MIXING PROPERTIES AND STATISTICAL LIMIT THEOREMS FOR SINGULAR HYPERBOLIC FLOWS WITHOUT A SMOOTH STABLE FOLIATION

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Abstract. Over the last 10 years or so, advanced statistical properties, including exponential decay of correlations, have been established for certain classes of singular hyperbolic flows in three dimensions. The results apply in particular to the classical Lorenz attractor. However, many of the proofs rely heavily on the smoothness of the stable foliation for the flow.

In this paper, we show that many statistical properties hold for singular hyperbolic flows with no smoothness assumption on the stable foliation. These properties include existence of SRB measures, central limit theorems and associated invariance principles, as well as results on mixing and rates of mixing. The properties hold equally for singular hyperbolic flows in higher dimensions provided the center-unstable subspaces are two-dimensional.

1. Introduction

Singular hyperbolicity is a far-reaching generalization of Smale's notion of Axiom A [51] that allows for the inclusion of equilibria (also known as singular points or steady-states) and incorporates the classical Lorenz attractor [31] as well as the geometric Lorenz attractors of [1, 24]. For three-dimensional flows, singular hyperbolic attractors are precisely the ones that are robustly transitive, and they reduce to Axiom A attractors when there are no equilibria [40].

For the classical Lorenz attractor, strong statistical properties such as exponential decay of correlations, the central limit theorem (CLT), and associated invariance principles have been established in [6, 7, 8, 27]. However the proofs rely heavily on the existence of a smooth stable foliation for the flow. Various issues regarding the existence and smoothness of the stable foliation are clarified in [7]; a topological foliation always exists, and an analytic proof of smoothness of the foliation for the classical Lorenz attractor (and nearby attractors) is given in [7, 8].

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Even for three-dimensional flows, the stable foliation for a singular hyperbolic attractor need not be better than Hölder. In this paper, we consider statistical properties for singular hyperbolic attractors that do not have a smooth stable foliation. We do not restrict to three-dimensional flows, but our main results assume that the stable foliation has codimension two.

Main results. For codimension two singular hyperbolic attracting sets, we prove that the stable foliation is at least Hölder continuous (Theorem 6.2), and using Pesin theory [14] we deduce that the stable holonomies are absolutely continuous with Hölder Jacobians (Theorem 6.3). As a consequence of this, we obtain that the stable holonomies for the associated Poincaré map are $C^{1+\epsilon}$ (since they are one-dimensional with Hölder Jacobians). This extends results of [1, 48, 50] who obtain a C^1 result for geometric Lorenz attractors (see the discussion after equation (6) in [43]). Quotienting out by the stable foliation, we obtain a $C^{1+\epsilon}$ one-dimensional expanding map. We can now proceed following [10] to obtain a spectral decomposition for the singular hyperbolic attracting set (Theorem 9.2).

To study statistical properties, we focus attention on the transitive components of a singular hyperbolic attracting set; these are called singular hyperbolic attractors. In the Axiom A case, the CLT and associated invariance principles are well-known [17, 38, 46] and we extend these results to general (codimension two) singular hyperbolic attractors. In particular, as described in Section 9.1, the (functional) CLT and related results follow from [13] using the results in this paper. Moreover, many strong limit laws are obtained for the associated Poincaré maps in Theorem 8.7.

Mixing and rates of mixing for Axiom A attractors are less well-understood even today, but an open and dense set of Axiom A attractors have superpolynomial decay of correlations [18, 20]. Theorem 9.5 shows that the same result holds for singular hyperbolic attractors. As a consequence, Corollary 9.6, we obtain the CLT and almost sure invariance principle for the time-one map of the flow for this open and dense set of singular hyperbolic attractors and sufficiently smooth observables. (We note that such results are much more delicate for time-one maps than for the flow and for Poincaré maps.)

In fact, for singular hyperbolic attractors containing at least one equilibrium and with a smooth stable foliation, mixing [32], superpolynomial decay of correlations [8], and exponential decay of correlations [6] are automatic subject to a certain indecomposability condition (locally eventually onto). Theorem 9.7 yields a similar result on automatic mixing when there is not a smooth stable foliation. However, automatic rates of mixing, or any results on exponential decay of correlations, seems beyond current techniques when the stable foliation is not smooth.

Example. In a recent paper, Ovsyannikov & Turaev [43] (see also previous work of [19]) give an analytic proof of singular hyperbolic attractors in the extended Lorenz model

$$\dot{x} = y, \quad \dot{y} = -\lambda y + \gamma x(1 - z) - \delta x^3, \quad \dot{z} = -\alpha z + \beta x^2.$$

The attractors contain precisely one equilibrium, namely the origin, and are of geometric Lorenz type [1, 24]. The eigenvalues of the linearized equations at the equilibrium are close to -1, -1 and 1 (up to a scaling) for the parameters considered in [43], so the standard q-bunching condition [7, 26] guaranteeing a C^q stable foliation holds only for q close to zero. In this situation it is anticipated that the foliation fails to be C^1 except in pathological cases. In particular, previous results on statistical properties for singular hyperbolic flows do not apply. However, the results in the present paper do not require a smooth foliation. It follows that the attractors in [43] satisfy the statistical limit laws described in this paper. Moreover, there is an open set $\mathcal{U}$ within the space of C^2 flows on $\mathbb{R}^3$, containing the extended Lorenz examples of [43], that satisfy these statistical limit laws. In addition, an open and dense set of flows in $\mathcal{U}$ have superpolynomial decay of correlations.

Spectral decompositions. Whereas the results on statistical properties for singular hyperbolic flows in this paper are completely new, we note that there are existing results on spectral decompositions [10, 30]. The decomposition in [10] is for three-dimensional flows and our method extends [10] in the more general codimension two situation. The method in [30] works directly with the flow and does not require the codimension two restriction. However [10, 30] both make liberal use of Pesin theory, including results that seem currently unavailable in the literature. The main issue, as clarified in [7], is that *a priori* the stable lamination over a partially hyperbolic attracting set Λ need not cover a neighborhood of Λ. The stable bundle extends to an invariant contracting bundle over a neighborhood $U \supset \Lambda$ and this integrates to a topological foliation of U. However, the complementary center-unstable bundle does not extend invariantly, so the resulting extended splitting is not invariant. This means that the application of Pesin theory in [10, 30] is inaccurate. It is likely that the desired results hold (some aspects were extended to noninvariant splittings already in [7]) but currently the arguments seem incomplete.

