{H}_m(s) = \hat{g}(s)\hat{\rho}_{v,\partial_t^m w}(s) = s^{-m}\hat{\rho}_{v,\partial_t^m w}(s)$. ∎

4 Dolgopyat-type estimates

In this section, we recall estimates of [9] extended to the nonuniformly hyperbolic setting [26, 27]. These are required to control $(I - \hat{R}(s))^{-1}$ for $s = ib$, b large. The arguments need some modification here to allow for the possibility that $\varphi \notin L^1$.

We recall that the twisted transfer operators $\hat{R}_0(s) : L^1(Y) \to L^1(Y)$ satisfy for $n \geq 1$,

$$(\hat{R}_0(s)^n v)(y) = \sum_{a \in \alpha_n} e^{g_n(y_a)} e^{-s\varphi_n(y_a)} v(y_a), \tag{4.1}$$

where y_a denotes the unique preimage $y_a \in a \cap F^{-n}y$ and $\varphi_n = \sum_{j=0}^{n-1} \varphi \circ F^j$.

Lemma 4.1 *For every $\epsilon > 0$, there exist constants $C \geq 1$ and $\theta, \tau \in (0,1)$ such that for every $v \in F_\theta(Y)$, $b \in \mathbb{R}$,*

- *(a) $|\hat{R}_0(ib)v|_\infty \leq |v|_\infty$,*
- *(b) $|\hat{R}_0(ib)^n v|_\theta \leq C\{(1 + |b|^\epsilon \int_Y \varphi^\epsilon \, d\mu)|v|_\infty + \theta^n |v|_\theta\}$.*
- *(c) $\|R^n v - \int_Y v \, d\mu\|_\theta \leq C\tau^n \|v\|_\theta$.*

Proof Note that $\hat{R}_0(s)^n v = R^n(e^{-s\varphi_n} v)$. Since $|R|_\infty = 1$, it follows that part (a) is valid. Full branch Gibbs-Markov maps are mixing, so R has no eigenvalues on the unit circle except for the simple eigenvalue at 1. Part (c) follows from this together with quasicompactness [1, Section 4.7].

It remains to prove (b). Our argument improves [7] where it is assumed that $\varphi \in L^1(Y)$. Let $y, y' \in Y$. Then

$$(\hat{R}_0(ib)^n(1_a v))(y) - (\hat{R}_0(ib)^n(1_a v))(y') = D_1 + D_2 + D_3,$$

where

$$D_1 = (e^{g_n(y_a)} - e^{g_n(y'_a)}) e^{ib\varphi_n(y_a)} v(y_a), \quad D_2 = e^{g_n(y'_a)}(e^{ib\varphi_n(y_a)} - e^{ib\varphi_n(y'_a)}) v(y_a),$$
$$D_3 = e^{g_n(y'_a)} e^{ib\varphi_n(y'_a)} (v(y_a) - v(y'_a)).$$

By the estimates (2.1),

$$|D_1| \leq C_1 \mu(a) |v|_\infty d_\theta(y, y'), \quad |D_3| \leq C_1 \mu(a) |v|_\theta d_\theta(y_a, y'_a) = C_1 \theta^n \mu(a) |v|_\theta d_\theta(y, y').$$

Summing over $a \in \alpha_n$, we obtain that the terms of type D_1 and D_3 contribute $C_1 |v|_\infty$ and $C_1 \theta^n |v|_\theta$ respectively to $|\hat{R}_0(ib)^n v|_\theta$.

Next,

$$|D_2| \leq C_1 \mu(a) \sum_{j=0}^{n-1} |e^{ib\varphi(F^j y_a)} - e^{ib\varphi(F^j y'_a)}| |v|_\infty.$$

Recall that by assumption $1_a \varphi \in F_{\theta_0}(Y)$ for some $\theta_0 \in (0,1)$. Using the inequality $|e^{ix} - 1| \leq \min\{2, |x|\} \leq 2|x|^\epsilon$ for $x \in \mathbb{R}$, $\epsilon \in [0,1]$,

$$|D_2| \leq 2C_1 \mu(a) |b|^\epsilon \sum_{j=0}^{n-1} |1_{F^j a} \varphi|_{\theta_0}^\epsilon d_{\theta_0}(F^j y_a, F^j y'_a)^\epsilon |v|_\infty.$$

Let $\theta = \theta_0^\epsilon$. Then $d_{\theta_0}(F^j y_a, F^j y'_a)^\epsilon = d_\theta(F^j y_a, F^j y'_a) = \theta^{n-j} d_\theta(y, y')$. By (A1),

$$|D_2| \le 2C_1 C_2^\epsilon \mu(a)|b|^\epsilon \sum_{j=0}^{n-1} \inf(1_{F^j a}\varphi)^\epsilon \theta^{n-j} d_\theta(y, y')|v|_\infty.$$

Hence, summing over $a \in \alpha_n$, the D_2 terms contribute $2C_1 C_2^\epsilon |b|^\epsilon |v|_\infty S$, where

$$\begin{aligned} S &= \sum_{a\in\alpha_n} \mu(a) \sum_{j=0}^{n-1} \theta^{n-j} \inf(1_{F^j a}\varphi)^\epsilon = \sum_{j=0}^{n-1} \sum_{d\in\alpha_{n-j}} \sum_{a\in\alpha_n : F^j a = d} \mu(a)\theta^{n-j} \inf(1_{F^j a}\varphi)^\epsilon \\ &= \sum_{j=0}^{n-1} \theta^{n-j} \sum_{d\in\alpha_{n-j}} \inf(1_d\varphi)^\epsilon \sum_{a\in\alpha_n : F^j a = d} \mu(a) = \sum_{j=0}^{n-1} \theta^{n-j} \sum_{d\in\alpha_{n-j}} \inf(1_d\varphi)^\epsilon \mu(d) \\ &\le \sum_{j=0}^{n-1} \theta^{n-j} \int_Y \varphi^\epsilon \, d\mu \le \theta(1-\theta)^{-1} \int_Y \varphi^\epsilon \, d\mu. \end{aligned}$$

This completes the proof of part (b). ∎

For $b \in \mathbb{R}$, define $M_b : L^\infty(Y) \to L^\infty(Y)$, $M_b v = e^{ib\varphi} v \circ F$.

Definition 4.2 There are *approximate eigenfunctions* on a subset $Z \subset Y$ if there exist constants $A > 0$ arbitrarily large, $\beta > 0$ and $C \ge 1$, and sequences $|b_k| \to \infty$, $\psi_k \in [0, 2\pi)$, $u_k \in F_\theta(Y)$ with $|u_k| \equiv 1$, such that setting $n_k = [\beta \ln |b_k|]$,

$$|M_{b_k}^{n_k} u_k(y) - e^{i\psi_k} u_k(y)| \le C|b_k|^{-A},$$

for all $y \in Z$ and all $k \ge 1$.

A subset $Z_0 \subset Y$ is called a *finite subsystem* if $Z_0 = \bigcap_{n\ge 0} F^{-n} Z$ where Z is a finite union of partition elements $a \in \alpha$.

Proposition 4.3 *There exists a finite subsystem Z_0 such that there are no approximate eigenfunctions on Z_0.*

Proof By (A2), we can fix two periodic orbits with periods τ_1 and τ_2 such that τ_1/τ_2 is Diophantine. Let Z be the union of the partition elements $a \in \alpha$ intersected by the periodic orbits and define $Z_0 = \bigcap_{n\ge 0} F^{-n} Z$. It follows from [9, Section 13] that there are no approximate eigenfunctions on Z_0. ∎

Lemma 4.4 *There exists $A > 0$ and $C \ge 1$ such that $\|(I - \hat{R}(ib))^{-1}\|_\theta \le C|b|^A$ for all $b \in \mathbb{R}$ with $|b| \ge 1$.*

Proof By (3.1), it suffices to prove this with $\hat{R}(ib)$ replaced by $R_0(ib) : F_\theta(Y) \to F_\theta(Y)$. By Proposition 4.3, there is a finite subsystem on which there are no approximate eigenfunctions. The estimate for $(I - \hat{R}_0)^{-1}$ follows from this by exactly the argument used in [26, Lemmas 3.12 and 3.13]. Lemma 4.1 plays the role of [26, Proposition 3.7]. ∎

Remark 4.5 We shall consider various families of linear operators acting on various function spaces. Throughout, $\| \, \|_\theta$ denotes the operator norm on either $F_\theta(Y)$ or $F_\theta(\tilde{Y})$. Similarly, $\| \, \|_\infty$ denotes the operator norm on either $L^\infty(Y)$ or $L^\infty(\tilde{Y})$. Whether the space is Y or $\tilde{Y}$ should be clear from the context.

Part II

Infinite measure systems

In this part of the paper, we prove our main results in the infinite measure context. Throughout, we assume the setup from Section 2, so $F : Y \to Y$ is a full branch Gibbs-Markov map and $\varphi : Y \to \mathbb{R}^+$ is a roof function satisfying assumptions (A1) and (A2). In addition, we make the standing assumption throughout this part of the paper that φ is nonintegrable and $\mu(\varphi > t) \sim \ell(t)t^{-\beta}$ where $\beta \in (\frac{1}{2}, 1]$ and $\ell(t)$ is slowly varying.

Section 5 contains various operator-theoretic estimates. Our result on first order asymptotics (mixing) stated in Theorem 2.2(a) is proved in Section 6. Our result on second order asymptotics and rates of mixing, Theorem 2.2(b), is proved in Section 7.

5 Functional analytic estimates

In this section, we carry out various operator-theoretic estimates. Most of these are fairly straightforward generalisations of the estimates in [28] which built upon [2, 14]. However, the estimates in Lemma 5.2 are considerably more complicated than in the discrete time case.

5.1 Estimates for $\tilde{R}$

In this subsection, we prove a key technical estimate that we have not seen elsewhere in the literature (though Lemma 5.2 has a similar flavour to estimates in [35]).

We have the estimate $\mu(E > t) = O(\ell(t)t^{-\beta})$ for various functions $E : \alpha \to \mathbb{R}$ related to φ including the locally constant functions $E(a) = \inf_a \varphi$ and hence $E(a) = |1_a\varphi|_\infty$ and $E(a) = |1_a\varphi|_{\theta_0}$ by condition (A1).

We use the following resummation argument extensively.

Proposition 5.1 *Let $\omega, E : \alpha \to \mathbb{R}$ be such that $\mu(E > t) = O(\ell(t)t^{-\beta})$, ω is a bounded function, and $\omega(a) \le GE(a)$. Then $\sum_{a\in\alpha} \mu(a)\omega(a) \ll \ell(1/G)G^\beta$.*

Proof For $L \ge 1$, write

$$\sum_{a\in\alpha} \mu(a)\omega(a) \le G \sum_{a:E(a)\le L} \mu(a)E(a) + \sum_{a:E(a)>L} \mu(a)|\omega|_\infty = GK + O(\ell(L)L^{-\beta}),$$

where

$$
\begin{aligned}
K &= \sum_{a:E(a)\le L} \mu(a)E(a) \le \sum_{j=1}^{L} \sum_{a:E(a)\in(j-1,j]} \mu(a)j = \sum_{j=1}^{L} \sum_{a:E(a)>j-1} \mu(a)j - \sum_{j=0}^{L} \sum_{a:E(a)>j} \mu(a)j \\
&= \sum_{j=0}^{L-1} \sum_{a:E(a)>j} \mu(a)(j+1) - \sum_{j=0}^{L} \sum_{a:E(a)>j} \mu(a)j \le \sum_{j=0}^{L-1} \sum_{a:E(a)>j} \mu(a) \\
&= \sum_{j=0}^{L-1} \mu(E>j) \ll \ell(L)L^{1-\beta}.
\end{aligned}
$$

Taking $L \approx 1/G$ yields the result. ∎

Lemma 5.2 *Let $\epsilon \in (0,\beta)$. There exists $\theta \in (0,1)$, $C>0$ such that*

$$\|\tilde{R}(ib_1) - \tilde{R}(ib_2)\|_\theta \le C\big\{\ell(|b_1-b_2|^{-1})|b_1-b_2|^\beta + \ell(|b_1-b_2|^{\epsilon/\beta-1}|b_2|^{-\epsilon/\beta})|b_2|^\epsilon|b_1-b_2|^{\beta-\epsilon}\big\}.$$

Proof By (3.1), it suffices to prove the result for $\hat{R}_0(ib)$. We show that

$$
\begin{aligned}
|\hat{R}_0(ib_1) - \hat{R}_0(ib_2)v|_\theta \le C\big\{&\ell(|b_1-b_2|^{-1})|b_1-b_2|^\beta \\
&+ \ell(|b_1-b_2|^{\epsilon/\beta-1}|b_2|^{-\epsilon/\beta})|b_2|^\epsilon|b_1-b_2|^{\beta-\epsilon}\big\}\|v\|_\theta.
\end{aligned}
$$

A simpler argument which we omit shows that $|(\hat{R}_0(ib_1)-\hat{R}_0(ib_2))v|_\infty \le C\ell(|b_1-b_2|^{-1})|b_1-b_2|^\beta|v|_\infty$ and the result follows.

The structure of the calculation begins as in [2, Theorem 2.4]. Let $DR = \hat{R}_0(ib_1) - \hat{R}_0(ib_2)$ and $\Delta(y) = e^{ib_1\varphi(y)} - e^{ib_2\varphi(y)}$. Recalling formula (4.1), we have $(DRv)(y) = \sum_{a\in\alpha}\Delta(y_a)e^{g(y_a)}v(y_a)$ and so

$$
\begin{aligned}
(DRv)(y) - (DRv)(y') &= \sum_{a\in\alpha} \Delta(y_a)e^{g(y_a)}v(y_a) - \Delta(y'_a)e^{g(y'_a)}v(y'_a) \\
&= \sum_{a\in\alpha} \Delta(y_a)[e^{g(y_a)}v(y_a) - e^{g(y'_a)}v(y'_a)] + [\Delta(y_a)-\Delta(y'_a)]e^{g(y'_a)}v(y'_a).
\end{aligned}
$$

By the estimates (2.1),

$$|(DRv)(y) - (DRv)(y')| \le C_1|v|_\theta J_1 + C_1|v|_\infty J_2,$$

where

$$J_1 = \sum_{a\in\alpha}\mu(a)|\Delta(y_a)|d_\theta(y,y'), \qquad J_2 = \sum_{a\in\alpha}\mu(a)|\Delta(y_a)-\Delta(y'_a)|.$$

Now Δ is bounded and also $|\Delta(y_a)| \le |b_1-b_2||1_a\varphi|_\infty = GE(a)$ where $G = |b_1-b_2|$ and $\mu(E>t) = O(\ell(t)t^{-\beta})$. By Proposition 5.1, $J_1 \ll \ell(|b_1-b_2|^{-1})|b_1-b_2|^\beta d_\theta(y,y')$.

Next,

$$|\Delta(y_a)-\Delta(y'_a)| \le |e^{i(b_1-b_2)\varphi(y_a)} - e^{i(b_1-b_2)\varphi(y'_a)}| + |e^{i(b_1-b_2)\varphi(y'_a)}-1||e^{ib_2\varphi(y_a)} - e^{ib_2\varphi(y'_a)}|,$$

so $J_2 \le J_2' + J_2''$ where

$$J_2' = \sum_{a\in\alpha} \mu(a)|e^{i(b_1-b_2)\varphi(y_a)} - e^{i(b_1-b_2)\varphi(y_a')}|,$$
$$J_2'' = \sum_{a\in\alpha} \mu(a)|e^{i(b_1-b_2)\varphi(y_a')} - 1||e^{ib_2\varphi(y_a)} - e^{ib_2\varphi(y_a')}|,$$

Now $J_2' = \sum_{a\in\alpha} \mu(a)\omega(a)$ where ω is bounded and $|\omega(a)| \le GE(a)$ where $G = |b_1 - b_2|d_{\theta_0}(y,y')$ and $E(a) = |1_a\varphi|_{\theta_0}$. It follows from (A1) that $\mu(E > t) = O(\ell(t)t^{-\beta})$. By Proposition 5.1, $J_2' \ll \ell((|b_1-b_2|d_{\theta_0}(y,y'))^{-1})(|b_1-b_2|d_{\theta_0}(y,y'))^\beta$. By Potter's bounds (see for instance [5]), $J_2' \ll \ell(|b_1-b_2|^{-1})|b_1-b_2|^\beta d_{\theta_0}(y,y')^{\beta-\epsilon}$. Choosing $\theta \ge \theta_0^{\beta-\epsilon}$, we obtain $J_2' \ll \ell(|b_1-b_2|)^{-1}|b_1-b_2|^\beta d_\theta(y,y')$.

Finally, write $J_2'' = \sum_{a\in\alpha}\mu(a)\omega(a)$ where ω is bounded. The inequality $|e^{ix}-1| \le 2|x|^\delta$ holds for all $x \ge 0$, $\delta \in [0,1]$, so

$$|\omega(a)| \le 4\big\{|b_1-b_2||1_a\varphi|_\infty\big\}^{1-\epsilon/\beta}\big\{|b_2||1_a\varphi|_{\theta_0}d_{\theta_0}(y,y')\big\}^{\epsilon/\beta} = 4GE(a),$$

where $G = |b_1-b_2|^{1-\epsilon/\beta}|b_2|^{\epsilon/\beta}d_{\theta_0{}^{\epsilon/\beta}}(y,y')$ and $E(a) = |1_a\varphi|_\infty^{1-\epsilon/\beta}|1_a\varphi|_{\theta_0}^{\epsilon/\beta}$. Again $\mu(E > t) = O(\ell(t)t^{-\beta})$ so it follows from Proposition 5.1 and Potter's bounds that

$$J_2'' \ll \ell(G^{-1})G^\beta = \ell(|b_1-b_2|^{\epsilon/\beta-1}|b_2|^{-\epsilon/\beta}d_{\theta_0{}^{\epsilon/\beta}}(y,y')^{-1})|b_1-b_2|^{\beta-\epsilon}|b_2|^\epsilon d_{\theta_0{}^\epsilon}(y,y')$$
$$\ll \ell(|b_1-b_2|^{\epsilon/\beta-1}|b_2|^{-\epsilon/\beta})|b_1-b_2|^{\beta-\epsilon}|b_2|^\epsilon d_{\theta_0{}^{\epsilon/2}}(y,y').$$

Choosing $\theta \ge \theta_0^{\epsilon/2}$, we obtain

$$J_2'' \ll \ell(|b_1-b_2|^{\epsilon/\beta-1}|b_2|^{-\epsilon/\beta})|b_1-b_2|^{\beta-\epsilon}|b_2|^\epsilon d_\theta(y,y'),$$

completing the proof. ∎

Remark 5.3 If $\sup_{a\in\alpha}|1_a\varphi|_\theta < \infty$, then the proof simplifies considerably [2] and we obtain that $\|\hat R(ib_1) - \hat R(ib_2)\|_\theta \le C\ell(|b_1-b_2|^{-1})|b_1-b_2|^\beta$. However, such a condition is too restrictive for the inducing step in Subsection 2.1 and in particular is not satisfied for Example 2.6.

Proposition 5.4 $\|\hat R(s) - \hat R(0)\|_\infty \ll \ell(1/|s|)|s|^\beta$ *for all* $s \in \overline{\mathbb{H}}$.

Proof Again it suffices to prove the result for the operators $\hat R_0(s)$. It follows from the proof of Lemma 5.2 that $\|\hat R_0(ib) - \hat R_0(0)\|_\infty \ll \ell(|b|)|b|^\beta$. An identical argument shows that $\|\hat R_0(ib+h) - \hat R_0(ib)\|_\infty \ll \ell(h)h^\beta$ for all $b \in \mathbb{R}$ and $h > 0$ (the restriction to $h > 0$ guarantees that the function $1 - e^{-h\varphi}$ is bounded). ∎

5.2 Estimates for $(I - \hat R)^{-1}$

Let $c_\beta = i\int_0^\infty e^{-i\sigma}\sigma^{-\beta}\,d\sigma$.

Lemma 5.5 *Viewing* $(I-\hat R(s))^{-1}$ *as a family of linear operators on* $F_\theta(\tilde Y)$,

$$(I-\hat R(ib))^{-1}v \sim c_\beta^{-1}\ell(1/b)^{-1}b^{-\beta}\int_Y v(y,\cdot)\,d\mu(y), \quad \textit{as } b\to 0^+.$$

Proof Since $\hat R(ib)v(y,u)=(\hat R_0(ib)v^u)(y)$, where $v^u(y)=v(y,u)$, it suffices to prove that $((I-\hat R_0(ib))^{-1}v)\sim c_\beta^{-1}\ell(1/b)^{-1}b^{-\beta}\int_Y v\,d\mu$ as $b\to 0^+$, for all $v\in F_\theta(Y)$.

By Lemma 5.2, the map $b\mapsto \hat R_0(ib)$ is continuous. By Proposition 3.5(a), $\hat R_0(0)$ has 1 as a simple eigenvalue, so there exists $\delta>0$ and a continuous family $\lambda(ib)$ of simple eigenvalues of $\hat R_0(ib)$ for $b\in(-\delta,\delta)$ with $\lambda(0)=1$. Let $P(ib)$ denote the corresponding family of spectral projections with complementary projections $Q(ib)=I-P(ib)$. Also, let $v(ib)$ denote the corresponding family of eigenfunctions normalized so that $\int_Y v(ib)\,d\mu=1$. In particular, $v(0)\equiv 1$ and $P(0)w=\int_Y w\,d\mu$ for all $w\in L^1(Y)$.

Following Gouëzel [18] (a simplification of [2]), we write

$$\lambda(ib)=\int_Y \lambda(ib)v(ib)\,d\mu=\int_Y R_0(e^{-ib\varphi}v(ib))\,d\mu=\int_Y e^{-ib\varphi}\,d\mu+V(ib),$$

where $V(ib)=\int_Y(\hat R_0(ib)-\hat R_0(0))(v(ib)-v(0))\,d\mu$.

By the argument in [14] (see also [2, 28]), $1-\int_Y e^{-ib\varphi}\,d\mu\sim c_\beta\ell(1/b)b^\beta$ as $b\to 0^+$. By Lemma 5.2, $\hat R_0$ and hence v are $C^{\beta-2\epsilon}$ (say), so $|V(ib)|=O(b^{2(\beta-2\epsilon)})$. Hence,

$$1-\lambda(ib)\sim c_\beta\ell(1/b)b^\beta \ \text{ as } b\to 0^+.$$

Next, for $b\in(-\delta,\delta)$,

$$\begin{aligned}(I-\hat R_0(ib))^{-1}=(1-\lambda(ib))^{-1}P(0)-(1-\lambda(ib))^{-1}(P(ib)-P(0))\\ +(I-\hat R_0(ib))^{-1}Q(ib).\end{aligned}$$

By Proposition 3.5(a), $\|(I-\hat R_0(ib))^{-1}Q(ib)\|_\theta=O(1)$. By Lemma 5.2, $\|P(ib)-P(0)\|_\theta\ll b^{\beta-2\epsilon}$. Hence

$$(I-\hat R_0(ib))^{-1}=(1-\lambda(ib))^{-1}(P(0)+o(1))+O(1)\sim c_\beta^{-1}\ell(1/b)^{-1}b^{-\beta}P(0) \ \text{ as } b\to 0^+,$$

as required. ∎

Proposition 5.6 $|(I-\hat R(s))^{-1}v|_\infty\ll \ell(1/|s|)^{-1}|s|^{-\beta}\|v\|_\theta$ *for all* $s\in\overline{\mathbb H}$ *with* $|s|$ *sufficiently small, and all* $v\in F_\theta(\tilde Y)$.

Proof As in the proof of Lemma 5.5, for $s\in\overline{\mathbb H}$ close to zero, we have the decomposition

$$(I-\hat R_0(s))^{-1}v=(1-\lambda(s))^{-1}P(0)v+(1-\lambda(s))^{-1}(P(s)-P(0))v+(I-\hat R_0(s))^{-1}Q(s)v,$$

where the last term is bounded. (As before, $\lambda(s)$ is the leading eigenvalue for $\hat R_0(s)$ with spectral projection $P(s)$ and $Q(s)=I-P(s)$.) By Proposition 5.4, $P(s)-P(0)=o(1)$ as $s\to 0$, so it remains to estimate $(1-\lambda(s))^{-1}$. Again, write

$$\lambda(s)=\int_Y e^{-s\varphi}\,d\mu+V(s),\quad V(s)=\int_Y(\hat R_0(s)-\hat R_0(0))(v(s)-v(0))\,d\mu,$$

where $v(s)$ are the normalised eigenfunctions for $\lambda(s)$. By Proposition 5.4, $\|\hat{R}_0(s) - \hat{R}_0(0)\|_{L^\infty(Y)} \ll \ell(1/|s|)|s|^\beta$ and this estimate is inherited by $v(s)$ so that $|V(s)|_\infty \ll |s|^{2\beta-\epsilon}$.

Let $G(x) = \mu(\varphi < x)$. Following and using the proof of [29, Lemma 2.4],

$$1 - \int_Y e^{-s\varphi}\,d\mu = \int_0^\infty (1 - e^{-sx})\,dG(x) = s\int_0^\infty e^{-sx}(1 - G(x))\,dx = \ell(1/|s|)s^\beta J(s)$$

where $J(s) \to \Gamma(1-\beta)$ as $s \to 0$. Hence $1 - \lambda(s) \sim c\ell(1/|s|)s^\beta$ with $c = \Gamma(1-\beta)$. ∎

Lemma 5.7 *For $b \in (0, 1]$, $\|(I - \hat{R}(ib))^{-1}\|_\theta \ll \ell(1/b)^{-1}b^{-\beta}$.*

Proof The proof of Lemma 5.5 shows that there exists $\delta > 0$ so that the result holds for $b \in (-\delta, \delta)$. Proposition 3.5(b) guarantees that $\|(I - \hat{R}(ib))^{-1}\|_\theta = O(1)$ for $b \in (\delta, 1]$. ∎

5.3 Estimates for $\hat{U}$

In this subsection, we obtain estimates for the family of operators $\hat{U}(s)$ that appeared in the renewal equation.

Lemma 5.8 *(a) $\hat{U}(0)v(y, u) = \int_0^u v(y, \tau)\,d\tau + \int_u^1 (\tilde{R}v)(y, \tau)\,d\tau$.*

(b) The family of linear maps $\hat{U}(ib) : L^\infty(\tilde{Y}) \to L^1(\tilde{Y})$, $b \in \mathbb{R}$, is uniformly bounded (indeed $\|\hat{U}(ib)\| \le 2$ for all $b \in \mathbb{R}$) and $\|\hat{U}(i(b + h)) - \hat{U}(ib)\| \ll \ell(1/h)h^\beta$ for all $h > 0$.

Proof (a) Write $\int_{\tilde{Y}} \hat{U}(0)v\,w\,d\tilde{\mu} = \int_{\tilde{Y}} \int_0^1 U_t v\,w\,dt\,d\tilde{\mu} + \int_{\tilde{Y}} \int_1^\infty U_t v\,w\,dt\,d\tilde{\mu}$. Using Proposition 3.4,

$$\begin{aligned}\int_{\tilde{Y}} \int_0^1 U_t v\,w\,dt\,d\tilde{\mu} &= \int_{\tilde{Y}} \int_0^1 1_{\{t,1)}(u)v(y, u-t)w(y, u)\,dt\,d\tilde{\mu} \\ &= \int_{\tilde{Y}} \Big\{\int_0^u v(y, u-t)\,dt\Big\} w(y, u)\,d\tilde{\mu} = \int_{\tilde{Y}} \Big\{\int_0^u v(y, \tau)\,d\tau\Big\} w(y, u)\,d\tilde{\mu},\end{aligned}$$

and

$$\begin{aligned}\int_{\tilde{Y}} \int_1^\infty U_t v\,w\,dt\,d\tilde{\mu} &= \int_{\tilde{Y}} \int_1^\infty 1_{\{t<\tilde{\varphi}(y)<t+1-u\}} v(y, u - t + \tilde{\varphi}(y))w(Fy, u)\,dt\,d\tilde{\mu} \\ &= \int_{\tilde{Y}} \Big\{\int_{\tilde{\varphi}-1+u}^{\tilde{\varphi}} v(y, u - t + \tilde{\varphi}(y))\,dt\Big\} w(Fy, u)\,d\tilde{\mu} \\ &= \int_{\tilde{Y}} \Big\{\int_u^1 v(y, \tau)\,d\tau\Big\} w \circ \tilde{F}(y, u)\,d\tilde{\mu} = \int_{\tilde{Y}} \Big\{\int_u^1 (\tilde{R}v)(y, \tau)\,d\tau\Big\} w(y, u)\,d\tilde{\mu}.\end{aligned}$$

This completes the proof of part (a).

(b) By Proposition 3.4, $|U_t v|_1 \le |v|_\infty$ for $0 < t < 1$, and $|U_t v|_1 \le \tilde{\mu}\{(y, u) : t < \tilde{\varphi}(y, u) < t + 1 - u\}|v|_\infty \le \mu\{t < \varphi < t + 1\}|v|_\infty$. Hence,

$$\begin{aligned}|\hat{U}(ib)v|_1 &= \Big|\int_0^\infty e^{-ibt}U_t v\,dt\Big|_1 \le \int_0^\infty |U_t v|_1\,dt \le \Big(1 + \int_1^\infty \mu(t < \varphi < t + 1)\,dt\Big)|v|_\infty \\ &= \Big(1 + \int_1^\infty (\mu(\varphi > t) - \mu(\varphi > t + 1))\,dt\Big)|v|_\infty = \Big(1 + \int_1^2 \mu(\varphi > t))\,dt\Big)|v|_\infty \le 2|v|_\infty.\end{aligned}$$

Also,

$$
\begin{aligned}
|(\hat{U}(i(b+h)) - \hat{U}(ib))v|_1 &= \Big|\int_0^\infty e^{-ibt}(e^{-iht}-1)U_t v\,dt\Big|_1 \\
&\le |v|_\infty\Big(h + \int_1^\infty |e^{-iht}-1|\mu(t<\varphi<t+1)\,dt\Big).
\end{aligned}
$$

But

$$
\begin{aligned}
\int_1^\infty |e^{-iht}-1|\mu(t<\varphi<t+1)\,dt \le h\int_1^L & t\mu(t<\varphi<t+1)\,dt \\
&+ 2\int_L^\infty \mu(t<\varphi<t+1)\,dt.
\end{aligned}
$$

Also, note that

$$
\begin{aligned}
&\int_1^L t\mu(t<\varphi<t+1)\,dt = \int_1^L t\mu(\varphi>t)\,dt - \int_1^L t\mu(\varphi>t+1)\,dt \\
&\quad = \int_1^L t\mu(\varphi>t)\,dt - \int_2^{L+1}(t-1)\mu(\varphi>t)\,dt \le 1 + \int_2^L \mu(\varphi>t)\,dt \ll \ell(L)L^{1-\beta},
\end{aligned}
$$

by Karamata's Theorem (see for instance [5]). Similarly,

$$
\begin{aligned}
\int_L^\infty \mu(t<\varphi<t+1)\,dt &= \int_L^\infty \mu(\varphi>t)\,dt - \int_{L+1}^\infty \mu(\varphi>t)\,dt \\
&= \int_L^{L+1} \mu(\varphi>t)\,dt \le \mu(\varphi>L) = \ell(L)L^{-\beta}.
\end{aligned}
$$

Putting these together, $|\hat{U}(i(b+h)) - \hat{U}(ib)|_1 \ll h + h\ell(L)L^{1-\beta} + \ell(L)L^{-\beta}$. The conclusion follows by taking $L = 1/h$. ∎

Remark 5.9 It is immediate from the proof that $\|\hat{U}(s)\|_{L^\infty(\tilde{Y})\mapsto L^1(\tilde{Y})} \le 2$ for all $s \in \overline{\mathbb{H}}$.

5.4 Estimates for $\hat{T}$

In this subsection, we combine our estimates from the previous subsections to estimate $\hat{T} = \hat{U}(I-\hat{R})^{-1}$. Recall that $c_\beta = i\int_0^\infty e^{-i\sigma}\sigma^{-\beta}\,d\sigma$.

Corollary 5.10 *There exists $A>0$ such that for all $\epsilon \in (0,\beta/2)$, the family of linear maps $\hat{T}(ib) : F_\theta(\tilde{Y}) \to L^1(\tilde{Y})$ satisfies the following properties.*

(a) $\hat{T}(ib)v \sim c_\beta^{-1}\ell(1/b)^{-1}b^{-\beta}\int_{\tilde{Y}} v\,d\tilde{\mu}$ *as* $b \to 0^+$.

(b) $\|\hat{T}(ib)\| \ll \begin{cases} \ell(1/b)^{-1}b^{-\beta}, & 0<b<1 \\ b^A, & b \ge 1 \end{cases}.$

(c) For $0 < b_1 < b_2 < 1$*,*

$$\begin{aligned}\|\hat T(ib_1)-\hat T(ib_2)\| \ll & \ell(1/b_1)^{-2}b_1^{-2\beta}\big\{\ell(|b_1-b_2|^{-1})|b_1-b_2|^{\beta} \\ & \qquad + \ell(|b_1-b_2|^{\epsilon/\beta-1}|b_2|^{-\epsilon/\beta})|b_2|^{\epsilon}|b_1-b_2|^{\beta-\epsilon}\big\}.\end{aligned}$$

For $1 < b_1 < b_2 < b_1+1$*,* $\|\hat T(ib_1)-\hat T(ib_2)\| \ll b_2^A|b_1-b_2|^{\beta-2\epsilon}$.

Proof (a) By continuity of $\hat U$ (Lemma 5.8(b)), we have

$$\begin{aligned}\lim_{b\to 0^+}\ell(1/b)b^{\beta}\hat T(ib)v &= \lim_{b\to 0^+}\hat U(ib)\ell(1/b)b^{\beta}(I-\hat R(ib))^{-1}v \\ &= \hat U(0)\lim_{b\to 0^+}\ell(1/b)b^{\beta}(I-\hat R(ib))^{-1}v.\end{aligned}$$

By Lemma 5.8(a),

$$\begin{aligned}\Big(\lim_{b\to 0^+}\ell(1/b)b^{\beta}\hat T(ib)v\Big)(y,u) =& \int_0^u \Big(\lim_{b\to 0^+}\ell(1/b)b^{\beta}(I-\hat R(ib))^{-1}v\Big)(y,\tau)\,d\tau \\ & + \int_u^1 \tilde R\Big(\lim_{b\to 0^+}\ell(1/b)b^{\beta}(I-\hat R(ib))^{-1}v\Big)(y,\tau)\,d\tau.\end{aligned}$$

Hence, by Lemma 5.5,

$$\begin{aligned}\Big(\lim_{b\to 0^+}\ell(1/b)b^{\beta}\hat T(ib)v\Big)(y,u) &= c_\beta^{-1}\int_0^u\int_Y v(y,\tau))\,d\mu\,d\tau + c_\beta^{-1}\int_u^1\Big(\tilde R\int_Y v(y,\tau)\,d\mu\Big)\,d\tau \\ &= c_\beta^{-1}\int_0^1\int_Y v(y,\tau)\,d\tilde\mu, = c_\beta^{-1}\int_{\tilde Y} v\,d\tilde\mu,\end{aligned}$$

where we have used also the fact that $\tilde R$ fixes functions that are independent of y. This proves part (a).
(b) This follows from Lemma 5.7 and Lemma 5.8(b) for $0 < b < 1$ and from Lemma 4.4 and Lemma 5.8(b) for $b \ge 1$.
(c) We give the details for $0 < b_1 \le b_2 \le 1$. Recall that $\hat T(ib) = \hat U(ib)S(ib)$ where $S(ib) = (I-\hat R(ib))^{-1}$. By the resolvent inequality,

$$\begin{aligned}&\|S(ib_1)-S(ib_2)\|_\theta \le \|S(ib_1)\|_\theta\|\hat R(ib_1)-\hat R(ib_2)\|_\theta\|S(ib_2)\|_\theta \\ &\quad \ll \ell(1/b_1)^{-2}b_1^{-2\beta}\big\{\ell(|b_1-b_2|^{-1})|b_1-b_2|^{\beta}+\ell(|b_1-b_2|^{\epsilon/\beta-1}b_2^{-\epsilon/\beta})b_2^{\epsilon}|b_1-b_2|^{\beta-\epsilon}\big\},\end{aligned}$$

using Lemma 5.2 and Lemma 5.7. Combining this with Lemma 5.8(b),

$$\begin{aligned}&\|\hat T(ib_1)-\hat T(ib_2)\|_{F_\theta\mapsto L^1} \le \|\hat U(ib_1)-\hat U(ib_2)\|_{L^\infty\mapsto L^1}\|S(ib_1)\|_\theta \\ &\qquad\qquad\qquad + \|\hat U(ib_1)\|_{L^\infty\mapsto L^1}\|S(ib_1)-S(ib_2)\|_\theta \\ &\ll \ell(|b_1-b_2|^{-1})|b_1-b_2|^{\beta}\ell(1/b_1)^{-1}b_1^{-\beta} \\ &\quad + \ell(1/b_1)^{-2}b_1^{-2\beta}\{\ell(|b_1-b_2|^{-1})|b_1-b_2|^{\beta}+\ell(|b_1-b_2|^{\epsilon/\beta-1}b_2^{-\epsilon/\beta})b_2^{\epsilon}|b_1-b_2|^{\beta-\epsilon}\}.\end{aligned}$$

The argument for $1 < b_1 < b_2 < b_1+1$ is similar but simpler because we establish a cruder estimate. The slowly varying functions are taken care of by $\epsilon' s$ in the exponents, and by increasing the value of A. ∎

6 First order asymptotics (mixing)

In this section, we prove Theorem 2.2(a).