In this paper, we make the approach in [10] completely rigorous by bypassing the issue of noninvariance of the extended splitting. Theorem 5.1 below shows that *a posteriori* the stable bundle restricted to Λ integrates to a topological foliation. This relies heavily on the special structure associated to a codimension two singular hyperbolic attracting set and uses also the information about the extended bundle [7]. Consequently, we can work with the nonextended splitting which is invariant and Pesin theory applies. Also, using [45] we show that the foliation is Hölder which simplifies the arguments in [10].

Sectional hyperbolicity. Finally, we remark on the restriction to singular hyperbolic attracting sets that are codimension two. The natural setting in general is to consider *sectional* hyperbolic attracting sets [39] (in the codimension two case, sectional and singular hyperbolicity are the same). The proof of Theorem 5.1 (specifically Proposition 5.2) relies on the restriction to codimension two. Nevertheless, we expect that in the sectional hyperbolic setting, our results on the stable foliation should go through largely unchanged (after adapting various arguments to deal with the noninvariant splitting). However, the quotient map is higher-dimensional and so Pesin theory only gives a Hölder Jacobian; the map itself is no better than Hölder. Hence the arguments in Section 8 and 9 on spectral decompositions and statistical properties break down; this remains the subject of future work.

The remainder of the paper is organized as follows. In Section 2, we review background material on partially hyperbolic attracting sets and singular hyperbolicity, and recall results on stable foliations from [6]. In Section 3, we construct a global Poincaré map f associated to any partially hyperbolic attracting set, following (and modifying) the construction in [10]. Section 4 establishes that f is uniformly hyperbolic (with singularities) when the attracting set is singular hyperbolic.

In Section 5, we show that the stable lamination over an attracting codimension two singular hyperbolic set is a topological foliation. In Section 6, we establish Hölder regularity and absolute continuity of the stable foliation, and show that the stable holonomies have Hölder Jacobians. Using this, we obtain a uniformly expanding piecewise $C^{1+\epsilon}$ quotient map $\bar{f}$ in Section 7.

Finally, in Sections 8 and 9, we prove results on spectral decompositions, statistical limit laws, and rates of mixing, for $\bar{f}$, f, and the underlying flow.

Notation. Let (M,d) be a metric space and $\eta \in (0,1)$. Given $v : M \to \mathbb{R}$, define $\|v\|_{C^\eta} = |v|_\infty + |v|_{C^\eta}$ where $|v|_{C^\eta} = \sup_{x\neq x'} |v(x) - v(x')|/d(x,x')^\eta$. We say that v is C^η and write $v \in C^\eta(M)$ if $\|v\|_{C^\eta} < \infty$.

2. Singular hyperbolic attracting sets

In this section, we define what is understood as a singular hyperbolic attracting set. Throughout this paper, we restrict mainly to the case where the center-unstable subspace is two-dimensional.

Let M be a compact Riemannian manifold and $\mathfrak{X}^r(M)$, $r > 1$, be the set of C^r vector fields on M. Let Z_t denote the flow generated by $G \in \mathfrak{X}^r(M)$. Given a compact invariant set Λ for $G \in \mathfrak{X}^r(M)$, we say that Λ is *isolated* if there exists an open set $U \supset \Lambda$ such that $\Lambda = \bigcap_{t\in\mathbb{R}} Z_t(U)$. If U can be chosen so that $Z_t(U) \subset U$ for all $t > 0$, then we say that Λ is an *attracting set*.

Definition 2.1. Let Λ be a compact invariant set for $G \in \mathfrak{X}^r(M)$. We say that Λ is *partially hyperbolic* if the tangent bundle over Λ can be written as a continuous DZ_t-invariant sum

$$T_\Lambda M = E^s \oplus E^{cu},$$

where $d_s = \dim E^s_x \geq 1$ and $d_{cu} = \dim E^{cu}_x = 2$ for $x \in \Lambda$, and there exist constants $C > 0$, $\lambda \in (0,1)$ such that for all $x \in \Lambda$, $t \geq 0$, we have

- uniform contraction along E^s:

$$\|DZ_t|E^s_x\| \leq C\lambda^t; \tag{2.1}$$

- domination of the splitting:

$$\|DZ_t|E^s_x\| \cdot \|DZ_{-t}|E^{cu}_{Z_t x}\| \leq C\lambda^t. \tag{2.2}$$

We refer to E^s as the stable bundle and to E^{cu} as the center-unstable bundle. A *partially hyperbolic attracting set* is a partially hyperbolic set that is also an attracting set.

Definition 2.2. The center-unstable bundle E^{cu} is *volume expanding* if there exists $K, \theta > 0$ such that $|\det(DZ_t|E^{cu}_x)| \geq Ke^{\theta t}$ for all $x \in \Lambda$, $t \geq 0$.

If $\sigma \in M$ and $G(\sigma) = 0$, then σ is called an *equilibrium*. An invariant set is *nontrivial* if it is neither a periodic orbit nor an equilibrium.

Definition 2.3. Let Λ be a compact nontrivial invariant set for $G \in \mathfrak{X}^r(M)$. We say that Λ is a *singular hyperbolic set* if all equilibria in Λ are hyperbolic, and Λ is partially hyperbolic with volume expanding center-unstable bundle. A singular hyperbolic set which is also an attracting set is called a *singular hyperbolic attracting set*.

Remark 2.4. A singular hyperbolic attracting set contains no isolated periodic orbits. For such a periodic orbit would have to be a periodic sink, violating volume expansion.

A subset $\Lambda \subset M$ is *transitive* if it has a full dense orbit, that is, there exists $x \in \Lambda$ such that $\operatorname{cl}\{Z_t x : t \geq 0\} = \Lambda = \operatorname{cl}\{Z_t x : t \leq 0\}$.

Definition 2.5. A *singular hyperbolic attractor* is a transitive singular hyperbolic attracting set.

Proposition 2.6. *Suppose that Λ is a singular hyperbolic attractor with $d_{cu} = 2$, and let $\sigma \in \Lambda$ be an equilibrium. Then σ is* Lorenz-like. *That is, $DG(\sigma)|E^{cu}_\sigma$ has real eigenvalues λ^s, λ^u satisfying $-\lambda^u < \lambda^s < 0 < \lambda^u$.*

Proof. It follows from Definition 2.3 that σ is a hyperbolic saddle and that at most two eigenvalues have positive real part. If there is only one such eigenvalue $\lambda^u > 0$ then the constraints on λ^s follow from volume expansion.