6.1 The case $\beta \in (\frac{1}{2}, 1)$

Proposition 6.1 *There exists $\delta > 0$ such that $|\hat{\rho}(s)| \ll \ell(1/|s|)^{-1}|s|^{-\beta}$ for $s \in \overline{\mathbb{H}}$ satisfying $|s| \le \delta$.*

Proof This follows from Theorem 3.3, Proposition 5.6 and Remark 5.9. ∎

Proposition 6.2 $\rho(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ibt}\hat{\rho}(ib)\,db = \frac{1}{\pi}\operatorname{Re}\int_0^{\infty} e^{ibt}\hat{\rho}(ib)\,db.$

Proof Since $\hat{\rho}$ is analytic on $\mathbb{H}$, we can invert the Laplace transform by computing $\rho(t) = \frac{1}{2\pi i}\int_{\Gamma_1} e^{st}\hat{\rho}(s)\,ds$ where Γ_1 is the contour $\operatorname{Re} s = 1$ traversed upwards. As noted in (3.2), $\hat{\rho}(ib)$ is well-defined and continuous on the imaginary axis except for the singularity at zero, so by Cauchy's Theorem we can move the contour to a contour Γ_0 which consists of the segments of the imaginary axis $\{s = ib : -\infty < b < -\delta\} \cup \{s = ib : \delta < b < \infty\}$ together with a semicircle $\Gamma_\delta = \{s = \delta e^{i\psi} : -\pi/2 < \psi < \pi/2\}$ where $\delta > 0$ is arbitrarily small.

Let $\epsilon \in (0, 1-\beta)$. It follows from Proposition 6.1 that $\int_{\Gamma_\delta} e^{st}\hat{\rho}(s)\,ds = O(e^{\delta t}\delta^{1-\beta-\epsilon})$ and $\int_{-\delta}^{\delta} e^{ibt}\hat{\rho}(ib)\,db = O(\delta^{1-\beta-\epsilon})$. Letting $\delta \to 0$, we obtain $\rho(t) = \frac{1}{2\pi i}\int_{\Gamma_1} e^{st}\hat{\rho}(s)\,ds = \frac{1}{2\pi i}\int_{\Gamma_0} e^{st}\hat{\rho}(s)\,ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ibt}\hat{\rho}(ib)\,db$ as required. ∎

Recall that $c_\beta = i\int_0^\infty e^{-i\sigma}\sigma^{-\beta}\,d\sigma$.

Proposition 6.3 *For any $a > 0$,*

$$\lim_{t\to\infty} \ell(t)t^{1-\beta}\int_0^{a/t} e^{ibt}\hat{\rho}_{v,w}(ib)\,db = c_\beta^{-1}\int_0^a e^{i\sigma}\sigma^{-\beta}\,d\sigma \int_{\tilde{Y}} v\,d\mu \int_{\tilde{Y}} w\,d\mu.$$

Proof It follows from Proposition 3.2 and Corollary 5.10(a) that $\hat{\rho}(ib) = c_\beta^{-1}\ell(1/b)^{-1}b^{-\beta}h(b)\int_{\tilde{Y}} v\,d\tilde{\mu}\int_{\tilde{Y}} w\,d\tilde{\mu}$ where $\lim_{b\to 0^+} h(b) = 1$. The result follows from the dominated convergence theorem as in [28, Lemma 5.2]. ∎

Proposition 6.4 *Let $\beta' \in (\frac{1}{2}, \beta)$. For all $2\pi < a < t$,*

$$\int_{a/t}^1 e^{ibt}\hat{\rho}_{v,w}(ib)\,db = O(\ell(t)^{-1}t^{-(1-\beta)}a^{-(2\beta'-1)}).$$

Proof We modify [28, Lemma 5.1] to deal with the ϵ in Corollary 5.10(c). Let $b, b_1, b_2 \in (0, 1]$, $b_1 < b_2$. By Proposition 3.2 and Corollary 5.10(b,c),

$$\begin{aligned} |\hat{\rho}(ib)| &\ll \ell(1/b)^{-1}b^{-\beta}\|v\|_\theta|w|_\infty, \\ |\hat{\rho}(ib_1) - \hat{\rho}(ib_2)| &\ll \ell(1/b_1)^{-2}b_1^{-2\beta}\big\{\ell(|b_1-b_2|^{-1})|b_1-b_2|^{\beta} \\ &\qquad + \ell(|b_1-b_2|^{\epsilon/\beta-1}b_2^{-\epsilon/\beta})b_2^{\epsilon}|b_1-b_2|^{\beta-\epsilon}\big\}\|v\|_\theta|w|_\infty. \end{aligned}$$

Write

$$I=\int_{a/t}^{1} e^{ibt}\hat\rho(ib)\,db = -\int_{(a+\pi)/t}^{1+\pi/t} e^{ibt}\hat\rho(i(b-\pi/t))\,db.$$

Then $2I=J_1+J_2+J_3$, where

$$J_1=-\int_1^{1+\pi/t} e^{ibt}\hat\rho(i(b-\pi/t))\,db, \qquad J_2=\int_{a/t}^{(a+\pi)/t} e^{ibt}\hat\rho(ib)\,db,$$
$$J_3=\int_{(a+\pi)/t}^{1} e^{ibt}(\hat\rho(ib)-\hat\rho(i(b-\pi/t)))\,db.$$

We suppress the factor $\|v\|_\theta|w|_\infty$ from now on. Clearly $J_1=O(t^{-1})$, and by Potter's bounds,

$$\begin{aligned}|J_2| &\ll \int_{a/t}^{(a+\pi)/t} \ell(1/b)^{-1}b^{-\beta}\,db = \ell(t)^{-1}t^{-(1-\beta)}\int_a^{a+\pi}[\ell(t)/\ell(t/\sigma)]\sigma^{-\beta}\,d\sigma \\ &\ll \ell(t)^{-1}t^{-(1-\beta)}\int_a^{a+\pi}\sigma^{-\beta'}\,d\sigma \ll \ell(t)^{-1}t^{-(1-\beta)}a^{-\beta'}.\end{aligned}$$

Finally,

$$\begin{aligned}|J_3| &\ll \ell(t)t^{-\beta}\int_{a/t}^{1}\ell(1/b)^{-2}b^{-2\beta}\,db + t^{-\beta+\epsilon}\int_{a/t}^{1}\ell(1/b)^{-2}b^{-2\beta+\epsilon}\ell(t^{1-\epsilon/\beta}b^{-\epsilon/\beta})\,db \\ &= J_3'+J_3''.\end{aligned}$$

By Potter's bounds,

$$\begin{aligned}J_3' = \ell(t)^{-1}t^{\beta-1}\int_a^t[\ell(t)/\ell(\sigma/t)]^2\sigma^{-2\beta}\,d\sigma &\ll \ell(t)^{-1}t^{\beta-1}\int_a^\infty \sigma^{-2\beta'}\,d\sigma \\ &\ll \ell(t)^{-1}t^{\beta-1}a^{-(2\beta'-1)},\end{aligned}$$

and shrinking ϵ if necessary so that $\epsilon<2(\beta-\beta')$,

$$\begin{aligned}J_3'' &= \ell(t)^{-1}t^{\beta-1}\int_a^t[\ell(t)/\ell(\sigma/t)]^2[\ell(t/\sigma^{\epsilon/\beta})/\ell(t)]\sigma^{-(2\beta-\epsilon)}\,d\sigma \\ &\ll \ell(t)^{-1}t^{\beta-1}\int_a^\infty\sigma^{-2\beta'}\,d\sigma \ll \ell(t)^{-1}t^{\beta-1}a^{-(2\beta'-1)},\end{aligned}$$

as required. ∎

Proposition 6.5 *For any $\epsilon\in(0,\beta)$, there exist $\theta\in(0,1)$, $m\ge 1$, such that $\int_1^\infty e^{ibt}\hat\rho_{v,w}(ib)\,db = O(t^{-(\beta-\epsilon)})$, for all $v\in F_\theta(\tilde Y)$, $w\in L^{\infty,m}(\tilde Y)$.*

Proof Choose $m>2A+\epsilon+1$. By Proposition 3.7, $\hat\rho_{v,w}(s)=\hat P_m(s)+\hat H_m(s)$, where $\hat P_m(s)$ is a linear combination of s^{-j}, $j=1,\dots,m$, and $\hat H_m(s)=s^{-m}\hat\rho_{v,\partial_t^m w}(s)$.

Since $\hat{P}_m$ is analytic on $\operatorname{Im} s \ge 1$, we can write

$$i\int_1^\infty e^{ibt}\hat{P}_m(ib)\,db = -\int_0^1 e^{(i-a)t}\hat{P}_m(i-a)\,da + i\int_1^\infty e^{(-1+ib)t}\hat{P}_m(-1+ib)\,db$$
$$= O(t^{-1}) + O(e^{-t}) = O(t^{-1}).$$

It remains to estimate the contribution from $\hat{H}_m(ib) = b^{-m}\hat{\rho}_{v,\partial_t^m w}(ib)$. Modifying the proof of Proposition 6.4 (using the fact that $b \mapsto \hat{\rho}_{v,\partial_t^m w}(ib)$ satisfies the other estimates in Corollary 5.10(b,c)), we have that for any $\epsilon > 0$ and any $\epsilon' > \epsilon$,

$$|\hat{H}_m(ib_1) - \hat{H}_m(ib_2)| \le b_2^{-(m-2A-\epsilon')}|b_1 - b_2|^{\beta-\epsilon}\|v\|_\theta|\partial_t^m w|_\infty.$$

Hence,

$$\Big|2\int_1^\infty e^{ibt}\hat{H}_m(ib)\,db\Big| \le \int_1^\infty |\hat{H}_m(ib) - \hat{H}_m(i(b-\pi/t))|\,db$$
$$+\int_1^{1+\pi/t} |\hat{H}_m(i(b-\pi/t))|\,db$$
$$\ll t^{-(\beta-\epsilon)}\int_1^\infty b^{-(m-2A-\epsilon')}\,db + O(t^{-1}) = O(t^{-(\beta-\epsilon)}),$$

as required. ∎

Proof of Theorem 2.2(a) Combining Propositions 6.3, 6.4 and 6.5 (with $\epsilon < 2\beta - 1$),

$$\lim_{t\to\infty} \ell(t)t^{1-\beta}\int_0^\infty e^{ibt}\hat{\rho}_{v,w}(ib)\,db = c_\beta^{-1}\int_0^a e^{i\sigma}\sigma^{-\beta}\,d\sigma\int_{\tilde{Y}} v\,d\tilde{\mu}\int_{\tilde{Y}} w\,d\tilde{\mu} + O(a^{-(2\beta'-1)}).$$

Since a is arbitrary and $\beta' > 1/2$,

$$\lim_{t\to\infty} \ell(t)t^{1-\beta}\int_0^\infty e^{ibt}\hat{\rho}_{v,w}(ib)\,db = c_\beta^{-1}\int_0^\infty e^{i\sigma}\sigma^{-\beta}\,d\sigma\int_{\tilde{Y}} v\,d\tilde{\mu}\int_{\tilde{Y}} w\,d\tilde{\mu}.$$

A standard calculation shows that $\operatorname{Re}(c_\beta^{-1}\int_0^\infty e^{i\sigma}\sigma^{-\beta}\,d\sigma) = \sin\beta\pi$. Hence the result follows from Proposition 6.2. ∎

6.2 The case $\beta = 1$

We sketch the differences for $\beta = 1$. Here $\mu(\varphi > t) = \ell(t)t^{-1}$ where $\int_1^\infty \ell(t)t^{-1}\,dt = \infty$. By Karamata's Theorem, $\tilde{\ell}(t) = \int_1^t \ell(s)s^{-1}\,ds$ is slowly varying and $\ell(t)/\tilde{\ell}(t) \to 0$ as $t \to \infty$. In particular, $\tilde{\ell}$ is monotone increasing and $\lim_{t\to\infty}\tilde{\ell}(t) = \infty$.

Many of the basic estimates change in a mild way. The estimates on the imaginary axis ($s = ib$) in Section 5 are unchanged except that all occurrences of $\ell(1/b)$ on the right-hand-side are replaced by $\tilde{\ell}(1/b)$.

The major alteration is that $\hat{\rho}(ib)$ is not integrable near zero. As in [28, Section 6], we replace $\int_{-\infty}^\infty e^{ibt}\hat{\rho}(ib)\,db$ by the expression

$$\int_{-\infty}^\infty e^{ibt}\operatorname{Re}\hat{\rho}(ib)\,db = 2\int_0^\infty \cos bt\ \operatorname{Re}\hat{\rho}(ib)\,db.$$

In addition to the modified estimates for $\hat\rho(ib)$ (which are inherited by $\operatorname{Re}\hat\rho(ib)$), we have the improved asymptotics

$$\operatorname{Re}\hat\rho(ib) \sim \frac{\pi}{2}\ell(1/b)\tilde\ell(1/b)^{-2}b^{-1}\int_{\tilde Y} v\,d\tilde\mu\int_{\tilde Y} w\,d\tilde\mu \text{ as } b\to 0^+,$$

from which it follows that

$$\lim_{t\to\infty}\tilde\ell(t)\int_0^{a/t}\cos tb\,\operatorname{Re}\hat\rho(ib)\,db = \frac{\pi}{2}\int_{\tilde Y} v\,d\tilde\mu\int_{\tilde Y} w\,d\tilde\mu.$$

We omit the details of these last two assertions which follow from straightforward modifications of the calculations for $\beta\in(\frac12,1)$ (cf. [28, Section 6]). It now follows exactly as in Subsection 6.1 that

$$\lim_{t\to\infty}\tilde\ell(t)\int_0^{\infty}\cos tb\,\operatorname{Re}\hat\rho(ib)\,db = \frac{\pi}{2}\int_{\tilde Y} v\,d\tilde\mu\int_{\tilde Y} w\,d\tilde\mu.$$

Hence to prove Theorem 2.2(a) for $\beta=1$, it remains to prove the following result.

Proposition 6.6 $\rho(t)=\frac{1}{\pi}\int_{-\infty}^{\infty}e^{ibt}\operatorname{Re}\hat\rho(ib)\,db.$

Proof Write $s=a+ib$. Define $g:\mathbb{R}\to\mathbb{R}$ to be even with $g(t)=e^{-at}\rho(t)$ for $t>0$. Then $\operatorname{Re}\hat\rho(s)=\frac12\int_{-\infty}^{\infty}e^{-ib\tau}g(\tau)\,d\tau$. By the Fourier inversion formula,

$$\int_{-\infty}^{\infty}e^{ibt}\operatorname{Re}\hat\rho(s)\,db=\frac12\int_{-\infty}^{\infty}e^{ibt}\Big\{\int_{-\infty}^{\infty}e^{-ib\tau}g(\tau)\,d\tau\Big\}\,db=\pi g(t).$$

Hence, restricting to $t>0$,

$$\rho(t)=e^{at}g(t)=\frac{1}{\pi}\int_{-\infty}^{\infty}e^{st}\operatorname{Re}\hat\rho(s)\,db=\frac{1}{\pi}\int_{\Gamma_1}e^{st}\operatorname{Re}\hat\rho(s)\,ds,$$

where Γ_1 is the contour $\operatorname{Re}s=1$ traversed upwards.

As in Proposition 6.2, we can move the contour to the contour Γ_0 consisting again of two segments of the imaginary axis and a semicircle Γ_δ of radius δ around the origin. By [28, Proposition 6.1], $|\int_{-\delta}^{\delta}e^{ibt}\hat\rho(ib)\,db|\ll\int_0^{\delta}\ell(1/b)\tilde\ell(1/b)^{-2}b^{-1}\,db\ll\tilde\ell(1/\delta)^{-1}\to 0$ as $\delta\to 0$. Using the estimates in [28, Lemma 6.4], it can be shown that $|\int_{\Gamma_\delta}e^{ibt}\hat\rho(ib)\,db|\to 0$ as $\delta\to 0$. Hence the contour can be moved to the imaginary axis completing the proof. ∎

7 Second order asymptotics and rates of mixing

In this section, we prove Theorem 2.2(b). Choose $\delta>0$ such that $\lambda(ib)$ is well defined for $b\in(0,\delta)$.

Proposition 7.1 *There are constants $e_1,e_2\in\mathbb{C}$ such that $1-\lambda(ib)=e_1b^{\beta}(1-e_2b^{1-\beta}+O(b^{q-\beta}))$ for $b\in(0,\delta)$.*

Proof From the proof of Lemma 5.5, we recall that $\lambda(ib) = \int_Y e^{-ib\varphi}\, d\mu + V(ib)$, where $|V(ib)| = O(b^{2\beta-\epsilon})$. Here, $\epsilon > 0$ is arbitrarily small so $|V(ib)| = O(b^q)$. By [28, Lemma 3.2], $\int_Y e^{-ib\varphi}\, d\mu = 1 - e_1 b^\beta + e_1 e_2 b + O(b^q)$ and the result follows. (Note that much of the proof of [28, Lemma 3.2] is not required. The functions H and H_1 introduced there coincide in the continuous time case. Moreover, the four terms involving v_θ^s in [28, Lemma 3.2] have been subsumed into the $V(ib)$ term.) ∎

Corollary 7.2 *There are constants $c_j \in \mathbb{C}$ such that*

$$\hat\rho(ib) = \sum_j c_j b^{-((j+1)\beta-j)} \int_{\tilde Y} v\, d\tilde\mu \int_{\tilde Y} w\, d\tilde\mu + O(b^{-(2\beta-q)})\|v\|_\theta\, |w|_\infty,$$

for $b \in (0,\delta)$, $v \in F_\theta(\tilde Y)$, $w \in L^\infty(\tilde Y)$, where the sum is over those $j \geq 0$ with $(j+1)\beta - j \geq 2\beta - q$.

Proof Recall that

$$(I - \hat R_0(ib))^{-1} = (1-\lambda(ib))^{-1}P(0) - (1-\lambda(ib))^{-1}(P(ib)-P(0)) + (I - \hat R_0(ib))^{-1}Q(ib),$$

for $b \in (0,\delta)$. It follows from Proposition 7.1 that

$$(1-\lambda(ib))^{-1} = \sum_j c_j b^{-((j+1)\beta-j)} + O(b^{-(2\beta-q)}),$$

for constants $c_0, c_1, \ldots \in \mathbb{C}$. By Proposition 3.5(a), $\|(I - \hat R_0(ib))^{-1}Q(ib)\|_\theta = O(1)$. By Lemma 5.2, $\|\hat R_0(ib) - R\|_\theta \ll b^\beta$ and it follows that $\|P(ib) - P(0)\|_\theta \ll b^\beta$. Hence

$$(I - \hat R_0(ib))^{-1}v_0 = \sum_j c_j b^{-((j+1)\beta-j)} \int_Y v_0\, d\mu + O(b^{-(2\beta-q)})\|v_0\|_\theta,$$

for all $v_0 \in F_\theta(Y)$. By Lemma 5.8(a),

$$\hat U(0)(I - \hat R(ib))^{-1}v = \sum_j c_j b^{-((j+1)\beta-j)} \int_{\tilde Y} v\, d\tilde\mu + O(b^{-(2\beta-q)})\|v\|_\theta,$$

By Lemma 5.8(b), $\|\hat U(ib) - \hat U(0)\|_\theta \ll b^\beta$ and so

$$\hat T(ib)v = \hat U(ib)(I - \hat R(ib))^{-1}v = \sum_j c_j b^{-((j+1)\beta-j)} \int_{\tilde Y} v\, d\tilde\mu + O(b^{-(2\beta-q)})\|v\|_\theta.$$

The result follows from Proposition 3.2. ∎

Proof of Theorem 2.2(b) By Proposition 6.4, for all $\beta' \in (\frac12, \beta)$, $\int_{a/t}^1 e^{ibt}\hat\rho_{v,w}(ib)\, db = O(t^{-(1-\beta)}a^{-(2\beta'-1)})$ where $\beta - \beta'$ is arbitrarily small.

A calculation (see for example [28, Proposition 9.5]) shows that $\int_0^{a/t} b^{-((j+1)\beta-j)}e^{-itb}db = \text{const}\, t^{-(j+1)(1-\beta)}(1 + O(a^{-((j+1)\beta-j)}))$. Also, $2\beta - q \in (0,1)$ so that $\int_0^{a/t} b^{-(2\beta-q)}\, db = O((a/t)^{1-2\beta+q})$.

Choosing $a = t^{1-q^{-1}\beta-\epsilon'}$, we obtain from Corollary 7.2 that

$$\int_0^1 e^{ibt}\hat\rho_{v,w}(ib)\,db = \sum_j d_j t^{-j(1-\beta)} \int_{\tilde Y} v\,d\tilde\mu \int_{\tilde Y} w\,d\tilde\mu + O(t^{-\beta(1-q^{-1}(2\beta-1)-\epsilon)}).$$

Also, by Proposition 6.5, $\int_1^\infty e^{ibt}\hat\rho_{v,w}(ib)\,db = O(t^{-(\beta-\epsilon)})$. Hence the result follows from Proposition 6.2. ∎

Part III
Finite measure systems

In this part of the paper, we prove our main results in the finite measure context. Throughout, we continue to assume the setup from Section 2, so $F : Y \to Y$ is a full branch Gibbs-Markov map and $\varphi : Y \to \mathbb{Z}^+$ is a roof function satisfying assumptions (A1) and (A2). In addition, we make the standing assumption throughout this part of the paper that $\mu(\varphi > t) = O(t^{-\beta})$ where $\beta > 1$.

In Section 8, we decompose the family of operators $\hat T_0(s)$ into various pieces and formulate Lemmas 8.4, 8.5, 8.6 that provide estimates for each of the pieces. Theorem 2.4(a) thereby reduces to proving these lemmas.

In Section 9, we prove Lemma 8.4. Section 10 contains an operator-theoretic estimate, and in Section 11, we prove Lemma 8.5. Section 12 contains estimates on derivatives of various families of operators. In Section 13, we derive a continuous time version of the "first main lemma" that was crucial in [17, 33]. In Section 14, we prove Lemma 8.6 completing the proof of Theorem 2.4(a). In Section 15, we prove Theorem 2.4(b).

In this part of the paper, $k \ge 1$ is fixed but chosen sufficiently large. All implied constants are allowed to depend on k unless stated otherwise.

8 Decomposition for $\hat T_0$

Let $k \ge 1$ and define $\varphi^* = \varphi \wedge k$, $\bar\varphi^* = \int_Y \varphi^*\,d\mu$. Recall that $P(0) : L^1(Y) \to L^1(Y)$ is the projection $P(0)v = \int_Y v\,d\mu$ and define

$$P_\varphi = (1/\bar\varphi)P(0), \quad P^*_\varphi = (1/\bar\varphi^*)P(0), \quad \hat R^*_0(s)v = R(e^{-s\varphi^*}v),$$
$$\hat T^*_0(s) = (I - \hat R^*_0(s))^{-1} = s^{-1}P^*_\varphi + \hat H^*(s), \quad \hat B(s) = s\hat T_0(s).$$

Also define

$$\hat C^*(s) = s^{-1}(\hat R^*_0(s) - \hat R_0(s)), \quad \hat D^*(s) = \hat C^*(0) - \hat C^*(s).$$

We note that $\hat C^*(0)v = R((\varphi - \varphi^*)v)$ and hence $\int_Y \hat C^*(0)1_Y\,d\mu = \bar\varphi - \bar\varphi^*$.

Proposition 8.1 *Let $c_2 = (C_2+1)^{-1}$ where C_2 is as in (A1). If $a \in \alpha$ such that $|1_a\varphi|_\infty > t$, then $\varphi(y) > c_2 t$ for all $y \in a$.*

Proof Choose $y_0 \in a$ with $\varphi(y_0) > t$. By assumption (A1), for all $y \in a$,

$$t - \varphi(y) < \varphi(y_0) - \varphi(y) \le |1_a\varphi|_{\theta_0} d_{\theta_0}(y, y_0) < |1_a\varphi|_{\theta_0} \le C_2\varphi(y),$$

and the result follows. ∎

Proposition 8.2 *$\|\hat{C}^*(s)\|_\infty \le C_1 \int_{\{\varphi > c_2 k\}} \varphi \, d\mu$ for $s \in \overline{\mathbb{H}}$. In particular, if $\varphi \in L^1$, then $\hat{C}^*$ extends continuously to $\overline{\mathbb{H}}$ and $\|\hat{C}^*(s)\|_\infty \to 0$ as $k \to \infty$ uniformly on $\overline{\mathbb{H}}$.*

Proof Recalling (4.1), we have

$$(\hat{C}^*(s)v)(y) = s^{-1} \sum_{a \in \alpha : |1_a\varphi|_\infty \ge k} e^{g(y_a)} v(y_a)(e^{-s\varphi^*(y_a)} - e^{-s\varphi(y_a)}), \tag{8.1}$$

so by (2.1) and Proposition 8.1,

$$\begin{aligned} |(\hat{C}^*(s)v)(y)| &\le C_1 |v|_\infty \sum_{a : |1_a\varphi|_\infty \ge k} \mu(a)(\varphi(y_a) - \varphi^*(y_a)) \\ &\le C_1 |v|_\infty \sum_{a : |1_a\varphi|_\infty \ge k} \mu(a)\varphi(y_a) \le C_1 |v|_\infty \int_{\{\varphi > c_2 k\}} \varphi \, d\mu, \end{aligned}$$

as required. ∎

Proposition 8.3 *For k sufficiently large, $\hat{T}_0(s) = \hat{T}_{0,1}(s) + \hat{T}_{0,2}(s) + \hat{T}_{0,3}(s) + \hat{T}_{0,4}(s)$ for all $s \in \mathbb{H}$, where*

$$\hat{T}_{0,1}(s) = s^{-1}P_\varphi, \qquad \hat{T}_{0,2}(s) = s^{-1}P_\varphi \hat{D}^* P_\varphi, \qquad \hat{T}_{0,3}(s) = s^{-1}(I - P_\varphi \hat{D}^*)^{-1}(P_\varphi \hat{D}^*)^2 P_\varphi,$$
$$\hat{T}_{0,4}(s) = (I - P_\varphi \hat{D}^*)^{-1}(I - P_\varphi \hat{C}^*(0))\hat{H}^*(I - \hat{C}^*\hat{B}).$$

Proof Since $\int_Y \hat{C}^*(0)1_Y \, d\mu = \bar{\varphi} - \bar{\varphi}^*$, it follows that

$$(I - P_\varphi \hat{C}^*(0))P_\varphi^* = P_\varphi, \quad (I - P_\varphi \hat{C}^*(0))(I + P_\varphi^* \hat{C}^*) = I - P_\varphi \hat{D}^*. \tag{8.2}$$

Using the identity $\hat{T}_0 = \hat{T}_0^* - \hat{T}_0^*(\hat{R}_0^* - \hat{R}_0)\hat{T}_0$, it follows that

$$\begin{aligned} \hat{T}_0 = \hat{T}_0^* - \hat{T}_0^* \hat{C}^* \hat{B} &= s^{-1}P_\varphi^* + \hat{H}^* - s^{-1}P_\varphi^* \hat{C}^* \hat{B} - \hat{H}^* \hat{C}^* \hat{B} \\ &= s^{-1}P_\varphi^* - P_\varphi^* \hat{C}^* \hat{T}_0 + \hat{H}^*(I - \hat{C}^* \hat{B}). \end{aligned}$$

Hence $(I + P_\varphi^* \hat{C}^*)\hat{T}_0 = s^{-1}P_\varphi^* + \hat{H}^*(I - \hat{C}^*\hat{B})$. Multiplying throughout by $(I - P_\varphi \hat{C}^*(0))$, and using (8.2), we obtain

$$(I - P_\varphi \hat{D}^*)\hat{T}_0 = s^{-1}P_\varphi + (I - P_\varphi \hat{C}^*(0))\hat{H}^*(I - \hat{C}^*\hat{B}).$$

For k sufficiently large, we can invert $I - P_\varphi \hat{D}^*$ by Proposition 8.2 and the result follows. ∎

Substituting into (3.2), we obtain that $\hat\rho(s)=\sum_{i=1}^4\hat\rho_i(s)$, where

$$\hat\rho_i(s)=\int_{\tilde Y}\hat U(s)\hat T_i(s)v\,w\,d\mu^\varphi=(1/\bar\varphi)\int_{\tilde Y}\hat U(s)\hat T_i(s)v\,w\,d\tilde\mu, \tag{8.3}$$
$$(\hat T_i(s)v)(y,u)=(\hat T_{0,i}(s)v^u)(y),\quad i=1,2,3,4.$$

Theorem 2.4 is an immediate consequence of the next three lemmas. We recall that $\zeta(t)$ and $\xi_{\beta,\epsilon}(t)$ were defined in (2.2).

Lemma 8.4 *Suppose that $\mu(\varphi>t)=O(1/t^\beta)$ for some $\beta>1$. Then*

(a) $\rho_1(t)-\bar v\bar w=O\big(|v|_\infty|w|_\infty t^{-\beta}\big)$, *and*

(b) $\rho_2(t)=(1/\bar\varphi)\zeta(t)\bar v\bar w+O\big(v|_\infty|w|_\infty t^{-\beta}\big)$,

for all $v,w\in L^\infty(\tilde Y)$, $t>0$, $k\ge1$.

Lemma 8.5 *Suppose that $\mu(\varphi>t)=O(1/t^\beta)$ for some $\beta>1$. Then for any $\epsilon>0$ and for all k sufficiently large, there is a constant $C>0$ such that*

$$|\rho_3(t)|\le C|v|_\infty|w|_\infty\xi_{\beta,\epsilon}(t),$$

for all $v,w\in L^\infty(\tilde Y)$, $t>0$.

Lemma 8.6 *Assume conditions (A1) and (A2) and suppose that $\mu(\varphi>t)=O(t^{-\beta})$ where $\beta>1$. Then for any $\epsilon>0$ and for all k sufficiently large, there exists $\theta\in(0,1)$, $m\ge1$, $C>0$, such that*

$$|\rho_4(t)|\le C\|v\|_\theta|w|_{\infty,m}t^{-(\beta-\epsilon)},$$

for all $v\in F_\theta(\tilde Y)$, $w\in L^{\infty,m}(\tilde Y)$, $t>0$.

9 Proof of Lemma 8.4

We require the following preliminary result.

Proposition 9.1 *Let $\hat r(s)=s^{-1}\int_{\tilde Y}\hat U(s)v\,w\,d\tilde\mu$. Then*

$$r(t)=\int_{\tilde Y}\int_0^u v(y,\tau)\,d\tau\,w(y,u)\,d\tilde\mu+\int_{\tilde Y}\int_u^1 v(y,\tau)\,d\tau\,w(Fy,u)\,d\tilde\mu+O(|v|_\infty|w|_\infty\mu(\varphi>t)).$$

Proof By Proposition 3.4,

$$s^{-1}(\hat U(s)v)(y,u)=s^{-1}\int_0^1 e^{-s\tau}v(y,u-\tau)1_{[\tau,1]}(u)\,d\tau+s^{-1}\int_1^\infty e^{-s\tau}(\tilde Rv_\tau)(y,u)\,d\tau,$$

with inverse Laplace transform

$$\int_0^u 1_{[\tau,\infty)}(t)v(y,u-\tau)\,d\tau+\int_1^\infty 1_{[\tau,\infty)}(t)(\tilde Rv_\tau)(y,u)\,d\tau.$$

Hence $r(t) = r_1(t) + r_2(t)$ where

$$r_1(t) = \int_{\tilde{Y}} \int_0^u 1_{[\tau,\infty)}(t) v(y, u-\tau)\, d\tau\, w\, d\tilde{\mu}, \qquad r_2(t) = \int_{\tilde{Y}} \int_1^\infty 1_{[\tau,\infty)}(t)(\tilde{R}v_\tau)(y,u)\, d\tau\, w\, d\tilde{\mu}.$$

For $t > 1$,

$$r_1(t) = \int_{\tilde{Y}} \int_0^u v(y, u-\tau)\, d\tau\, w\, d\tilde{\mu} = \int_{\tilde{Y}} \int_0^u v(y,\tau)\, d\tau\, w\, d\tilde{\mu}.$$

Also,

$$\begin{aligned} r_2(t) &= \int_{\tilde{Y}} \int_1^\infty 1_{[\tau,\infty)}(t) 1_{\{\tau<\varphi<\tau+1-u\}} v(y, u-\tau+\varphi)\, d\tau\, w \circ \tilde{F}\, d\tilde{\mu} \\ &= \int_{\tilde{Y}} \int_{\varphi-1+u}^{\varphi} 1_{[\tau,\infty)}(t) v(y, u-\tau+\varphi)\, d\tau\, w \circ \tilde{F}\, d\tilde{\mu} \\ &= \int_{\tilde{Y}} \int_u^1 1_{[u-\tau+\varphi,\infty)}(t) v(y,\tau)\, d\tau\, w \circ \tilde{F}\, d\tilde{\mu} \\ &= \int_{\tilde{Y}} \int_u^1 v(y,\tau)\, d\tau\, w \circ \tilde{F}\, d\tilde{\mu} - E(t), \end{aligned}$$

where

$$E(t) = \int_{\tilde{Y}} \int_u^1 1_{[0,u-\tau+\varphi]}(t) v(y,\tau)\, d\tau\, w \circ \tilde{F}\, d\tilde{\mu} = \int_{\tilde{Y}} \int_u^1 1_{\{\varphi>t+\tau-u\}} v(y,\tau)\, d\tau\, w \circ \tilde{F}\, d\tilde{\mu}.$$

Finally, note that $|E(t)| \le |v|_\infty |w|_\infty \mu(\varphi > t)$. ∎

Proof of Lemma 8.4(a) We have $\hat{\rho}_1(s) = (1/\bar{\varphi}) \int_{\tilde{Y}} \hat{U}(s)\hat{T}_1(s) v\, w\, d\tilde{\mu}$, where

$$(\hat{T}_1(s)v)(y,u) = (\hat{T}_{0,1}(s)v^u)(y) = (1/\bar{\varphi}) s^{-1} \int_Y v^u\, d\mu.$$

By Proposition 9.1,

$$\begin{aligned} \rho_1(t) = (1/\bar{\varphi})^2 & \int_{\tilde{Y}} \int_0^u \Big(\int_Y v^\tau\, d\mu \Big)\, d\tau\, w(y,u)\, d\tilde{\mu} \\ & + (1/\bar{\varphi})^2 \int_{\tilde{Y}} \int_u^1 \Big(\int_Y v^\tau\, d\mu \Big)\, d\tau\, w(Fy,u)\, d\tilde{\mu} + O(\mu(\varphi > t)). \end{aligned}$$

Note that $\int_Y v^\tau\, d\mu$ is independent of y and $\tilde{F}$ acts trivially on the second coordinate, so the second term reduces to

$$(1/\bar{\varphi})^2 \int_{\tilde{Y}} \Big\{ \int_u^1 \Big(\int_Y v^\tau\, d\mu \Big)\, d\tau\, w \Big\} \circ \tilde{F}\, d\tilde{\mu} = (1/\bar{\varphi})^2 \int_{\tilde{Y}} \int_u^1 \Big(\int_Y v^\tau\, d\mu \Big)\, d\tau\, w\, d\tilde{\mu}.$$

Hence

$$\rho_1(t) = (1/\bar{\varphi}) \int_{\tilde{Y}} \int_0^1 \Big(\int_Y v^\tau\, d\mu \Big)\, d\tau\, w(y,u)\, d\mu^\varphi + O(\mu(\varphi > t)).$$

But

$$\int_0^1 \Big(\int_Y v^\tau\, d\mu\Big)\, d\tau = \int_Y \int_0^1 v(y,\tau)\, d\tau\, d\mu = \int_{\tilde Y} v\, d\tilde\mu = \bar\varphi \int_{\tilde Y} v\, d\mu^\varphi,$$

and the result follows. ∎

Recall that $C^*(t)$ is the inverse Laplace transform of $\hat C^*(s)$.