Let γ be the local stable manifold for σ. It remains to rule out the case $\dim \gamma = \dim M - 2$. In this case, $T_p\gamma = E^s_p$ for all $p \in \gamma \cap \Lambda$ and in particular $G(p) \in E^s_p$. Also, $G(p) \in E^{cu}_p$ (see for example [9, Lemma 6.1]), so we deduce that $G(p) = 0$ for all $p \in \gamma \cap \Lambda$ and hence that $\gamma \cap \Lambda = \{\sigma\}$.

On the other hand, Λ is transitive and nontrivial, so there exists $x \in \Lambda \setminus \{\sigma\}$ such that $\sigma \in \omega(x)$. By the local behavior of orbits near hyperbolic saddles, there exists $p \in (\gamma \setminus \{\sigma\}) \cap \omega(x) \subset (\gamma \setminus \{\sigma\}) \cap \Lambda$ which as we have seen is impossible. $\square$

We end this section by recalling/extending some results from [7]. These results hold for general $d_{cu} \geq 2$.

Proposition 2.7. *Let Λ be a partially hyperbolic attracting set. The stable bundle E^s over Λ extends to a continuous uniformly contracting DZ_t-invariant bundle E^s over an open neighborhood of Λ.*

Proof. See [7, Proposition 3.2]. $\square$

Let $\mathcal{D}^k$ denote the k-dimensional open unit disk and let $\mathrm{Emb}^r(\mathcal{D}^k, M)$ denote the set of C^r embeddings $\phi : \mathcal{D}^k \to M$ endowed with the C^r distance.

Proposition 2.8. *Let Λ be a partially hyperbolic attracting set. There exists a positively invariant neighborhood U_0 of Λ, and constants $C > 0$, $\lambda \in (0, 1)$, such that the following are true:*

(a) For every point $x \in U_0$ there is a C^r embedded d_s-dimensional disk $W^s_x \subset M$, with $x \in W^s_x$, such that

(1) $T_x W^s_x = E^s_x$.

(2) $Z_t(W^s_x) \subset W^s_{Z_t x}$ for all $t \geq 0$.

(3) $d(Z_t x, Z_t y) \leq C\lambda^t d(x, y)$ for all $y \in W^s_x$, $t \geq 0$.

(b) The disks W^s_x depend continuously on x in the C^0 topology: there is a continuous map $\gamma : U_0 \to \mathrm{Emb}^0(\mathcal{D}^{d_s}, M)$ such that $\gamma(x)(0) = x$ and $\gamma(x)(\mathcal{D}^{d_s}) = W^s_x$. Moreover, there exists $L > 0$ such that $\mathrm{Lip}\,\gamma(x) \leq L$ for all $x \in U_0$.

(c) The family of disks $\{W^s_x : x \in U_0\}$ defines a topological foliation of U_0.

Proof. See [7, Theorem 4.2 and Lemma 4.8]. $\square$

The splitting $T_\Lambda M = E^s \oplus E^{cu}$ extends continuously to a splitting $T_{U_0} M = E^s \oplus E^{cu}$ where E^s is the invariant uniformly contracting bundle in Proposition 2.7. (In general, E^{cu} is not invariant.) Given $a > 0$, we define the *center-unstable cone field*,

$$\mathcal{C}^{cu}_x(a) = \{v = v^s + v^{cu} \in E^s_x \oplus E^{cu}_x : \|v^s\| \leq a\|v^{cu}\|\}, \quad x \in U_0.$$

Proposition 2.9. *Let Λ be a partially hyperbolic attracting set. There exists $T_0 > 0$ such that for any $a > 0$, after possibly shrinking U_0,*

$$DZ_t \cdot \mathcal{C}^{cu}_x(a) \subset \mathcal{C}^{cu}_{Z_t x}(a) \quad \textit{for all } t \geq T_0,\ x \in U_0.$$

Proof. See [7, Proposition 3.1]. $\square$

Proposition 2.10. *Let Λ be a singular hyperbolic attracting set. After possibly increasing T_0 and shrinking U_0, there exist constants $K, \theta > 0$ such that $|\det(DZ_t|E^{cu}_x)| \geq K\, e^{\theta t}$ for all $x \in U_0$, $t \geq 0$.*

Proof. Let $K_0, \theta_0 > 0$ be the constants from Definition 2.2. Fix $a > 0$ and T_0 as in Proposition 2.9. We may suppose without loss that $K_0 < 2$ and that $K_0 e^{\theta_0 T_0} > 2$.

By continuity, we may assume that for every $x \in U_0$ there exists $y \in \Lambda$ such that

$$|\det(DZ_t|P)| \geq \tfrac{1}{2}|\det(DZ_t|E^{cu}_y)| \geq \tfrac{1}{2}K_0 e^{\theta_0 t},$$

for all $t \in [0, T_0]$ and every d_{cu}-dimensional subspace $P \subset \mathcal{C}^{cu}_x(a)$.

Write $t = mT_0 + r$ where $m \in \mathbb{N}$, $r \in (0, T_0]$. Since $Z_{jT_0}x \in U_0$ for all $j \geq 0$ by invariance of U_0, and since $DZ_{jT_0}P \subset \mathcal{C}^{cu}_{Z_{jT_0}x}(a)$ for all $j \geq 0$ by Proposition 2.9, it follows inductively that

$$|\det(DZ_t|P)| \geq (\tfrac{1}{2}K_0 e^{\theta_0 r})(\tfrac{1}{2}K_0 e^{\theta_0 T_0})^m \geq (\tfrac{1}{2}K_0)^{1+t/T_0} e^{\theta_0 t} = K e^{\theta t},$$

where $\theta = T_0^{-1}\log(\frac{1}{2}K_0 e^{\theta_0 T_0}) > 0$ and $K > 0$. Taking $P = E^{cu}_x$ yields the desired result. □

3. Global Poincaré map $f : X \to X$

In this section, we suppose that Λ is a partially hyperbolic attracting set, and recall how to construct a piecewise smooth Poincaré map $f : X \to X$ preserving a contracting stable foliation $\mathcal{W}^s(X)$. This largely follows [10] (see also [9, Chapter 6]) but with slight modifications; the details enable us to establish notation required for later sections. Mainly for notational convenience we restrict to the case $d_{cu} = 2$.

3.1. **Construction of the global cross-section X.** Let $y \in \Lambda$ be a regular point (not an equilibrium). There exists an open set (flow box) $V_y \subset U_0$ containing y such that the flow on V_y is diffeomorphic to a linear flow. More precisely, let $\mathcal{D}$ denote the $(\dim M - 1)$-dimensional unit disk and fix $\epsilon_0 \in (0, 1)$ small. There is a diffeomorphism $\chi : \mathcal{D} \times (-\epsilon_0, \epsilon_0) \to V_y$ with $\chi(0, 0) = y$ such that $\chi^{-1} \circ Z_t \circ \chi(z, s) = (z, s + t)$. Define the cross-section $\Sigma_y = \chi(\mathcal{D} \times \{0\})$.

For each $x \in \Sigma_y$, let $W^s_x(\Sigma_y) = \bigcup_{|t|<\epsilon_0} Z_t(W^s_x) \cap \Sigma_y$. This defines a topological foliation $\mathcal{W}^s(\Sigma_y)$ of Σ_y.

We can identify Σ_y diffeomorphically with $(-1, 1) \times \mathcal{D}^{d_s}$. The stable boundary $\partial^s\Sigma_y \cong \{\pm 1\} \times \mathcal{D}^{d_s}$ consists of two stable leaves. Let $\mathcal{D}^{d_s}_{1/2}$ denote the open disk of radius $\frac{1}{2}$ in $\mathbb{R}^{d_s}$. Define the *subcross-section* $\Sigma'_y \cong (-1, 1) \times \mathcal{D}^{d_s}_{1/2}$, and the corresponding subflow box $V'_y \cong \Sigma'_y \times (-\epsilon_0, \epsilon_0)$ consisting of trajectories in V_y that pass through Σ'_y.

For each equilibrium $\sigma \in \Lambda$, we let V_σ be an open neighborhood of σ on which the flow is linearizable. Let γ^s_σ and γ^u_σ denote the local stable and unstable manifolds of σ within V_σ; trajectories starting in V_σ remain in V_σ for all future time if and only if they lie in γ^s_σ.

Remark 3.1. Note that W^s_σ denotes the strong stable manifold of σ. In general, $\dim \gamma^s_\sigma \geq \dim W^s_\sigma = d_s$. (In the case of a Lorenz-like singularity, $\dim \gamma^s_\sigma = d_s + 1$.)

Define $V_0 = \bigcup_\sigma V_\sigma$. We shrink the neighborhoods V_σ so that (i) they are disjoint, (ii) $\Lambda \not\subset V_0$, and (iii) $\gamma^u_\sigma \cap \partial V_\sigma \subset V'_y$ for some regular point $y = y(\sigma)$.

By compactness of Λ, there exists $\ell \geq 1$ and regular points $y_1, \ldots, y_\ell \in \Lambda$ such that $\Lambda \setminus V_0 \subset \bigcup_{j=1}^{\ell} V'_{y_j}$. We enlarge the set $\{y_j\}$ to include the points $y(\sigma)$ mentioned in (iii) above. Adjust the positions of the cross-sections Σ_{y_j} if necessary so that they are disjoint, and define the global cross-section

$$X = \bigcup_{j=1}^{\ell} \Sigma_{y_j}.$$

In the remainder of the paper, we often modify the choices of U_0 and T_0. However, the choices of V_{y_j}, Σ_{y_j} and X remain unchanged from now on and correspond to our current choice of U_0 and T_0. To avoid confusion, all subsequent choices will be labelled $U_1 \subset U_0$ and $T_1 \geq T_0$. In particular, we suppose from now on that $U_1 \subset V_0 \cup \bigcup_{j=1}^{\ell} V'_{y_j}$.

3.2. **Definition of the Poincaré map.** By Proposition 2.8, for any $\delta > 0$ we can choose $T_1 \geq T_0$ such that

$$\operatorname{diam} Z_t(W^s_x(\Sigma_{y_j})) < \delta, \quad \text{for all } x \in \Sigma_{y_j},\ j = 1, \ldots, \ell,\ t > T_1. \tag{3.1}$$

Define

$$\Gamma_0 = \{x \in X : Z_{T_1+1}(x) \in \textstyle\bigcup_\sigma \gamma^s_\sigma\}, \qquad X' = X \setminus \Gamma_0.$$

If $x \in X'$, then $Z_{T_1+1}(x)$ cannot remain inside V_0 so there exists $t > T_1 + 1$ and $j = 1, \ldots, \ell$ such that $Z_t x \in V'_{y_j}$. Since $\epsilon_0 < 1$, there exists $t > T_1$ such that $Z_t x \in \Sigma'_{y_j}$. Hence for $x \in X'$, we can define

$$f(x) = Z_{\tau(x)}(x) \qquad \text{where} \qquad \tau(x) = \inf\{t > T_1 : Z_t x \in \textstyle\bigcup_{j=1}^{\ell} \operatorname{cl} \Sigma'_{y_j}\}.$$

In this way we obtain a piecewise C^r global Poincaré map $f : X' \to X$ with piecewise C^r roof function $\tau : X' \to [T_1, \infty)$.

Lemma 3.2. *If Λ contains no equilibria (so $\Gamma_0 = \emptyset$), then $\tau \leq T_1 + 2$. In general, there exists a constant $C > 0$ such that*

$$\tau(x) \leq -C \log \operatorname{dist}(x, \Gamma_0) \quad \textit{for all } x \in X'.$$

Proof. This is a standard result so we sketch the arguments.

If $Z_{T_1+1}x \in V'_{y_j}$ for some j, then $Z_t x \in \Sigma'_{y_j}$ for some $t \in (T_1 + 1 - \epsilon_0, T_1 + 1 + \epsilon_0)$ so $\tau(x) \leq T_1 + 2$. Otherwise, $Z_{T_1+1}x \in V_\sigma \subset V_0$ for some equilibrium σ, and we define

$$\tau_0(x) = \sup\{t \in [0, T_1 + 1] : Z_t x \notin V_\sigma\}, \qquad \tau_1(x) = \sup\{t \geq T_1 + 1 : Z_t x \in V_\sigma\}.$$

Note that $Z_{\tau_1(x)}(x) \in \bigcup_j V'_{y_j}$ so $\tau(x) \leq \tau_1(x) + 1 \leq \tau_1(x) - \tau_0(x) + T_1 + 2$.

By the Hartman-Grobman Theorem, the flow in V_σ is homeomorphic (by a time-preserving conjugacy) to the linearized flow $\dot{x} = DG(\sigma)x = (A \oplus E)x$ where A has eigenvalues with negative real part and E has eigenvalues with positive real part. After writing E in Jordan normal form, a standard and elementary argument shows that the "time of flight" of trajectories in V_σ satisfies $\tau_1(x) - \tau_0(x) \leq -C' \log \operatorname{dist}(Z_{\tau_0(x)}(x), \Gamma')$ where Γ' denotes the local stable manifold of σ in the linear flow.