Proposition 9.2 *$\int_Y \hat C^*(0)1_Y\, d\mu = \bar\varphi - \bar\varphi^* = \zeta(k)$ and $\int_Y C^*(t)1_Y\, d\mu = 1_{\{t>k\}}\mu(\varphi > t)$.*

Proof Let $G(x) = \mu(\varphi < x)$ denote the distribution function of φ. For the first statement,

$$\begin{aligned}\int_Y \hat C^*(0)1_Y\, d\mu &= \int_Y (\varphi - \varphi^*)\, d\mu = \int_0^\infty x\, dG - \int_0^k x\, dG^* = \int_k^\infty x\, dG - k\mu(\varphi > k) \\ &= -\int_k^\infty x\, d(1 - G(x)) - k\mu(\varphi > k) \\ &= -x(1 - G(x))\Big|_{x=k}^{x=\infty} + \int_k^\infty (1 - G(x))\, dx - k\mu(\varphi > k) = \int_k^\infty \mu(\varphi > x)\, dx.\end{aligned}$$

For the second statement, $\int_Y \hat C^*(s)1_Y\, d\mu = \int_Y s^{-1}(e^{-s\varphi^*} - e^{-s\varphi})\, d\mu$, so

$$\begin{aligned}\int_Y C^*(t)1_Y\, d\mu &= \int_Y (1_{[\varphi^*,\infty)}(t) - 1_{[\varphi,\infty)}(t))\, d\mu = \mu(\varphi^* < t) - \mu(\varphi < t) \\ &= \mu(\varphi > t) - \mu(\varphi^* > t) = \mu(\varphi > t) - \mu(\varphi \wedge k > t).\end{aligned}$$

But $\mu(\varphi \wedge k > t) = 0$ if $k < t$ and $\mu(\varphi \wedge k > t) = \mu(\varphi > t)$ if $k > t$. ∎

In the next proof, $a \star b$ denotes the convolution $(a \star b)(t) = \int_0^t a(\tau)b(t-\tau)\, d\tau$ of real-valued functions of $t \in [0,\infty)$. (In subsequent sections, we speak also of the convolution of operator-valued functions of t.)

Proof of Lemma 8.4(b) We have $\hat\rho_2(s) = (1/\bar\varphi)\int_{\tilde Y} \hat U(s)\hat T_2(s)v\, w\, d\tilde\mu$, where

$$\begin{aligned}(\hat T_2(s)v)(y,u) = (\hat T_{0,2}(s)v^u)(y) &= (1/\bar\varphi)^2 s^{-1} P\hat D^*(s)P(0)v^u \\ &= (1/\bar\varphi)^2 s^{-1} \int_Y \hat D^*(s)1_Y\, d\mu \int_Y v^u\, d\mu.\end{aligned}$$

Comparing with the proof of part (a), we observe that

$$\bar\varphi\hat\rho_2(s) = \hat\rho_1(s)\int_Y \hat D^*(s)1_Y\, d\mu = \hat\rho_1(s)\int_Y \hat C^*(0)1_Y\, d\mu - \hat\rho_1(s)\int_Y \hat C^*(s)1_Y\, d\mu.$$

By Proposition 9.2, for $t > k$,

$$\begin{aligned}\bar\varphi\rho_2(t) &= \zeta(k)\,\rho_1(t) - (1_{\{t>k\}}\mu(\varphi > t)) \star (\rho_1(t)) \\ &= \zeta(k)\big(\bar v\bar w + O(\mu(\varphi > t))\big) - \int_k^t \mu(\varphi > \tau)\rho_1(t-\tau)\, d\tau \\ &= \zeta(k)\bar v\bar w + O\big(\zeta(k)\mu(\varphi > t)\big) - \bar v\bar w(\zeta(k) - \zeta(t)) \\ &\qquad + O\Big(\int_k^t \mu(\varphi > \tau)\mu(\varphi > t-\tau)\, d\tau\Big) \\ &= \zeta(t)\bar v\bar w + O\big(\zeta(k)\mu(\varphi > t)\big) + O\big(\mu(\varphi > t) \star \mu(\varphi > t)\big).\end{aligned}$$

The result follows. ∎

10 An estimate for $(I - P_\varphi \hat{D}^*)^{-1}$

In Proposition 8.2, we showed that for k sufficiently large the family of operators $(I - P_\varphi \hat{D}^*(s))^{-1}$ on $L^\infty(Y)$ is analytic on $\mathbb{H}$ with a continuous extension to $\overline{\mathbb{H}}$. Moreover, $\|\hat{D}^*\|_\infty$ is uniformly small on $\overline{\mathbb{H}}$ for k large.

In this section, we obtain an estimate on the decay of its inverse Laplace transform. This is required in a diluted form in Section 11 and in its full strength in Section 14.

Let $\mathcal{B}$ be a Banach space and suppose that $S : [0,\infty) \to \mathcal{B}$ lies in L^1 with Laplace transform $\hat{S} : \overline{\mathbb{H}} \to \mathcal{B}$. We write $\hat{S} \in \mathcal{R}(a(t))$ if $\|S(t)\| \le Ca(t)$ for all $t \ge 0$.

Proposition 10.1 *$\|C^*(t)\|_\infty \le C_1 1_{\{t\ge k\}}\mu(\varphi > c_2 t)$, for all $k \ge 1$. In particular, $\hat{C}^* \in \mathcal{R}(\mu(\varphi > c_2 t))$.*

Proof Starting from formula (8.1) for $\hat{C}^*$, the inverse Laplace transform is given by

$$(C^*(t)v)(y) = \sum_{a\in\alpha} e^{g(y_a)} v(y_a) 1_{[\varphi^*(y_a),\varphi(y_a)]}(t) = \sum_{a\in\alpha} e^{g(y_a)} v(y_a) 1_{\{\varphi(y_a)>k\}} 1_{[k,\varphi(y_a)]}(t)$$

$$= 1_{\{t\ge k\}} \sum_{a\in\alpha} e^{g(y_a)} v(y_a) 1_{[0,\varphi(y_a)]}(t).$$

Hence by Proposition 8.1,

$$|(C^*(t)v)(y)| \le C_1 1_{\{t\ge k\}} |v|_\infty \sum_{a\in\alpha:|1_a\varphi|_\infty > t} \mu(a) \le C_1 1_{\{t\ge k\}} |v|_\infty \mu(\varphi > c_2 t),$$

as required. ∎

Remark 10.2 Since $\hat{D}^*(s) = \hat{C}^*(0) - \hat{C}^*(s)$, it follows from Proposition 10.1 that formally we have $D^*(t) = C^*(0)\delta_0(t) - C^*(t)$ where $\|C^*(t)\|_\infty \le C_1 1_{\{t\ge k\}}\mu(\varphi > c_2 t)$. To avoid such formal expressions, we restrict to estimating expressions like $\hat{D}^*(s)\hat{E}(s)$ where $\hat{E}(s)$ has no constant terms.

Corollary 10.3 *Let $\mathcal{B}$ be a Banach space. Let $\beta > 1$. Suppose that $\mu(\varphi > t) = O(1/t^\beta)$ and that $\hat{E} : \mathcal{B} \to L^\infty(Y)$ lies in $\mathcal{R}(1/t^\beta)$. Then $\hat{D}^*(s)\hat{E}(s) : \mathcal{B} \to L^\infty(Y)$ lies in $\mathcal{R}(1/t^\beta)$.*

Proof We have $\hat{D}^*(s)\hat{E}(s) = \hat{C}^*(0)\hat{E}(s) - \hat{C}^*(s)\hat{E}(s)$ with inverse Laplace transform $\hat{C}^*(0)E(t) - (C^* \star E)(t) \in \mathcal{R}(1/t^\beta)$. ∎

Proposition 10.4 *Let $\mathcal{B}$ be a Banach space. Let $\beta > 1$, $\epsilon > 0$ such that $\beta - \epsilon > 1$. Suppose that $\mu(\varphi > t) = O(1/t^\beta)$ and that $\hat{E} : \mathcal{B} \to L^\infty(Y)$ lies in $\mathcal{R}(1/t^{\beta-\epsilon})$. Then for k sufficiently large, $(I - P_\varphi\hat{D}^*(s))^{-1}\hat{E}(s) : \mathcal{B} \to L^\infty(Y)$ lies in $\mathcal{R}(1/t^{\beta-\epsilon})$.*

Proof By Proposition 8.2, we can choose k so large that $\|P_\varphi\hat{C}^*(s)\|_\infty \le \frac{1}{3}$ for all $s \in \overline{\mathbb{H}}$ and hence we can write

$$\hat{Q}(s) = (1 - P_\varphi\hat{D}^*(s))^{-1}\hat{E}(s) = \sum_{n=0}^\infty \hat{Q}_n(s), \quad \hat{Q}_n(s) = (P_\varphi\hat{C}^*(0) - P_\varphi\hat{C}^*(s))^n\hat{E}(s).$$

Given $\hat{S} \in \mathcal{R}(1/t^\beta)$, $\beta > 1$, we define $\|\hat{S}\|_{\mathcal{R}_\beta} = \int_0^\infty \|S(t)\|_\infty \, dt + \sup_{t\ge 0} \|S(t)\|_\infty t^\beta$. This makes $\mathcal{R}(1/t^\beta)$ into a Banach algebra under composition and we can rescale the norm so that $\|\hat{S}_1\hat{S}_2\|_{\mathcal{R}_\beta} \le \|\hat{S}_1\|_{\mathcal{R}_\beta}\|\hat{S}_2\|_{\mathcal{R}_\beta}$. In particular,

$$\begin{aligned}\|\hat{Q}_n\|_{\mathcal{R}_{\beta-\epsilon}} &\le \sum_{j=0}^n \binom{n}{j} \|P_\varphi \hat{C}^*(0)\|^j \big(\|P_\varphi \hat{C}^*\|_{\mathcal{R}_{\beta-\epsilon}}\big)^{n-j} \|\hat{E}\|_{\mathcal{R}_{\beta-\epsilon}} \\ &\le \sum_{j=0}^n \binom{n}{j} \Big(\frac{1}{3}\Big)^j \big(\|P_\varphi \hat{C}^*\|_{\mathcal{R}_{\beta-\epsilon}}\big)^{n-j} \|\hat{E}\|_{\mathcal{R}_{\beta-\epsilon}}.\end{aligned}$$

By Proposition 10.1,

$$\|P_\varphi \hat{C}^*\|_{\mathcal{R}_{\beta-\epsilon}} \le C_1 \int_k^\infty \mu(\varphi > c_2 t)\, dt + C_1 \sup_{t>k} \mu(\varphi > c_2 t) t^{\beta-\epsilon} \ll k^{-(\beta-\epsilon-1)} + k^{-\epsilon},$$

so we can ensure that $\|P_\varphi \hat{C}^*\|_{\mathcal{R}_{\beta-\epsilon}} < \frac{1}{3}$ by choosing k sufficiently large. Then

$$\|\hat{Q}_n\|_{\mathcal{R}_{\beta-\epsilon}} \le \sum_{j=0}^n \binom{n}{j} \Big(\frac{1}{3}\Big)^j \Big(\frac{1}{3}\Big)^{n-j} \|\hat{E}\|_{\mathcal{R}_{\beta-\epsilon}} = \Big(\frac{2}{3}\Big)^n \|\hat{E}\|_{\mathcal{R}_{\beta-\epsilon}}.$$

Hence $\hat{Q} = (I - P_\varphi \hat{D}^*)^{-1}\hat{E} \in \mathcal{R}(1/t^\beta)$ as required. ∎

Remark 10.5 Equally we can consider products of the form $\hat{E}(s)(I - P_\varphi \hat{D}^*(s))^{-1} : L^\infty(Y) \to \mathcal{B}$ where $\hat{E} : L^\infty(Y) \to \mathcal{B}$ lies in $\mathcal{R}(1/t^{\beta-\epsilon})$ and the conclusion of Proposition 10.4 is unchanged.

11 Proof of Lemma 8.5

We have $\hat{\rho}_3(s) = (1/\bar{\varphi}) \int_{\tilde{Y}} \hat{U}(s)\hat{T}_3(s)v\, w\, d\tilde{\mu}$, where

$$\begin{aligned}(\hat{T}_3(s)v)(y,u) &= (\hat{T}_{0,3}(s)v^u)(y) \\ &= (1/\bar{\varphi})^3 s^{-1}\Big(1 - (1/\bar{\varphi})\int_Y \hat{D}^*(s)1_Y d\mu\Big)^{-1}\Big(\int_Y \hat{D}^*(s)1_Y\, d\mu\Big)^2 \int_Y v^u\, d\mu.\end{aligned}$$

Proposition 11.1 *$\int_Y \hat{D}^*(s)1_Y\, d\mu$ extends continuously to $\overline{\mathbb{H}}$ and $\int_Y \hat{D}^*(s)1_Y\, d\mu = \int_k^\infty (1 - e^{-sx})\mu(\varphi > x)\, dx \le \zeta(k)$ for $s \in \overline{\mathbb{H}}$.*

Proof Let $G(x) = \mu(\varphi < x)$ and $G^*(x) = \mu(\varphi* < x)$ denote the distribution functions of φ and φ^*. Then

$$\begin{aligned} s\int_Y \hat{C}^*(s)1_Y\, d\mu &= \int_Y (e^{-s\varphi^*} - e^{-s\varphi})\, d\mu = \int_0^k e^{-sx}\, dG^* - \int_0^\infty e^{-sx}\, dG \\ &= e^{-sk}\mu(\varphi > k) - \int_k^\infty e^{-sx}\, dG = e^{-sk}\mu(\varphi > k) + \int_k^\infty e^{-sx}\, d(1 - G(x)) \\ &= e^{-sk}\mu(\varphi > k) + e^{-sx}(1 - G(x))\Big|_{x=k}^{x=\infty} + s\int_k^\infty e^{-sx}(1 - G(x))\, dx \\ &= s\int_k^\infty e^{-sx}\mu(\varphi > x)\, dx. \end{aligned}$$

Hence $\int_Y \hat{C}^*(s)1_Y\,d\mu = \int_k^\infty e^{-sx}\mu(\varphi > x)\,dx$ and the formula for $\int_Y \hat{D}^*(s)1_Y\,d\mu$ follows since $\hat{D}^*(s) = \hat{C}^*(0) - \hat{C}^*(s)$. ∎

Proposition 11.2 *Suppose that $\mu(\varphi > t) = O(1/t^\beta)$ for some $\beta > 1$. Define $\xi_\beta(t)$ as in* (1.2). *Then*

(a) $s^{-1}\int_Y \hat{D}^*(s)1_Y\,d\mu \in \mathcal{R}(1/t^{\beta-1})$.

(b) $s^{-1}(\int_Y \hat{D}^*(s)1_Y\,d\mu)^2 \in \mathcal{R}(\xi_\beta(t))$.

Proof (a) Let $\hat{q}(s) = s^{-1}\int_Y \hat{D}^*(s)1_Y\,d\mu = s^{-1}\int_Y \hat{C}^*(0)1_Y\,d\mu - s^{-1}\int_Y \hat{C}^*(s)1_Y\,d\mu$ with inverse Laplace transform $q(t) = \int_Y \hat{C}^*(0)1_Y\,d\mu + \int_0^t\int_Y C^*(\tau)1_Y\,d\mu\,d\tau$. By Proposition 9.2, for $t > k$,

$$q(t) = \zeta(k) + \int_0^t 1_{\{\tau>k\}}\mu(\varphi > \tau)\,d\tau = \zeta(k) + \int_k^t \mu(\varphi > \tau)\,d\tau = \zeta(t).$$

(b) Let $'$ denote $\frac{d}{ds}$. By Proposition 9.2, $\int_Y C^*(t)1_Y\,d\mu = 1_{\{t>k\}}\mu(\varphi > t) = O(t^{-\beta})$ and hence $\int_Y \hat{D}^{*\prime}(s)1_Y\,d\mu = -\int_Y \hat{C}^{*\prime}(s)1_Y\,d\mu \in \mathcal{R}(1/t^{\beta-1})$.

Let $\hat{q}(s) = s^{-1}(\int_Y \hat{D}^*(s)1_Y\,d\mu)^2$. Then

$$\hat{q}'(s) = -\Big(s^{-1}\int_Y \hat{D}^*(s)1_Y\,d\mu\Big)^2 + 2\Big(\int_Y \hat{D}^{*\prime}(s)1_Y\,d\mu\Big)\Big(s^{-1}\int_Y \hat{D}^*(s)1_Y\,d\mu\Big).$$

Each term is a product of two elements of $\mathcal{R}(1/t^{\beta-1})$. Hence $\hat{q}' \in \mathcal{R}(\{1/t^{\beta-1}\}\star\{1/t^{\beta-1}\})$. But $\hat{q}'(s)$ is the Laplace transform of $tq(t)$ so we obtain that

$$tq(t) = O(\{1/t^{\beta-1}\}\star\{1/t^{\beta-1}\}) = t\xi_\beta(t),$$

as required. ∎

Proposition 11.3 *$\hat{U} : L^\infty(\tilde{Y}) \to L^1(\tilde{Y})$ lies in $\mathcal{R}(\mu(\varphi > t))$.*

Proof Directly from the definition of $U(t)$, we have

$$|U(t)v|_1 = |T_t(1_{\{\tilde{\varphi}>t\}}v)|_1 = |1_{\{\tilde{\varphi}>t\}}v|_1 \le \tilde{\mu}(\tilde{\varphi} > t)|v|_\infty = \mu(\varphi > t)|v|_\infty.$$ ∎

Proof of Lemma 8.5 By Propositions 10.4 and 11.2(b), $\hat{T}_3 : L^\infty(\tilde{Y}) \to L^\infty(\tilde{Y})$ lies in $\mathcal{R}(\xi_{\beta,\epsilon}(t))$. By Proposition 11.3, $\hat{U} \in \mathcal{R}(1/t^\beta)$. Hence $\hat{\rho}_3 \in \mathcal{R}(\xi_{\beta,\epsilon}(t))$. ∎

12 Smoothness of some families of operators

Let $s \mapsto \hat{S}(s)$ be an analytic family of operators, $s \in \mathbb{H}$, such that the family extends continuously to $\overline{\mathbb{H}}$. If $p \ge 0$ is an integer, define

$$d_p\hat{S}(ib) = \max_{j=0,\dots,p}\|\hat{S}^{(j)}(ib)\|.$$

If $p > 0$ is not an integer, define

$$d_p\hat{S}(ib) = d_{[p]}\hat{S}(ib) + \sup_{h\neq 0}\|\hat{S}^{([p])}(i(b+h))) - \hat{S}^{([p])}(ib)\|/|h|^{p-[p]}.$$

Proposition 12.1 *Suppose that $\varphi \in L^p$ for some $p>0$, and let $\epsilon \in (0,p)$.*

(a) Viewed as a family of operators on $L^\infty(Y)$, $b \mapsto \hat R_0(ib)$ is C^p and there exists a constant $C>0$ such that $d_p\hat R_0(ib) \le C$ for all $b\in\mathbb{R}$.

(b) There exists $\theta\in(0,1)$ such that viewed as a family of operators on $F_\theta(Y)$, $b\mapsto \hat R_0(ib)$ is $C^{p-\epsilon}$ and $d_{p-\epsilon}\hat R_0(ib)\le C(1+|b|^\epsilon)$ for all $b\in\mathbb{R}$.