Finally, we can suppose without loss that ∂V_σ is smooth so that the initial transition $x \mapsto Z_{\tau_0(x)}(x)$ is a diffeomorphism in a neighborhood of Γ_0. Hence $\operatorname{dist}(Z_{\tau_0(x)}(x), \Gamma') \approx \operatorname{dist}(x, \Gamma_0)$ up to uniform constants. □

Remark 3.3. It is immediate from the proof of Lemma 3.2 that $\tau(x) \to \infty$ as $\operatorname{dist}(x, \Gamma_0) \to 0$.

Define the topological foliation $\mathcal{W}^s(X) = \bigcup_{j=1}^{\ell} \mathcal{W}^s(\Sigma_{y_j})$ of X with leaves $W^s_x(X)$ passing through each $x \in X$.

Proposition 3.4. *For T_1 sufficiently large, $f(W^s_x(X)) \subset W^s_{fx}(X)$ for all $x \in X'$.*

Proof. By definition of V'_{y_j}, it follows from (3.1) that we can choose T_1 large (and hence δ small) such that $W^s_{fx}(X) \subset V_{y_j}$ whenever $fx \in V'_{y_j}$. The result follows from this by definition of $\mathcal{W}^s(X)$ and flow invariance of $\mathcal{W}^s$. □

Define $\partial^s X = \bigcup_{j=1}^{\ell} \partial^s \Sigma_{y_j}$ and let

$$\Gamma = \Gamma_0 \cup \Gamma_1, \qquad \Gamma_1 = \{x \in X' : fx \in \partial^s X\}.$$

Proposition 3.5. Γ *is a finite union of stable disks $W^s_x(X)$, $x \in X$.*

Proof. It is clear that $W^s_x(X) \subset \Gamma$ for all $x \in \Gamma$. Also, if $x_0 \notin \Gamma$ then $fx_0 = Z_{\tau(x_0)}(x_0) \in \Sigma'$ for some $\Sigma' \in \{\Sigma'_{y_j}\}$. For x close to x_0, it follows from continuity of the flow that $fx \in \Sigma'$ (with $\tau(x)$ close to $\tau(x_0)$). Hence $x \notin \Gamma$ and so Γ is closed.

It remains to rule out the possibility that a sequence of stable disks $W^s_{x_n}(X)$, $x_n \in \Gamma$, accumulates on $W^s_{x_0}(X)$ where $x_0 = \lim_{n\to\infty} x_n$. In showing this, it is useful to note that if $Z_t x \in V'_y$ then $Z_s x \in \Sigma'_y$ for some $s \in (t-1, t+1)$. In particular, if $Z_t x \in V'_y$ for some $t \ge T_1 + 1$, then $\tau(x) \le t + 1$.

There are two cases to consider:

Case 1: $Z_{T_2} x_0 \in V'_y$ for some $T_2 \ge T_1 + 1$, $y \in \{y_1, \ldots, y_\ell\}$. In this case, restricting to large n we have $Z_{T_2} x_n \in V'_y$, and hence $\tau(x_n) \le T_2 + 1$. It follows that $\bigcup_n W^s_{x_n}(X) \subset X \cap \bigcup_j \bigcup_{t\in[0,T_2+1]} Z_{-t}(\partial^s \Sigma_{y_j})$. But this is a compact submanifold of X with the same dimension d_s as the stable disks, so $\{x_n\}$ is finite.

Case 2: $Z_t x_0 \in V'_\sigma$ for all $t \ge T_1 + 1$ for some equilibrium σ. Note that $Z_t x_0 \in \gamma^s_\sigma$ for all $t \ge T_1 + 1$. As in Case 1, we can easily rule out accumulations when $\tau(x_n) \le T_1 + 1$ so we can suppose that $\tau(x_n) > T_1 + 1$. Also, $\gamma^s_\sigma \cap Z_{T_1+1}(X)$ is a compact submanifold of dimension d_s, so $Z_{T_1+1} x_n \in V'_\sigma \setminus \gamma^s_\sigma$. Hence the trajectory through $Z_{T_1+1} x_n$ eventually leaves V'_σ close to γ^u_σ. Such trajectories immediately enter the flow box $V'_{y(\sigma)}$ and hence hit $\Sigma'_{y(\sigma)}$. In particular, $f(x_n) \in \Sigma'_{y(\sigma)}$ and $x_n \notin \Gamma$. □

Let $X'' = X \setminus \Gamma$. Then $X'' = S_1 \cup \cdots \cup S_m$ for some $m \ge 1$, where each S_i is homeomorphic to $(-1, 1) \times \mathcal{D}^{d_s}$. We call these regions *smooth strips*. Note that $f|_{S_i} : S_i \to X$ is a diffeomorphism onto its image and $\tau|_{S_i} : S_i \to [T_1, \infty)$ is smooth for each i. The foliation $\mathcal{W}^s(X)$ restricts to a foliation $\mathcal{W}^s(S_i)$ on each S_i.

Remark 3.6. In future sections, it may be necessary to increase T_1 leading to changes to f, τ, Γ and $\{S_i\}$ (and the constant C in Lemma 3.2). However the global cross-section $X = \bigcup \Sigma_{y_j}$ continues to remain fixed throughout the paper.

4. Uniform hyperbolicity of the Poincaré map

Let Λ be a singular hyperbolic attracting set. We continue to assume $d_{cu} = 2$ for notational simplicity. In this section, we show that for T_1 sufficiently large, the global Poincaré map $f : X' \to X$ constructed in Section 3 is uniformly hyperbolic (with singularities). (As noted in Remark 3.6, the global cross-section $X = \bigcup \Sigma_{y_j}$ is independent of T_1.)

Let $S \in \{S_i\}$ be one of the smooth strips from the end of Section 3. There exist cross-sections Σ, $\widetilde{\Sigma} \in \{\Sigma_{y_j}\}$ such that $S \subset \Sigma$ and $f(\Sigma) \subset \widetilde{\Sigma}$.

The splitting $T_{U_0}M = E^s \oplus E^{cu}$ induces a continuous splitting $T\Sigma = E^s(\Sigma) \oplus E^u(\Sigma)$ defined by

$$E^s_x(\Sigma) = (E^s_x \oplus \mathbb{R}\{G(x)\}) \cap T_x\Sigma \quad \text{and} \quad E^u_x(\Sigma) = E^{cu}_x \cap T_x\Sigma, \quad x \in \Sigma.$$

The analogous definitions apply to $\widetilde{\Sigma}$.