Proof (a) By (4.1),

$$(\hat R_0^{(j)}(ib)v)(y) = \sum_{a\in\alpha} e^{g(y_a)} v(y_a)(i\varphi(y_a))^j e^{ib\varphi(y_a)}$$

and hence by (2.1) and assumption (A1),

$$\begin{aligned}|(\hat R_0^{(j)}(ib)v)(y)| &\le C_1\sum_{a\in\alpha}\mu(a)|v|_\infty\varphi(y_a)^j \le C_1(C_2+1)^j|v|_\infty\sum_{a\in\alpha}\mu(a)\inf_a\varphi^j\\ &\le C_1(C_2+1)^j|v|_\infty|\varphi|_j^j.\end{aligned}$$

Also, for p not an integer,

$$(\{\hat R_0^{([p])}(i(b+h)) - \hat R_0^{([p])}(ib)\}v)(y) = \sum_{a\in\alpha} e^{g(y_a)}v(y_a)(i\varphi(y_a))^{[p]}e^{ib\varphi(y_a)}(e^{ih\varphi(y_a)}-1)$$

and hence using the inequality $|e^{ix}-1|\le|x|^\delta$ for all $x\in\mathbb{R}$, $\delta\in[0,1]$,

$$\begin{aligned}|(\{\hat R_0^{([p])}(i(b+h)) - \hat R_0^{([p])}(ib)\}v)(y)| &\le C_1\sum_{a\in\alpha}\mu(a)|v|_\infty\varphi(y_a)^{[p]}|e^{ih\varphi(y_a)}-1|\\ &\le C_1|v|_\infty\sum_{a\in\alpha}\mu(a)\varphi(y_a)^{[p]}|h|^{p-[p]}\varphi(y_a)^{p-[p]} = C_1|v|_\infty|h|^{p-[p]}\sum_{a\in\alpha}\mu(a)\varphi(y_a)^p\\ &\le C_1(C_2+1)^p|v|_\infty|h|^{p-[p]}|\varphi|_p^p.\end{aligned}$$

Hence $d_p\hat R_0(ib)\ll|\varphi|_p^p$.

(b) We give the details for p not an integer, and $\epsilon<p-[p]$. Set $\theta=\theta_0^\epsilon$. Let $j\in\{0,1,\dots,[p]\}$ and write $(\hat R_0^{(j)}(ib)v)(y)-(\hat R_0^{(j)}(ib)v)(y') = I+II+III+IV$, where

$$\begin{aligned}I &= \sum_{a\in\alpha}(e^{g(y_a)}-e^{g(y_a')})v(y_a)(i\varphi(y_a))^je^{ib\varphi(y_a)},\\ II &= \sum_{a\in\alpha}e^{g(y_a')}(v(y_a)-v(y_a'))(i\varphi(y_a))^je^{ib\varphi(y_a)},\\ III &= \sum_{a\in\alpha}e^{g(y_a')}v(y_a')i^j(\varphi(y_a)^j-\varphi(y_a')^j)e^{ib\varphi(y_a)},\\ IV &= \sum_{a\in\alpha}e^{g(y_a')}v(y_a')(i\varphi(y_a'))^j(e^{ib\varphi(y_a)}-e^{ib\varphi(y_a')}).\end{aligned}$$

We have

$$
\begin{aligned}
|I| &\le C_1 \sum_{a\in\alpha} \mu(a) d_\theta(y,y')|v|_\infty \varphi(y_a)^j \le C_1(C_2+1)^j|v|_\infty|\varphi|_j^j d_\theta(y,y'),\\
|II| &\le C_1 \sum_{a\in\alpha} \mu(a)|v|_\theta d_\theta(y,y')\varphi(y_a)^j \le C_1(C_2+1)^j|v|_\theta|\varphi|_j^j d_\theta(y,y'),\\
|III| &\le C_1 j \sum_{a\in\alpha} \mu(a)|v|_\infty \varphi(y_a)^{j-1}|1_a\varphi|_\theta d_\theta(y,y') \le C_1(C_2+1)^j j|v|_\infty|\varphi|_j^j d_\theta(y,y'),\\
|IV| &\le C_1 \sum_{a\in\alpha} \mu(a)|v|_\infty \varphi(y'_a)^j|b|^\epsilon|1_a\varphi|_{\theta_0}^\epsilon d_{\theta_0}(y,y')^\epsilon \le C_1(C_2+1)^{j+\epsilon}|b|^\epsilon|v|_\infty|\varphi|_{j+\epsilon}^{j+\epsilon} d_\theta(y,y'),
\end{aligned}
$$

so that

$$
|\hat R_0^{(j)}(ib)v|_\theta \ll (1+|b|^\epsilon)|\varphi|_p^p\|v\|_\theta. \tag{12.1}
$$

Finally,

$$
\begin{aligned}
&(\{\hat R_0^{([p])}(i(b+h)) - \hat R_0^{([p])}(ib)\}v)(y) - (\{\hat R_0^{([p])}(i(b+h)) - \hat R_0^{([p])}(ib)\}v)(y')\\
&= \sum_{a\in\alpha} e^{g(y_a)} v(y_a)(i\varphi(y_a))^{[p]} e^{ib\varphi(y_a)}(e^{ih\varphi(y_a)}-1)\\
&\qquad - \sum_{a\in\alpha} e^{g(y'_a)} v(y'_a)(i\varphi(y'_a))^{[p]} e^{ib\varphi(y'_a)}(e^{ih\varphi(y'_a)}-1)\\
&= I + II + III + IV + V,
\end{aligned}
$$

where

$$
\begin{aligned}
I &= \sum_{a\in\alpha} (e^{g(y_a)} - e^{g(y'_a)}) v(y_a)(i\varphi(y_a))^{[p]} e^{ib\varphi(y_a)}(e^{ih\varphi(y_a)}-1),\\
II &= \sum_{a\in\alpha} e^{g(y'_a)}(v(y_a) - v(y'_a))(i\varphi(y_a))^{[p]} e^{ib\varphi(y_a)}(e^{ih\varphi(y_a)}-1),\\
III &= \sum_{a\in\alpha} e^{g(y'_a)} v(y'_a) i^{[p]}(\varphi(y_a)^{[p]} - \varphi(y'_a)^{[p]}) e^{ib\varphi(y_a)}(e^{ih\varphi(y_a)}-1),\\
IV &= \sum_{a\in\alpha} e^{g(y'_a)} v(y'_a)(i\varphi(y'_a))^{[p]}(e^{ib\varphi(y_a)} - e^{ib\varphi(y'_a)})(e^{ih\varphi(y_a)}-1),\\
V &= \sum_{a\in\alpha} e^{g(y'_a)} v(y'_a)(i\varphi(y'_a))^{[p]} e^{ib\varphi(y'_a)}(e^{ih\varphi(y_a)} - e^{ih\varphi(y'_a)}).
\end{aligned}
$$

These terms are estimated using the same techniques as the previous ones that arose in this proof. For example, we use the inequalities $|e^{ib\varphi(y_a)} - e^{ib\varphi(y'_a)}| \le |b|^\epsilon|1_a\varphi|_{\theta_0}^\epsilon d_\theta(y,y')$ and $|e^{ih\varphi(y_a)} - 1| \le |h|^{p-[p]-\epsilon}\varphi(y_a)^{p-[p]-\epsilon}$ to obtain $|IV| \le C_1(C_2+1)^p|b|^\epsilon|v|_\infty|\varphi|_p^p|h|^{p-[p]-\epsilon}d_\theta(y,y')$, and we use the inequality $|e^{ih\varphi(y_a)} - e^{ih\varphi(y'_a)}| \le |h|^{p-[p]}|1_a\varphi|_{\theta_0}^{p-[p]}d_\theta(y,y')$ to obtain $|V| \le C_1(C_2+1)^p|v|_\infty|\varphi|_p^p|h|^{p-[p]}d_\theta(y,y')$. Altogether, we obtain

$$
|\{\hat R_0^{([p])}(i(b+h)) - \hat R_0^{([p])}(ib)\}v|_\theta \ll (1+|b|^\epsilon)\|v\|_\theta|\varphi|_p^p h^{p-[p]-\epsilon}. \tag{12.2}
$$

The estimates (12.1) and (12.2) combined with the estimates in (a) yield the required result. ∎

Proposition 12.2 *Suppose that $\varphi \in L^p$ for some $p > 0$, and let $\epsilon > 0$. Viewed as a family of operators on $F_\theta(Y)$, $b \mapsto (I - \hat{R}_0(ib))^{-1}$ is $C^{p-\epsilon}$ and there exist $C, A > 0$ such that $d_{p-\epsilon}(I - \hat{R}_0(ib))^{-1} \le C|b|^A$ for all $|b| > 1$.*

Proof Again, we give the details for p not an integer, and $\epsilon < p - [p]$.

A straightforward induction argument shows that $\frac{d^j}{db^j}(I - \hat{R}_0(ib))^{-1}$ is a finite linear combination of finite products of factors $\hat{F}$ where

$$\hat{F} \in \{(I - \hat{R}_0)^{-1}, \quad \hat{R}_0^{(k)}, \ k = 1, \dots, j\},$$

for each $j \le p$. For each choice of $\hat{F}$, there exists $A_1 > 0$ such that $\|\hat{F}(ib)\| \ll |b|^{A_1}$, and moreover $d_{p-[p]-\epsilon}\hat{F}(ib) \ll |b|^{A_1}$, by Proposition 12.1 and Lemma 4.4. The required estimate is an immediate consequence. ∎

13 First main lemma

In this section we prove the following counterpart of the "first main lemma" of [33, 17]. We view $\hat{B}(s) = s(I - \hat{R}_0(s))^{-1}$ as a family of operators on $F_\theta(Y)$. Inverse Laplace transforms will be computed by moving the contour of integration to the imaginary axis (the functions in question are nonsingular on $\overline{\mathbb{H}}$) and hence can be viewed as inverse Fourier transforms. Recall that we defined $\mathcal{R}(a(t))$ to be the space of Laplace transforms of maps $S : [0, \infty) \to \mathcal{B}$ with $\|S(t)\| \le Ca(t)$. We now enlarge the definition of $\mathcal{R}(a(t))$ to include (operator-valued) functions defined on the imaginary axis with inverse Fourier transform dominated by $a(t)$.

Also, we write $\mathcal{R}(1/t^{p-})$ to denote domination by $1/t^q$ for all $q < p$. Similarly, an (operator-valued) function is C^{p-} if it is C^q for all $q < p$.

Lemma 13.1 *Suppose that $\varphi \in L^p$ for some $p > 1$. Let $\psi : \mathbb{R} \to \mathbb{R}$ be C^∞ with $\operatorname{supp} \psi \subset [-r, r]$ where $r \in (0, 1)$ is sufficiently small and such that $\psi \equiv 1$ on a neighborhood of 0. Then $\psi \hat{B} \in \mathcal{R}(1/t^{p-})$.*

First we derive an elementary calculus estimate.

Proposition 13.2 *Let $s(y) = (e^{iy} - 1)/y$. For any $n \ge 0$, there exists a constant $C > 0$ such that $|s^{(n)}(y)| \le C$ and $|s^{(n)}(y)| \le C/|y|$ for all $y \in \mathbb{R}$.*

Proof Define the analytic functions $q_n, r_n : \mathbb{C} \to \mathbb{C}$ for $n \ge 1$,

$$q_n(z) = e^z - \sum_{j=0}^{n-1} \frac{z^j}{j!}, \quad r_n(z) = \frac{q_n(z)}{z^n}.$$

By Taylor's theorem, there exists ξ between 0 and z such that

$$q_n(z) = \sum_{j=0}^{n-1} q_n^{(j)}(0) z^j / j! + q_n^{(n)}(\xi) z^n / n! = e^\xi z^n / n!,$$

so that $|q_n(iy)| \le |y|^n/n!$ Similarly,

$$q_n(z) = \sum_{j=0}^{n-2} q_n^{(j)}(0) z^j/j! + q_n^{(n-1)}(\xi) z^{n-1}/(n-1)! = (e^{\xi} - 1) z^{n-1}/(n-1)!,$$

so that $|q_n(iy)| \le |y|^{n-1}/(n-1)!$

Next, note by induction that

$$r_1^{(n)} \in \mathbb{R}\{e^z/z,\, e^z/z^2, \dots, e^z/z^n,\, (e^z-1)/z^{n+1}\}.$$

But $e^z/z^k - r_k \in \mathbb{R}\{1/z, \dots, 1/z^k\}$. Hence there exist constants $a_1, \dots, a_{n+1}$ and a polynomial p of degree at most n such that

$$r_1^{(n)}(z) = \sum_{k=1}^{n+1} a_k r_k(z) + p(z)/z^{n+1}.$$

Since all terms in this identity are analytic with the possible exception of the last one, we deduce that $p \equiv 0$. Hence

$$r_1^{(n)}(z) = \sum_{k=1}^{n+1} a_k r_k(z) = \sum_{k=1}^{n+1} a_k q_k(z)/z^k.$$

Since $s(y) = ir_1(iy)$, the result follows for each fixed n by substituting in the estimates for q_k. ∎

Lemma 13.3 *Suppose that $\varphi \in L^p$ for some $p > 0$, and let $\epsilon > 0$. Then there exists $\theta \in (0,1)$ such that viewed as an operator on $F_\theta(Y)$,*

$$\chi(b)\frac{\hat{R}_0(ib) - \hat{R}_0(0)}{b} \in \mathcal{R}(1/t^{p-\epsilon}),$$

for all C^∞ functions $\chi : \mathbb{R} \to [0,1]$ with $\operatorname{supp}\chi \subset [-3,3]$.

Proof Let $k \ge 0$ such that $p \in (k, k+1]$ and let $\epsilon \in (0, p-k)$. Set $\theta = \theta_0^\epsilon$.

Let $S(t)$ denote the inverse Fourier transform of $\chi(b)(\hat{R}_0(ib) - \hat{R}_0(0))/b$. We show that $\|S(t)\|_\theta \ll |\varphi|_p^p |t|^{-(p-\epsilon)}$. Let $v \in F_\theta(Y)$. By (4.1),

$$((\hat{R}_0(ib) - \hat{R}_0(0))v)(y) = \sum_{a\in\alpha} e^{g(y_a)} v(y_a)(e^{ib\varphi(y_a)} - 1).$$

Hence

$$(S(t)v)(y) = \sum_{a\in\alpha} e^{g(y_a)} v(y_a) \int_{-3}^{3} r(b, \varphi(y_a)) e^{ibt}\, db$$

where

$$r(b,x) = \chi(b)(e^{ibx} - 1)/b = \chi(b) x s(xb), \quad s(y) = (e^{iy} - 1)/y.$$

Let $n \geq 0$. By Proposition 13.2, $|s^{(n)}(y)| \ll 1$ and $|s^{(n)}(y)| \ll |y|^{-1}$. Hence

$$|s^{(n)}(y)| \ll |y|^{-1} \min\{1, |y|\} \leq |y|^{-(1-\epsilon)}, \tag{13.1}$$

for all $y \in \mathbb{R}$. It follows from (13.1) that $|\partial^n r(b,x)/\partial b^n| \ll x^{n+\epsilon}|b|^{-(1-\epsilon)}$ for all $x \geq 1$, $b \in \mathbb{R}$.

Integrating by parts n times,

$$\begin{aligned}\Big|\int_{-3}^{3} r(b,x)e^{ibt}\,db\Big| &= |t|^{-n}\Big|\int_{-3}^{3} \partial^n r(b,x)/\partial b^n\, e^{ibt}\,db\Big| \\ &\ll x^{n+\epsilon}|t|^{-n}\int_{-3}^{3} |b|^{-(1-\epsilon)}\,db \ll x^{n+\epsilon}|t|^{-n}.\end{aligned}$$

Applying this with $n = k$ and $n = k+1$,

$$\begin{aligned}\Big|\int_{-3}^{3} r(b,x)e^{ibt}\,db\Big| &\ll \min\{x^{k+\epsilon}|t|^{-k}, x^{k+1+\epsilon}|t|^{-(k+1)}\} = x^{k+\epsilon}|t|^{-k}\min\{1, x|t|^{-1}\} \\ &\leq x^{k+\epsilon+\delta}|t|^{-(k+\delta)},\end{aligned}$$

for all $\delta \in [0,1]$. Taking $\delta = p - k - \epsilon$, we obtain $|\int_{-3}^{3} r(b,x)e^{ibt}\,db| \leq x^p|t|^{-(p-\epsilon)}$. In particular,

$$\Big|\int_{-3}^{3} r(b,\varphi(y_a))e^{ibt}\,db\Big| \ll \varphi(y_a)^p|t|^{-(p-\epsilon)}. \tag{13.2}$$

Also, $r(b,x) - r(b,x') = \chi(b)e^{ibx}(x-x')s(b(x'-x))$ and it follows from (13.1) that

$$|(\partial^n r(b,x)/\partial b^n - (\partial^n r(b,x')/\partial b^n| \ll \sum_{j=0}^{n} x^{n-j}|x-x'|^{j+\epsilon}|b|^{-(1-\epsilon)},$$

for all $x, x' \geq 1$, $b \in \mathbb{R}$. Hence using the estimate $|1_a\varphi|_\theta \leq C_2 \inf_a \varphi \leq C_2|1_a\varphi|_\infty$ from assumption (A1),

$$\begin{aligned}|(\partial^n r(b,\varphi(y_a))/\partial b^n - (\partial^n r(b,\varphi(y'_a))/\partial b^n| &\ll |1_a\varphi|_\infty^{n+\epsilon} d_{\theta_0}(y,y')^\epsilon |b|^{-(1-\epsilon)} \\ &= |1_a\varphi|_\infty^{n+\epsilon} d_\theta(y,y')|b|^{-(1-\epsilon)}.\end{aligned}$$

Integrating by parts k and $k+1$ times,

$$\begin{aligned}\Big|\int_{-3}^{3}(r(b,\varphi(y_a)) - r(b,\varphi(y'_a)))e^{ibt}\,db\Big| &\ll |1_a\varphi|_\infty^{k+\epsilon} d_\theta(y,y')|t|^{-k}\min\{1, |1_a\varphi|_\infty|t|^{-1}\} \\ &\leq |1_a\varphi|_\infty^{p} d_\theta(y,y')|t|^{-(p-\epsilon)}.\end{aligned} \tag{13.3}$$

We are now ready to estimate $\|S(t)\|$. Using (13.2), we obtain that

$$|S(t)v)(y)| \ll \sum_{a\in\alpha}\mu(a)|v|_\infty\Big|\int_{-3}^{3} r(b,\varphi(y_a))e^{ibt}\,db\Big| \ll |v|_\infty\sum_{a\in\alpha}\mu(a)\varphi(y_a)^p|t|^{-(p-\epsilon)},$$

and so

$$|S(t)|_\infty \ll |\varphi|_p^p |t|^{-(p-\epsilon)} |v|_\infty. \tag{13.4}$$

Next,

$$(S(t)v)(y) - (S(t)v)(y') = I + II + III,$$

where

$$\begin{aligned}
I &= \sum_{a\in\alpha} (e^{g(y_a)} - e^{g(y'_a)}) v(y_a) \int_{-3}^{3} r(b,\varphi(y_a)) e^{ibt}\, db, \\
II &= \sum_{a\in\alpha} e^{g(y'_a)} (v(y_a) - v(y'_a)) \int_{-3}^{3} r(b,\varphi(y_a)) e^{ibt}\, db, \\
III &= \sum_{a\in\alpha} e^{g(y'_a)} v(y'_a) \int_{-3}^{3} (r(b,\varphi(y_a)) - r(b,\varphi(y'_a)) e^{ibt}\, db.
\end{aligned}$$

The first two terms are estimated using (13.2):

$$\begin{aligned}
|I| &\ll \sum_{a\in\alpha} \mu(a) d_\theta(y,y') |v|_\infty \varphi(y_a)^p |t|^{-(p-\epsilon)} \ll |\varphi|_p^p |t|^{-(p-\epsilon)} |v|_\infty d_\theta(y,y'), \\
|II| &\ll \sum_{a\in\alpha} \mu(a) |v|_\theta d_\theta(y,y') \varphi(y_a)^p |t|^{-(p-\epsilon)} \ll |\varphi|_p^p |t|^{-(p-\epsilon)} |v|_\theta d_\theta(y,y').
\end{aligned}$$

The third term is estimated using (13.3):

$$|III| \ll \sum_{a\in\alpha} \mu(a) |v|_\infty |1_a \varphi|_\infty^p d_\theta(y,y') |t|^{-(p-\epsilon)} \ll |\varphi|_p^p |t|^{-(p-\epsilon)} |v|_\infty d_\theta(y,y').$$

Combining the estimates for I, II, III we obtain

$$|S(t)v|_\theta \ll |\varphi|_p^p |t|^{-(p-\epsilon)} \|v\|_\theta. \tag{13.5}$$

By (13.4) and (13.5), $\|S(t)\|_\theta \ll |\varphi|_p^p |t|^{-(p-\epsilon)}$ as required. ∎

Proposition 13.4 *Suppose that $\varphi \in L^p$ for some $p > 0$, and let $\epsilon > 0$. Let $\delta > 0$. For all $r > 0$ sufficiently small, there exists a $C^{p-\epsilon}$ family $b \mapsto \tilde{R}_0(b)$ with a $C^{p-\epsilon}$ family of simple eigenvalues $\tilde{\lambda}(b) \in \{z \in \mathbb{C} : |z-1| < \delta\}$ such that*

(a) $\tilde{R}_0(b) \equiv \hat{R}_0(ib)$ *for* $|b| \le r$.