For each $y \in \widetilde{\Sigma}$, define the projection $\pi_y : T_yM = T_y\widetilde{\Sigma} \oplus \mathbb{R}\{G(y)\} \to T_y\widetilde{\Sigma}$. Also, for $x \in \Sigma$, define the projection $\hat{\pi}_x : E^s_x \oplus \mathbb{R}\{G(x)\} \to E^s_x$.

By finiteness of the set of cross-sections $\{\Sigma_{y_j}\}$, there is a universal constant $C_1 \geq 1$ such that

$$\begin{aligned} \|\pi_y v\| &\leq C_1\|v\| \quad \text{for all } v \in T_yM, \\ \|\hat{\pi}_x v\| &\leq C_1\|v\| \quad \text{for all } v \in E^s_x \oplus \mathbb{R}\{G(x)\}. \end{aligned} \tag{4.1}$$

Proposition 4.1. *(a) $Df \cdot E^s_x(\Sigma) = E^s_{fx}(\widetilde{\Sigma})$ for all $x \in S$, and $Df \cdot E^u_x(\Sigma) = E^u_{fx}(\widetilde{\Sigma})$ for all $x \in \Lambda \cap S$.*
(b) Let $\lambda_1 \in (0,1)$. For T_1 sufficiently large if $\inf \tau > T_1$, *then for all $S \in \{S_i\}$,*

$$\|Df|E^s_x(\Sigma)\| \leq \lambda_1 \quad \textit{and} \quad \|Df|E^u_x(\Sigma)\| \geq \lambda_1^{-1} \quad \textit{for all } x \in S.$$

Proof. (a) For $x \in S$, we have that $Df(x) : T_x\Sigma \to T_{fx}\widetilde{\Sigma}$ is given by

$$Df(x) = D(Z_{\tau(x)}(x)) = DZ_{\tau(x)}(x) + G(fx)D\tau(x). \tag{4.2}$$

Let $v \in E^s_x(\Sigma) \subset E^s_x + \mathbb{R}\{G(x)\}$. Then using DZ_t-invariance of E^s on U_0 and of the flow direction,

$$Df(x)v \in DZ_{\tau(x)}(x)E^s_x + DZ_{\tau(x)}(x)\mathbb{R}\{Gx\} + \mathbb{R}\{G(fx)\} \subset E^s_{fx} + \mathbb{R}\{G(fx)\},$$

so $Df(x)v \in (E^s_{fx} + \mathbb{R}\{G(fx)\}) \cap T_{fx}\widetilde{\Sigma} = E^s_{fx}(\widetilde{\Sigma})$.

Similarly, for $x \in \Lambda \cap S$ and $v \in E^u_x(\Sigma) \subset E^{cu}_x$, using DZ_t-invariance of E^{cu} on Λ and the fact that the flow direction lies in E^{cu},

$$Df(x)v \in DZ_{\tau(x)}(x)E^{cu}_x + \mathbb{R}\{G(fx)\} \subset E^{cu}_{fx},$$

so $Df(x)v \in E^{cu}_{fx} \cap T_{fx}\widetilde{\Sigma} = E^u_{fx}(\widetilde{\Sigma})$.

(b) By (4.2) and the definition of π_y,

$$Df(x) = \pi_{fx} Df(x) = \pi_{fx} DZ_{\tau(x)}(x) \quad \text{for } x \in S. \tag{4.3}$$

Using the definition of $\hat{\pi}_x$, for $v \in E^s_x(\Sigma) \subset E^s_x \oplus \mathbb{R}\{G(x)\}$,

$$\|Df(x)v\| = \|\pi_{fx} DZ_{\tau(x)}(x)\hat{\pi}_x v\| \le C_1^2 \|DZ_{\tau(x)}(x)|E^s_x\| \, \|v\|,$$

by (4.1). It follows that

$$\|Df|E^s_x(\Sigma)\| \le C_1^2 C\lambda^{\tau(x)} \le C_1^2 C\lambda^{T_1},$$

where $C > 0$, $\lambda \in (0,1)$ are as in (2.1). The first estimate in (b) is immediate for T_1 large enough.

For the second estimate, define $P = DZ_{\tau(x)} E^{cu}_x$ and write $DZ_{\tau(x)}(x) : E^{cu}_x \to P$ in coordinates corresponding to the splittings

$$E^{cu}_x = E^u_x(\Sigma) \oplus \mathbb{R}\{G(x)\}, \qquad P = (P \cap \widetilde{\Sigma}) \oplus \mathbb{R}\{G(fx)\}.$$

In these coordinates, it follows from invariance and neutrality of the flow direction that

$$DZ_{\tau(x)}(x) = \begin{pmatrix} a_{11}(x) & 0 \\ a_{21}(x) & a_{22}(x) \end{pmatrix},$$

where $\sup_x |a_{22}(x)| \le C_2$ for some constant $C_2 > 0$. Moreover, by (4.3),

$$a_{11}(x) = \pi_{fx} DZ_{\tau(x)}(x)|_{E^u_x(\Sigma)} = Df(x)|_{E^u_x(\Sigma)}.$$

Hence by Proposition 2.10,

$$\begin{aligned} |Df(x)|E^u_x(\Sigma)| = |a_{11}(x)| &\ge C_2^{-1} |\det DZ_{\tau(x)}(x)|E^{cu}_x| \\ &\ge C_2^{-1} K e^{\theta\tau(x)} \ge C_2^{-1} K e^{\theta T_1} \ge \lambda_1^{-1}, \end{aligned}$$

for T_1 sufficiently large. □

Next, for $a > 0$ we define the *unstable cone field*

$$\mathcal{C}^u_x(\Sigma, a) = \{w = w^s + w^u \in E^s_x(\Sigma) \oplus E^u_x(\Sigma) : \|w^s\| \le a\|w^u\|\}, \quad x \in \Sigma.$$

Proposition 4.2. *For any $a > 0$, $\lambda_1 \in (0,1)$, we can increase T_1 and shrink U_1 such that if $\inf \tau > T_1$ then for all $S \in \{S_i\}$*
(a) $Df(x) \cdot \mathcal{C}^u_x(\Sigma, a) \subset \mathcal{C}^u_{fx}(\Sigma, a)$ for all $x \in S$.
(b) $\|Df(x)w\| \ge \lambda_1^{-1}\|w\|$ for all $x \in S$, $w \in \mathcal{C}^u_x(\Sigma, a)$.

Proof. Let $w = w^s + w^u \in \mathcal{C}^u_x(\Sigma, a)$. The estimates in Proposition 4.1(b) hold with $\lambda_1 = 1$, so

$$\|Df(x)w^s\| \le \|w^s\| \le a\|w^u\| \le a\|Df(x)w^u\|,$$

proving (a).
(b) Let $\lambda_1 \in (0,1)$ be the constant in Proposition 4.1(b). For $w \in \mathcal{C}^u_x(\Sigma, a)$,

$$\|Df(x)w\| \geq (1-a)\lambda_1^{-1}\|w^u\| \geq (1-a)(1+a)^{-1}\lambda_1^{-1}\|w\|$$

Since λ_1 is arbitrarily small, the result follows with a new value of λ_1. □

Taking unions over smooth strips S and cross-sections Σ, we obtain a global continuous uniformly hyperbolic splitting

$$TX'' = E^s(X) \oplus E^u(X),$$

with the following properties:

Theorem 4.3. *The stable bundle $E^s(X)$ and the restricted splitting $T_\Lambda X'' = E^s_\Lambda(X) \oplus E^u_\Lambda(X)$ are Df-invariant.*

Moreover, for fixed $a > 0$, $\lambda_1 \in (0,1)$, we can arrange that

$$Df \cdot \mathcal{C}^u_x(X,a) \subset \mathcal{C}^u_{fx}(X,a) \quad \text{and} \quad \|Df(x)w\| \geq \lambda_1^{-1}\|w\|$$

for all $x \in X''$, $w \in \mathcal{C}^u_x(X,a)$. □

5. The stable lamination is a topological foliation

The stable manifold theorem guarantees the existence of an Z_t-invariant stable lamination consisting of smoothly embedded disks W^s_x through each point $x \in \Lambda$. For general partially hyperbolic attracting sets, there is no guarantee that $\{W^s_x : x \in \Lambda\}$ defines a topological foliation in an open neighborhood of Λ. However, in this section we show that this is indeed the case under our assumptions that Λ is singular hyperbolic with $d_{cu} = 2$:

Theorem 5.1. *Let Λ be a singular hyperbolic attracting set with $d_{cu} = 2$. Then the stable lamination $\{W^s_x : x \in \Lambda\}$ is a topological foliation of an open neighborhood of Λ.*

The method of proof is to show that $\{W^s_x : x \in \Lambda\}$ coincides with the topological foliation $\{W^s_x : x \in U_0\}$ in Proposition 2.8(c). In particular, we have *a posteriori* that $\Lambda \subset \operatorname{Int} \bigcup_{x \in \Lambda} W^s_x$. The proof shows that for every x in an open neighbourhood of Λ, there exists $z \in \Lambda$ such that $x \in W^s_z$ (and hence $W^s_x = W^s_z$).

Fix $a > 0$ as in Theorem 4.3. A smooth curve $\gamma : [0,1] \to \Sigma \subset X$ is called a *u-curve* if $D\gamma(t) \in \mathcal{C}^u_{\gamma(t)}(\Sigma, a)$ for all $t \in [0,1]$. We say that a u-curve γ contained in X *crosses* a smooth strip S if each stable leaf $W^s_x(S)$ intersects γ in a unique point.

Proposition 5.2. *For every u-curve γ_0 there exists $n \geq 1$ and a restriction $\hat\gamma \subset \gamma_0$ so that $f^n|_{\hat\gamma} : \hat\gamma \to f^n\hat\gamma$ is a diffeomorphism and $f^n\hat\gamma$ crosses S_j for some j.*

Proof. We choose $\lambda_1 \in (0, \frac{1}{4}]$. Let $S \in \{S_1, \ldots, S_m\}$ and let γ be a u-curve in S with length $|\gamma|$. We consider three possibilities:

(i) $f\gamma \subset S_i$ for some i. In this case $|f\gamma| \geq 4|\gamma|$ by Theorem 4.3.

(ii) $f\gamma$ intersects $\bigcup \partial S_i$ in precisely one point q. In this case at least one of the connected components of $f\gamma \setminus \{q\}$ has length at least $2|\gamma|$.

(iii) $f\gamma$ intersects $\bigcup \partial S_i$ in at least two points.

In case (iii), we are finished with $n = 1$. In the other cases, we can pass to a restriction $\tilde{\gamma}$ such that $\tilde{\gamma}$ and $f\tilde{\gamma}$ lie in smooth strips with $|f\tilde{\gamma}| \geq 2|\gamma|$.

By Theorem 4.3, $f\tilde{\gamma}$ is a u-curve so we can repeat the procedure. After one such repetition, either the process has terminated with $n = 2$ or there is a restriction $\tilde{\gamma}$ such that $\tilde{\gamma}$ and $f^2\tilde{\gamma}$ lie in smooth strips with $|f^2\tilde{\gamma}| \geq 4|\gamma|$. Since X is bounded, the process terminates in finitely many steps. □

Proposition 5.3. *There exists a finite set $\{p_1, \ldots, p_k\} \subset X \cap \Lambda$ such that each p_i is a periodic point for f and $\bigcup_{n\geq 0} f^{-n}\big(\bigcup_{i=1}^k W^s_{p_i}(X)\big)$ is dense in X.*

Proof. Let γ_0 be a u-curve lying in a smooth strip. By Proposition 5.2, $f^{n_1}\gamma_0$ crosses a smooth strip for some $n_1 \geq 1$. Moreover, there exists a restriction $\tilde{\gamma}_0 \subset \gamma_0$ such that f^{n_1} maps $\tilde{\gamma}_0$ diffeomorphically inside this strip. Applying Proposition 5.2 again, we obtain $n_2 > n_1$ such that $f^{n_2}\gamma_0$ crosses a strip. Inductively, we obtain $1 \leq n_1 < n_2 < \cdots$ such that $f^{n_j}\gamma_0$ crosses a strip for each n_j. Since the number of smooth strips is finite, there exists $1 \leq q_1 < q_2$ such that $f^{q_1}\gamma_0$ and $f^{q_2}\gamma_0$ cross the same smooth strip S.

Let $q = q_2 - q_1$, $\gamma = f^{q_1}\gamma_0$. Choose a restriction $\tilde{\gamma}$ of γ such that $f^q|_{\tilde{\gamma}} : \tilde{\gamma} \to f^q\tilde{\gamma}$ is a diffeomorphism and $f^q\tilde{\gamma}$ crosses S. Shrink γ and $\tilde{\gamma}$ if necessary so that γ and $f^q\tilde{\gamma}$ cross $\operatorname{cl} S$ and are contained in $\operatorname{cl} S$.