(b) $\tilde{R}_0(b) \equiv \hat{R}_0(0)$ *and* $\tilde{\lambda}(b) \equiv 1$ *for* $|b| \ge 2$.

(c) $\|\tilde{R}_0(b) - \hat{R}_0(0)\|_\theta < \delta$ *for all* $b \in \mathbb{R}$.

(d) For all $b \in \mathbb{R}$*, the spectrum of* $\tilde{R}_0(b)$ *consists of* $\tilde{\lambda}(b)$ *together with a subset of* $\{z : |z-1| \ge 3\delta\}$.

(e) $(1 - \tilde{\lambda}(b))/b$ *is bounded away from zero on* $[-r, r]$.

Proof Recall that $\hat{R}_0(0)$ has a simple eigenvalue at 1. Also there exists $\delta_0 > 0$ such that the remainder of the spectrum lies outside the disk $\{|z-1| < \delta_0\}$. We suppose without loss that $\delta < \delta_0/3$.

Choose $\delta_1 \in (0, \delta)$ with the property that if A is an operator on $F_\theta(Y)$ and $\|A - \hat{R}_0(0)\|_\theta < \delta_1$, then the spectrum of A consists of a simple eigenvalue within distance δ of 1 and the remainder of the spectrum lies outside the disk $\{|z-1| < 3\delta\}$.

By Proposition 12.1 there is a C^{p-} family $b \mapsto \lambda(b)$, defined for b sufficiently small, consisting of simple eigenvalues for $\hat{R}_0(ib)$ with $\lambda(0) = 1$. Moreover, $\lambda(b) = 1 + ib\bar{\varphi} + o(b)$ as $b \to 0$.

Choose $r_0 \in (0,1)$ small so that $[-r_0, r_0]$ lies inside the domain of definition of this C^{p-} family and such that $(1-\lambda(b))/b$ is bounded away from zero on $[-r_0, r_0]$. Fix $r \in (0, r_0)$.

Let $\psi_1, \psi_2, \psi_3 : \mathbb{R} \to [0,1]$ be even C^∞ functions such that $\psi_1 + \psi_2 + \psi_3 \equiv 1$ and such that restricted to $[0, \infty)$,

$$\psi_1 \equiv 1 \text{ on } [0, r], \ \operatorname{supp} \psi_1 \subset [0, r_0].$$

$$\psi_2 \equiv 1 \text{ on } [r_0, 1], \ \operatorname{supp} \psi_2 \subset [r, 2].$$

$$\psi_3 \equiv 1 \text{ on } [2, \infty), \ \operatorname{supp} \psi_2 \subset [1, \infty).$$

Define the C^{p-} family of operators

$$\tilde{R}_0(b) = \psi_1(b)\hat{R}_0(ib) + \psi_2(b)\hat{R}_0(i \operatorname{sgn}(b) r_0) + \psi_3(b)\hat{R}_0(0).$$

For $b \geq 0$ we have

$$\tilde{R}_0(b) = \begin{cases} \hat{R}_0(ib), & b \in [0, r] \\ \hat{R}_0(ir_0) + \psi_1(b)(\hat{R}_0(ib) - \hat{R}_0(ir_0)), & b \in [r, r_0] \\ \hat{R}_0(ir_0), & b \in [r_0, 1] \\ \hat{R}_0(0) + \psi_2(b)(\hat{R}_0(ir_0) - \hat{R}_0(0)), & b \in [1, 2] \\ \hat{R}_0(0), & b \in [2, \infty) \end{cases}.$$

Shrinking r_0 if necessary, we can ensure that

$$\|\hat{R}_0(ib) - \hat{R}_0(0)\|_\theta < \delta_1/2 \quad \text{for all } b \in [0, r_0], \quad \|\tilde{R}_0(b) - \hat{R}_0(0)\|_\theta < \delta_1 \quad \text{for all } b \in [1, 2].$$

Then choosing r sufficiently close to r_0, we can ensure that $\|\tilde{R}_0(b) - \hat{R}_0(ir_0)\|_\theta < \delta_1/2$ for all $b \in [r, r_0]$. Altogether, we have that $\|\tilde{R}_0(b) - \hat{R}_0(0)\|_\theta < \delta_1$ for all $b \geq 0$. A similar picture holds for $b \leq 0$ and so we obtain that

$$\|\tilde{R}_0(b) - \hat{R}_0(0)\|_\theta < \delta_1 < \delta, \quad \text{for all } b \in \mathbb{R}.$$

This verifies condition (c). Moreover, by definition of δ_1 we obtain the required spectral properties for $\tilde{R}_0$, namely the family of simple eigenvalues $\tilde{\lambda}$ (which is C^{p-} by standard perturbation theory) together with the estimate in condition (d). Finally, we observe that properties (a,b,e) are immediate consequences of the construction. ∎

Let $\tilde{P}$ be the spectral projection corresponding to $\tilde{\lambda}$. By Proposition 13.4, $b \mapsto \tilde{P}(b)$ is C^{p-}.

Proposition 13.5 *Suppose that $\varphi \in L^p$ for some $p > 0$, and let $\epsilon > 0$. For $\delta > 0$ small enough in Proposition 13.4,* $\dfrac{1-\tilde\lambda(b)}{b} \in \mathcal{R}(1/t^{p-\epsilon})$.

Proof Recall the formula

$$\frac{1-\tilde\lambda(b)}{b}\tilde P(b) = \frac{\tilde R_0(0)-\tilde R_0(b)}{b}\tilde P(b) + (I-\tilde R_0(0))\frac{\tilde P(b)-\tilde P(0)}{b}. \tag{13.6}$$

Let $\chi : \mathbb{R} \to [0,1]$ be a C^∞ function supported in $[-3,3]$ with $\chi \equiv 1$ on $[-2,2]$. By Proposition 13.4(a,b),

$$\frac{\tilde R_0(b)-\tilde R_0(0)}{b} = \chi(b)\frac{\tilde R_0(b)-\hat R_0(0)}{b} = \chi(b)\frac{\tilde R_0(b)-\hat R_0(ib)}{b} + \chi(b)\frac{\hat R_0(ib)-\hat R_0(0)}{b}.$$

The first term on the RHS vanishes near zero by Proposition 13.4(a) and hence is C^{p-}. Also it is compactly supported and so lies in $\mathcal{R}(1/t^{p-})$. The second term on the RHS lies in $\mathcal{R}(1/t^{p-})$ by Lemma 13.3. We deduce that $\dfrac{\tilde R_0(b)-\tilde R_0(0)}{b} \in \mathcal{R}(1/t^{p-})$. Moreover,

$$\frac{\tilde R_0(b)-\tilde R_0(0)}{b}\tilde P(b) = \frac{\tilde R_0(b)-\tilde R_0(0)}{b}\chi(b)\tilde P(b),$$

where $\chi\tilde P$ is C^{p-} and compactly supported. It follows that $\dfrac{\tilde R_0(b)-\tilde R_0(0)}{b}\tilde P(b) \in \mathcal{R}(1/t^{p-})$.

Let Γ be the circle of radius 2δ around 1. By Proposition 13.4(d), $\tilde P(b) = (1/2\pi i)\int_\Gamma (\xi - \tilde R_0(b))^{-1}\,d\xi$. By Proposition 13.4(b), $\tilde P(b) = \tilde P(0)$ for $|b| \ge 2$. Hence

$$\frac{\tilde P(b)-\tilde P(0)}{b} = \chi(b)\frac{\tilde P(b)-\tilde P(0)}{b} = \frac{1}{2\pi i}\int_\Gamma G_1(b,\xi)G_2(b)G_3(\xi)\,db,$$

where

$$G_1(b,\xi) = \chi(b)(\xi-\tilde R_0(b))^{-1}, \quad G_2(b) = \frac{\tilde R_0(b)-\tilde R_0(0)}{b}, \quad G_3(\xi) = (\xi - \tilde R_0(0))^{-1}.$$

We already showed that $G_2 \in \mathcal{R}(1/t^{p-})$. Also $b \mapsto G_1(b,\xi)$ is compactly supported and C^{p-} uniformly in ξ. Hence $G_1(b,\xi)G_2(b)G_3(\xi) \in \mathcal{R}(1/t^{p-})$ with norms uniform in ξ and so $\dfrac{\tilde P(b)-\tilde P(0)}{b} \in \mathcal{R}(1/t^{p-})$.

The above arguments together with (13.6) imply that $\dfrac{1-\tilde\lambda(b)}{b}\tilde P(b) \in \mathcal{R}(1/t^{p-})$. Hence $\dfrac{1-\tilde\lambda(b)}{b}u(\tilde P(b)) \in \mathcal{R}(1/t^{p-})$ for any bounded linear functional $u : F_\theta(Y) \to \mathbb{R}$. Choose u so that $u(\tilde P(0)) \ne 0$. By Proposition 13.4(c), we can ensure that $u(\tilde P(b))$ is bounded away from zero for all b. Then $\chi/u(\tilde P)$ is compactly supported and C^{p-}, so $\chi/u(\tilde P) \in \mathcal{R}(1/t^{p-})$. By Proposition 13.4(b),

$$\frac{1-\tilde\lambda(b)}{b} = \frac{1-\tilde\lambda(b)}{b}\chi(b) = \frac{1-\tilde\lambda(b)}{b}u(\tilde P(b))\,\frac{\chi(b)}{u(\tilde P(b))}.$$

Hence $\dfrac{1-\tilde\lambda(b)}{b}\in\mathcal{R}(1/t^{p-})$ as required. ∎

Proof of Lemma 13.1 By Proposition 13.4(a), $\psi\hat B=\psi\tilde B$ where $\tilde B(b)=b(I-\tilde R_0(b))^{-1}$. Write

$$\tilde B(b)=((1-\tilde\lambda(b))/b)^{-1}\tilde P(b)+b(I-\tilde R_0(b))^{-1}(I-\tilde P(b)).$$

The second term is C^{p-} and so lies in $\mathcal{R}(1/t^{p-})$ when multiplied by ψ. Hence it remains to show that $\psi(b)((1-\tilde\lambda(b))/b)^{-1}\tilde P(b)\in\mathcal{R}(1/t^{p-})$.

Let χ be a compactly supported C^∞ function with $\chi\equiv 1$ on the support of ψ. Then

$$\psi(b)((1-\tilde\lambda(b))/b)^{-1}\tilde P(b)=\psi(b)((1-\tilde\lambda(b))/b)^{-1}\chi(b)\tilde P(b),$$

and $\chi\tilde P\in\mathcal{R}(1/t^{p-})$. Hence it remains to show that $\psi(b)((1-\tilde\lambda(b))/b)^{-1}\in\mathcal{R}(1/t^{p-})$.

Now ψ is a compactly supported element of $\mathcal{R}(1/t^{p-})$. By Proposition 13.5, $(1-\tilde\lambda(b))/b\in\mathcal{R}(1/t^{p-})$. Moreover, $(1-\tilde\lambda(b))/b$ is bounded away from zero on the support of ψ by Proposition 13.4(e). By Lemma A.2, there exists $g\in\mathcal{R}(1/t^{p-})$ such that $\psi(b)=g(b)(1-\tilde\lambda(b))/b$. Hence $\psi(b)((1-\tilde\lambda(b))/b)^{-1}=g(b)\in\mathcal{R}(1/t^{p-})$, as required. ∎

14 Proof of Lemma 8.6

Finally, we deal with the term

$$\hat\rho_4=(1/\bar\varphi)\int_{\tilde Y}\hat U\hat T_4 v\,w\,d\mu,\qquad (\hat T_4 v)(y,u)=(\hat T_{0,4}v^u)(y),$$
$$\hat T_{0,4}=(I-P_\varphi\hat D^*)^{-1}(I-P_\varphi\hat C^*(0))\hat K^*,\qquad \hat K^*=\hat H^*(I-\hat C^*\hat B).$$

In Section 12, we introduced the notation $d_p\hat S$. We recall the following basic result.

Proposition 14.1 *(a) Suppose that the family $b\mapsto\hat S(ib)$ is C^p for some $p>0$ and that there is a constant $C>0$ such that $d_p\hat S(ib)\le C|b|^{-2}$ for $|b|>1$. Then $\hat S\in\mathcal{R}(1/t^p)$.*

(b) Suppose that $g:\mathbb{R}\to\mathbb{R}$ is C^∞, such that $g\equiv 0$ in a neighborhood of 0, and $g(b)\equiv 1$ for $|b|$ sufficiently large. Let $m\ge 1$. Then $g(b)/b^m\in\mathcal{R}(1/t^p)$ for all $p>0$.

Proof (a) For p an integer, $S(t)=\int_{-\infty}^\infty e^{ibt}\hat S(ib)\,db=t^{-p}\int_{-\infty}^\infty e^{ibt}\hat S^{(p)}(ib)\,db$ so that $|S(t)|\ll t^{-p}\int_{-\infty}^\infty(1+|b|^{-2})\,db\ll t^{-p}$.

For p not an integer, we still have $S(t)=t^{-[p]}\int_{-\infty}^\infty e^{ibt}\hat S^{([p])}(ib)\,db=-t^{-[p]}\int_{-\infty}^\infty e^{ibt}\hat S^{([p])}(i(b+\pi/t))\,db$ and so

$$\begin{aligned}2|S(t)|&\le t^{-[p]}\int_{-\infty}^\infty|\hat S^{([p])}(i(b+\pi/t))-\hat S^{([p])}(ib)|\,db\\&\ll t^{-[p]}\int_{-\infty}^\infty d_p\hat S(ib)|t|^{-(p-[p])}\,db\ll|t|^{-p}\int_{-\infty}^\infty(1+|b|)^{-2}\ll|t|^{-p},\end{aligned}$$

as required.

(b) If $m \ge 2$, then this is immediate from part (a). For $m = 1$, note that

$$\Big|\lim_{L\to\infty}\int_{-L}^{L} e^{ibt}g(b)/b\,db\Big| = t^{-1}\Big|\int_{-\infty}^{\infty} e^{ibt}h(b)\,db\Big|,$$

where $h(b) = (g(b)/b)' = (bg'(b) - g(b))/b^2$ satisfies the conditions of part (a). (Recall that $g' \equiv 0$ for b large.) ∎

Proposition 14.2 $\hat{H}^* : F_\theta(Y) \to F_\theta(Y)$ *is analytic on a neighborhood of* $\overline{\mathbb{H}}$.

Proof This is standard since the flow under the truncated roof function φ^* is uniformly expanding: $\hat{T}_0^*$ has a meromorphic extension across the imaginary axis with a simple pole at zero, and $\hat{H}^*(s) = \hat{T}_0^*(s) - s^{-1}P_\varphi^*$ is analytic on a neighborhood of $\overline{\mathbb{H}}$. ∎

Let $\psi : \mathbb{R} \to [0,1]$ be as in Lemma 13.1. Recall that ψ is C^∞ with $\operatorname{supp}\psi \in [-1,1]$, and $\psi \equiv 1$ on a neighborhood of 0. We have the following consequence of Lemma 13.3.

Corollary 14.3 *Suppose that* $\varphi \in L^p$ *for some* $p > 0$, *and let* $\epsilon > 0$. *Then* $\psi\hat{C}^* \in \mathcal{R}(1/t^{p-\epsilon})$.

Proof By Lemma 13.3, $\psi(b)\dfrac{\hat{R}_0(ib) - \hat{R}_0(0)}{b} \in \mathcal{R}(1/t^{p-})$. Since $\varphi^* \in L^\infty$, it follows from Lemma 13.3 that $\psi(b)\dfrac{\hat{R}_0^*(ib) - \hat{R}_0^*(0)}{b} \in \mathcal{R}(1/t^q)$ for all q. But $\hat{C}^*(ib) = \dfrac{\hat{R}_0^*(ib) - \hat{R}_0^*(0)}{b} - \dfrac{\hat{R}_0(ib) - \hat{R}_0(0)}{b}$ so the result follows. ∎

Proposition 14.4 $\psi(b)^3\hat{\rho}_4(ib) \in \mathcal{R}(\|v\|_\theta|w|_\infty 1/t^{\beta-})$.

Proof Regard the operators $\hat{H}^*$, $\hat{C}^*$, $\hat{B}$ in the expression $\hat{K}^* = \hat{H}^*(I - \hat{C}^*\hat{B})$ as operators on $F_\theta(Y)$. By Lemma 13.1, $\psi\hat{B} \in \mathcal{R}(1/t^{\beta-})$. Also, $\psi\hat{C}^* \in \mathcal{R}(1/t^{\beta-})$ by Corollary 14.3. By Proposition 14.2, $\psi\hat{H}^*$ is C^∞ and this together with Proposition 14.1(a) implies that $\psi\hat{H}^* \in \mathcal{R}(1/t^p)$ for all p. Hence $\psi^3\hat{K}^* = \psi^3\hat{H}^* - (\psi\hat{H}^*)(\psi\hat{C}^*)(\psi\hat{B}) \in \mathcal{R}(1/t^{\beta-})$. It follows that $\psi^3\hat{K}^* : F_\theta(Y) \to L^\infty(Y)$ lies in $\mathcal{R}(1/t^{\beta-})$.

By Proposition 8.2, $\hat{C}^*(0)$ is a bounded operator on $L^\infty(Y)$. Hence by Proposition 10.4, $\psi^3\hat{T}_{0,4} = \psi^3(I - P_\varphi\hat{D}^*)^{-1}(I - \hat{C}^*(0))\hat{K}^* : F_\theta(Y) \to L^\infty(Y)$ lies in $\mathcal{R}(1/t^{\beta-})$. Hence $\psi^3\hat{T}_4 : F_\theta(\tilde{Y}) \to L^\infty(\tilde{Y})$ lies in $\mathcal{R}(1/t^{\beta-})$.

By Proposition 11.3, $\hat{U} : L^\infty(\tilde{Y}) \to L^1(\tilde{Y})$ lies in $\mathcal{R}(1/t^\beta)$ and the result follows. ∎

Proposition 14.5 $(1 - \psi(b)^3)(\hat{\rho}(ib) - \hat{\rho}_4(ib)) \in \mathcal{R}(|v|_\infty|w|_\infty 1/t^{\beta-})$.

Proof Using (8.3), write $(1-\psi^3)(\hat{\rho}-\hat{\rho}_4) = (1-\psi^3)(\hat{\rho}_1+\hat{\rho}_2+\hat{\rho}_3) = b^{-1}(1-\psi^3)\int_{\tilde{Y}} \hat{U}\hat{Q}v\,w\,d\tilde{\mu}$ where $\hat{Q}(s) = s(\hat{T}_{0,1} + \hat{T}_{0,2} + \hat{T}_{0,3}) = (I - P_\varphi\hat{D}^*)^{-1}$.

Now $\hat{U} : L^\infty(\tilde{Y}) \to L^1(\tilde{Y})$ lies in $\mathcal{R}(1/t^\beta)$ by Proposition 11.3. Hence $\hat{U}\hat{Q} \in \mathcal{R}(1/t^{\beta-})$ by Proposition 10.4 and Remark 10.5. Also $b^{-1}(1 - \psi^3) \in \mathcal{R}(1/t^p)$ for all p by Proposition 14.1(b), so the result follows. ∎

Proposition 14.6 *For $w \in L^{\infty,m}(\tilde{Y})$, m sufficiently large, we have that $(1-\psi(b)^3)\hat{\rho}(ib) \in \mathcal{R}(\|v\|_\theta |w|_{\infty,m} 1/t^{\beta-})$.*

Proof Write $\rho_{v,w}$ to stress the dependence on v, w and similarly for $\hat{\rho}_{v,w}$. By Proposition 3.7, $\hat{\rho}_{v,w}(s) = \hat{P}_m(s) + \hat{H}_m(s)$, where $\hat{P}_m(s)$ is a linear combination of s^{-j}, $j = 1, \dots, m$, and $\hat{H}_m(s) = s^{-m}\hat{\rho}_{v,\partial_t^m w}(s)$.

By Proposition 14.1(b), $(1-\psi(b)^3)\hat{P}_m(ib) \in \mathcal{R}(1/t^p)$ for all $p > 0$. Next,

$$\hat{\rho}_{v,\partial_t^m w} = \int_{\tilde{Y}} \hat{U}(I-\hat{R})^{-1} v \, \partial_t^m w \, d\tilde{\mu},$$

where $\hat{U} : L^\infty(\tilde{Y}) \to L^1(\tilde{Y})$ lies in $\mathcal{R}(1/t^\beta)$ by Proposition 11.3. It remains to show that $Z(b) = b^{-m}(1-\psi(b)^3)(I-\hat{R}_0(ib))^{-1} : F_\theta(Y) \to F_\theta(Y)$ lies in $\mathcal{R}(1/t^{\beta-})$.

By Proposition 12.1(b), $\hat{R}_0$ is $C^{\beta-}$. Hence $(I-\hat{R}_0)^{-1}$ is $C^{\beta-}$ on $\mathbb{R} \setminus \{0\}$ and Z is $C^{\beta-}$ on $\mathbb{R}$. Moreover, by Proposition 12.2 and Lemma 4.4, there exists $C, A > 0$ such that $d_{\beta-}(I-\hat{R}_0(ib))^{-1} \le C|b|^A$ for $|b| > 1$. Hence for m sufficiently large, $d_{\beta-}Z(ib) \ll |b|^{-2}$ for $|b| > 1$. We conclude from Proposition 14.1(a) that $Z \in \mathcal{R}(1/t^{\beta-})$ as required. ∎

Proof of Lemma 8.6 This is immediate by Propositions 14.4, 14.5 and 14.6. ∎

15 Proof of Theorem 2.4(b)

In this section, we complete the proof of Theorem 2.4(b). For this it suffices to replace the estimate for $\rho_3(t)$ in Lemma 8.5 by the improved estimate in Lemma 15.2 below.

Define $\hat{\zeta}_j(s) = s^{-1}(\int_Y \hat{D}^*(s)1_Y \, d\mu)^j$. Then $\zeta_0 \equiv 1$ and it follows from the proof of Proposition 11.2(a) that $\zeta_1(t) = \zeta(t)$ for $t > k$.

Proposition 15.1 *Let $\beta > 1$, $j \ge 0$. Then $\hat{\zeta}_j \in \mathcal{R}(1/t^{j(\beta-1)})$ if $j(\beta-1) < \beta$ and $\hat{\zeta}_j \in \mathcal{R}(1/t^\beta)$ if $j(\beta-1) > \beta$.*

Proof This is proved by continuing inductively the argument in Proposition 11.2(b). The details are the same as in [17, Lemma 5.1] (with the simplification that there are no non-commutativity issues). ∎

Lemma 15.2 *Let $1 < \beta < 2$ and choose $m \ge 3$ least such that $m(\beta-1) > \beta$. Then*

$$\rho_3(t) = \bar{v}\bar{w} \sum_{j=2}^{m-1} (1/\bar{\varphi})^j \zeta_j(t) + O(|v|_\infty |w|_\infty 1/t^{\beta-}).$$

Proof Recall that

$$\hat{\rho}_j = (1/\bar{\varphi}) \int_{\tilde{Y}} \hat{U}\hat{T}_j v \, w \, d\tilde{\mu}, \quad (\hat{T}_j v)(y,u) = (\hat{T}_{0,j} v^u)(y),$$

where

$$\hat{T}_{0,1} = (1/\bar{\varphi}) s^{-1} P(0), \quad \hat{T}_{0,3} = (1/\bar{\varphi})^3 s^{-1} \Big(1 - (1/\bar{\varphi}) \int_Y \hat{D}^* 1_Y \, d\mu_Y\Big)^{-1} \Big(\int_Y \hat{D}^* 1_Y \, d\mu\Big)^2 P(0).$$

Hence

$$\begin{aligned}\hat\rho_3 &= (1/\bar\varphi)^2\hat\rho_1\Big(1-(1/\bar\varphi)\int_Y \hat D^*1_Y\,d\mu_Y\Big)^{-1}\Big(\int_Y \hat D^*1_Y\,d\mu\Big)^2 \\ &= \sum_{j=2}^{m-1}(1/\bar\varphi)^j\Big(\int_Y \hat D^*1_Y\,d\mu\Big)^j\hat\rho_1 + (1/\bar\varphi)^m\hat q\Big(\int_Y \hat D^*1_Y\,d\mu\Big)^m\hat\rho_1,\end{aligned}$$

where $\hat q = \big(1-(1/\bar\varphi)\int_Y \hat D^*1_Y\,d\mu_Y\big)^{-1}$. By Lemma 8.4(a), $\hat\rho_1(s) = s^{-1}\bar v\bar w + \hat h(s)$ where $\hat h \in \mathcal{R}(1/t^\beta)$. Hence

$$\begin{aligned}\hat\rho_3 = \bar v\bar w&\sum_{j=2}^{m-1}(1/\bar\varphi)^j\hat\zeta_j + \bar v\bar w(1/\bar\varphi)^m\hat q\hat\zeta_m \\ &+\sum_{j=2}^{m-1}(1/\bar\varphi)^j\Big(\int_Y \hat D^*1_Y\,d\mu\Big)^j\hat h + (1/\bar\varphi)^m\hat q\Big(\int_Y \hat D^*1_Y\,d\mu\Big)^m\hat h.\end{aligned}$$

Now apply Corollary 10.3 and Propositions 10.4 and 15.1. ∎

A Wiener lemma

This appendix contains material about a version of the Wiener lemma that is required in Section 13. We have chosen the notation here to conform with standard conventions in Fourier analysis. (In the application of this material, the roles of $f:\mathbb{R}\to\mathbb{C}$ and its Fourier transform $\hat f$ is reversed, with b and t playing the role of x and ξ respectively.)

Let $\mathbb{A}$ be the Banach algebra of 2π-periodic continuous functions $f:\mathbb{R}\to\mathbb{C}$ such that their Fourier coefficients $\hat f_n$ are absolutely summable, with norm $\|f\|_{\mathbb{A}} = \sum_{n\in\mathbb{Z}}|\hat f_n|$. Similarly, let $\mathcal{R}$ be the Banach algebra of continuous functions $f:\mathbb{R}\to\mathbb{C}$ such that their Fourier transform $\hat f:\mathbb{R}\to\mathbb{C}$ lies in $L^1(\mathbb{R})$, with norm $\|f\|_{\mathcal{R}} = \int_{-\infty}^{\infty}|\hat f(\xi)|\,d\xi$.

Given $\beta>1$, we define the Banach algebra $\mathbb{A}_\beta = \{f\in\mathbb{A} : \sup_{n\in\mathbb{Z}}|n|^\beta|\hat f_n|<\infty\}$ with norm $\|f\|_{\mathbb{A}_\beta} = \sum_{n\in\mathbb{Z}}|\hat f_n| + \sup_{n\in\mathbb{Z}}|n|^\beta|\hat f_n|$. Similarly, we define the Banach algebra $\mathcal{R}_\beta = \{f\in\mathcal{R} : \sup_{\xi\in\mathbb{R}}|\xi|^\beta|\hat f(\xi)|<\infty\}$ with norm $\|f\|_{\mathcal{R}_\beta} = \int_{-\infty}^{\infty}|\hat f(\xi)|\,d\xi + \sup_{\xi\in\mathbb{R}}|\xi|^\beta|\hat f(\xi)|$.

The following Wiener lemmas are standard.

Lemma A.1 *Let $\beta>1$ and let $f,f_1\in\mathcal{A}_\beta$. Suppose that f is bounded away from zero on the support of f_1.*

Then there exists $g\in\mathcal{A}_\beta$ such that $f_1 = fg$.

Lemma A.2 *Let $\beta>1$ and let $f,f_1\in\mathcal{R}_\beta$. Suppose f_1 is compactly supported and that f is bounded away from zero on the support of f_1.*

Then there exists $g\in\mathcal{R}_\beta$ such that $f_1 = fg$.

A statement and proof of Lemma A.1 can be found in [13, Theorem 1.2.12]. In this paper, we require Lemma A.2, but we could not find it stated in the literature. Hence we provide here a proof of Lemma A.2, using a standard argument to reduce to Lemma A.1.

Lemma A.3 *Let $\epsilon > 0$. Suppose that $f : \mathbb{R} \to \mathbb{C}$ is a continuous function with* $\operatorname{supp} f \subset [-\pi+\epsilon, \pi-\epsilon]$*. Let $h : \mathbb{R} \to \mathbb{C}$ denote the 2π-periodic continuous function such that $h|_{[-\pi,\pi]} = f|_{[-\pi,\pi]}$. Then $f \in \mathcal{R}_\beta$ if and only if $h \in \mathbb{A}_\beta$.*

Proof (cf. [21, Theorem 6.2, Ch. VIII, p. 242]) Fix a C^∞ function $\psi : \mathbb{R} \to \mathbb{R}$ supported in $[-\pi+\epsilon/2, \pi-\epsilon/2]$ and such that $\psi \equiv 1$ on $[-\pi+\epsilon, \pi-\epsilon]$. For $\alpha \in [-1,1]$ let $\psi_\alpha(x) = e^{i\alpha x}\psi(x)$. Then there is a constant $K_0 > 0$ such that

$$|(\widehat{\psi_\alpha})_n| \le K_0 n^{-\beta}, \quad \text{for all } \alpha \in [-1,1],\, n \in \mathbb{Z}.$$

In particular, $\psi_\alpha \in \mathbb{A}_\beta$ for all α and $\sup_{|\alpha|\le 1} \|\psi_\alpha\|_{\mathbb{A}_\beta} < \infty$.

Define $h_\alpha(x) = e^{i\alpha x}h(x)$. If $h \in \mathbb{A}_\beta$, then $h_\alpha = h\psi_\alpha \in \mathbb{A}_\beta$ and there is a constant $K > 0$ such that $\|h_\alpha\|_{\mathbb{A}_\beta} \le K\|h\|_{\mathbb{A}_\beta}$ for all $\alpha \in [-1,1]$.

Now,

$$(\widehat{h_\alpha})_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\alpha x}h(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i(n-\alpha)x}\,dx = \frac{1}{2\pi}\hat{f}(n-\alpha).$$

Hence $\int_{n-1}^{n} |\hat{f}(\xi)|\,d\xi = \int_0^1 |\hat{f}(n-\alpha)|\,d\alpha = 2\pi\int_0^1 |(\widehat{h_\alpha})_n|\,d\alpha$. It follows that

$$\|f\|_{\mathcal{R}} = 2\pi\sum_{n=-\infty}^{\infty}\int_0^1 |(\widehat{h_\alpha})_n|\,d\alpha = 2\pi\int_0^1 \|h_\alpha\|_{\mathbb{A}}\,d\alpha \le 2\pi K\|h\|_{\mathbb{A}_\beta}. \tag{A.1}$$

Next, we observe that any $\xi \in \mathbb{R}$ can be expressed as $\xi = (n-\alpha)\operatorname{sgn}\xi$ where $n \ge 1$, $\alpha \in [0,1]$. Hence

$$\begin{aligned}\sup_{\xi\in\mathbb{R}} |\xi|^\beta|\hat{f}(\xi)| &= \sup_{n\ge1,\,\alpha\in[0,1]} (n-\alpha)^\beta|\hat{f}((n-\alpha)\operatorname{sgn}\xi)| \le \sup_{n\ge1,\,\alpha\in[0,1]} n^\beta|\hat{f}((n-\alpha)\operatorname{sgn}\xi)| \\ &\le \sup_{n\in\mathbb{Z},\,\alpha\in[-1,1]} |n|^\beta|\hat{f}(n-\alpha)| = 2\pi \sup_{n\in\mathbb{Z},\,\alpha\in[-1,1]} |n|^\beta|(\widehat{h_\alpha})_n| \\ &\le 2\pi\sup_{\alpha\in[-1,1]} \|h_\alpha\|_{A_\beta} \le 2\pi K\|h\|_{A_\beta}. \end{aligned} \tag{A.2}$$

Combining (A.1) and (A.2), we obtain that $\|f\|_{\mathcal{R}_\beta} \le 4\pi K\|h\|_{A_\beta}$. Hence we have shown that $h \in \mathbb{A}_\beta$ implies that $f \in \mathcal{R}_\beta$.

Conversely, suppose $f \in \mathcal{R}_\beta$. Then $\sum_{n\in\mathbb{Z}}\int_0^1 |\hat{f}(n-\alpha)|\,d\alpha = \int_{-\infty}^{\infty} |\hat{f}(\xi)|\,d\xi < \infty$ and it follows from Fubini that $\sum_{n\in\mathbb{Z}} |\hat{f}(n-\alpha)| < \infty$ for almost every α. Fix such an α. Then $\sum_{n\in\mathbb{Z}} |(\widehat{h_\alpha})_n| = (1/2\pi)\sum_{n\in\mathbb{Z}} |\hat{f}(n-\alpha)| < \infty$ so that $h_\alpha \in \mathbb{A}$. Hence $h = (h_\alpha)_{-\alpha} \in \mathbb{A}$. Moreover,

$$\sup_{n\in\mathbb{Z}} |n|^\beta|\hat{h}_n| = (1/2\pi)\sup_{n\in\mathbb{Z}} |n|^\beta|\hat{f}(n)| \le (1/2\pi)\sup_{\xi\in\mathbb{R}} |\xi|^\beta|\hat{f}(\xi)| < \infty,$$

so that $h \in \mathbb{A}_\beta$. ∎

Proof of Lemma A.2 (cf. [21, Lemma 6.3, Ch. VIII, p. 242]) We make the standard abuse of notation that functions on $\mathbb{R}$ supported on a closed subset of $(-\pi,\pi)$ can be identified with 2π-periodic functions on $\mathbb{R}$. In particular, the conclusion of Lemma A.3 becomes $f \in \mathcal{R}_\beta$ if and only if $f \in \mathbb{A}_\beta$.

Without loss, we can suppose that $\operatorname{supp} f_1 \subset [-2, 2]$. By Lemma A.3, $f_1 \in \mathbb{A}_\beta$.

Choose a C^∞ function $\chi : \mathbb{R} \to \mathbb{R}$ such that $\operatorname{supp} \chi \subset [-3, 3]$ and $\chi \equiv 1$ on $[-2, 2]$. Then $\chi \in \mathcal{A}_\beta$ and $\chi \in \mathcal{R}_\beta$. In particular $\chi f \in \mathcal{R}_\beta$, and by Lemma A.3 $\chi f \in \mathbb{A}_\beta$.

Moreover $\chi f = f$ on $\operatorname{supp} f_1$ and hence is bounded away from zero on $\operatorname{supp} f_1$. By Lemma A.1, there exists $g_0 \in \mathbb{A}_\beta$ such that $f_1 = g_0(\chi f) = (g_0\chi)f$.

Since $g_0, \chi \in \mathbb{A}_\beta$, we deduce that $g = g_0\chi \in \mathbb{A}_\beta$. By Lemma A.3, $g \in \mathcal{R}_\beta$. Hence $f_1 = gf$ with $g \in \mathcal{R}_\beta$ as required. ∎

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