Define the surjection $g : \tilde{\gamma} \to \gamma$ such that $g(x)$ is the unique point where $W^s_{f^q x}(X)$ intersects γ. Since $W^s(X)$ restricts to a topological foliation of S, it follows that g is continuous. Also $\tilde{\gamma} \subset \gamma$ are one-dimensional curves, so by the intermediate value theorem g possesses a fixed point $x_0 \in \operatorname{cl} \tilde{\gamma}$.

Since $g(x_0) = x_0$ it follows that $f^q x_0 \subset W^s_{x_0}(X)$ and hence that $f^q(W^s_{x_0}(X)) \subset W^s_{x_0}(X)$. By (3.1), $f^q : W^s_{x_0}(X) \to W^s_{x_0}(X)$ is a strict contraction, so $f^q p = p$ for some $p \in W^s_{x_0}(X)$. In particular, p is a periodic point for f lying in $X \cap U_0$. Since Λ is an attracting set, $p \in X \cap \Lambda$. Moreover, $f^{q_1}\gamma_0$ intersects $W^s_p(X)$.

Starting with a new u-curve γ_0' and proceeding as before, either $f^n\gamma_0'$ crosses S and hence intersects $W^s_p(X)$ for some $n \geq 0$, or we can construct a new periodic orbit p' in a new smooth strip such that $f^n\gamma_0'$ intersects $W^s_{p'}(X)$. In this way we obtain periodic points $p_1, \ldots, p_k$ such that every u-curve eventually intersects $\bigcup_{i=1}^k W^s_{p_i}(X)$ under iteration. Since u-curves are dense and arbitrarily short, the result follows. □

Remark 5.4. The periodic points constructed in the proof of Proposition 5.3 lie in distinct smooth strips, so $k \leq m$. The proof does not show that each strip contains a periodic point.

Proposition 5.5. *For each $x \in X$ there exists $y \in X \cap \Lambda$ such that $x \in W^s_y(X)$.*

Proof. Define
$$E = \{x \in X : x \in W^s_y(X) \text{ for some } y \in X \cap \Lambda\}.$$
We show that $E = X$.

Suppose first that $x \in \bigcup_{n\ge 0} f^{-n}\big(\bigcup_{i=1}^k W^s_{p_i}(X)\big)$, so there exists $n \ge 0$, $i \in \{1,\dots,k\}$ and $y \in W^s_{p_i}(X)$ such that $f^n x = y$. Choose an open set V formed of a union of stable leaves and containing x such that $f^n|_V : V \to f^n V$ is a diffeomorphism. By Remark 2.4, periodic points are not isolated inside $X \cap \Lambda$, so there exists a sequence W_j of stable leaves inside $f^n V \cap E$ that converges to $W^s_{p_i}(X)$. Choose $y_j \in W_j$ such that $y_j \to y$. Let $x_j = f^{-n} y_j$ so $x_j \to x$.

Since $y_j \in E$, we have $y_j \in W^s_{y'_j}(X)$ for some $y'_j \in f^n V \cap \Lambda$. Write $y'_j = f^n x'_j$ where $x'_j \in V \cap \Lambda$. Since $f^n|_V$ is a diffeomorphism and $y_j \in W^s_{y'_j}(X)$, it follows that $x_j \in W^s_{x'_j}(X)$.

Passing to a subsequence if needed, we can assume that $x'_j \to x' \in X \cap \Lambda$ and so $x_j \to x \in W^s_{x'}(X) \subset E$.

We have shown that E contains $\bigcup_{n\ge 0} f^{-n}\big(\bigcup_{i=1}^k W^s_{p_i}(X)\big)$ and so E is dense in X by Proposition 5.3.

Now for $x \in \Sigma$ we take $x_k \in E$ so that $x_k \to x$. We know that $x_k = W^s_{y_k}(\Sigma)$ for $y_k \in A$ and passing to a subsequence we find $y \in A \cap \Sigma$ so that $y_k \to y$. Then $x \in W^s_y(\Sigma)$ and $x \in E$. □

Proof of Theorem 5.1. If $x \in W^u_\sigma$ for some equilibrium σ, then $x \in \Lambda$ and there is nothing to do. Otherwise, restricting to a smaller positively invariant neighborhood U_0, we can ensure that there always exists $t > 0$ such that $Z_{-t}x$ lies in one of the flow boxes V_{y_j}. But then there exists $t > 0$ such that $Z_{-t}x \in \Sigma_{y_j} \subset X$. By Proposition 5.5, $Z_{-t}x \in W^s_y(X)$ for some $y \in X \cap \Lambda$. Hence there exists $t > 0$ such that $Z_{-t}x \in W^s_y$, and so $x \in W^s_z$ where $z = Z_t y \in \Lambda$. □

We have shown that the stable lamination $\{W^s_x : x \in \Lambda\}$ coincides with the stable foliation $\{W^s_x : x \in U_0\}$. From now on, we refer to $\mathcal{W}^s = \{W^s_x : x \in \Lambda\}$ as the stable foliation.

6. Hölder regularity and absolute continuity of the stable foliation

In this section, we continue to assume that Λ is a singular hyperbolic attracting set, and show that the topological foliation $\mathcal{W}^s$ is in fact a Hölder foliation (bi-Hölder charts). Also we recall results on absolute continuity of the stable foliation. These results do not use explicitly the fact that $d_{cu} = 2$; it suffices that the conclusion of Theorem 5.1 holds.

A key ingredient is regularity of stable holonomies. Let $Y_0, Y_1 \subset U_0$ be two smooth disjoint d_{cu}-dimensional disks that are transverse to the stable foliation $\mathcal{W}^s$. Suppose that for all $x \in Y_0$, the stable leaf W^s_x intersects each of Y_0 and Y_1 in precisely one point. The *stable holonomy* $H : Y_0 \to Y_1$ is given by defining $H(x)$ to be the intersection point of W^s_x with Y_1.

Lemma 6.1. *There exists $\epsilon > 0$ such that the stable holonomies $H : Y_0 \to Y_1$ are C^ϵ. Moreover, if the angles between Y_i and stable leaves are bounded away from zero for $i = 0, 1$, then there is a constant $K > 0$ dependent on this bound but otherwise independent of the holonomy $H : Y_0 \to Y_1$ such that $d(H(y), H(y')) \le Kd(y, y')^\epsilon$ for all $y, y' \in Y_0$.*

Proof. By Theorem 5.1, we can view $\mathcal{W}^s$ as the stable lamination corresponding to the invariant splitting $T_\Lambda M = E^s \oplus E^{cu}$ for the partially hyperbolic diffeomorphism $f = Z_1$. Hence we can apply [45, Theorem A']. The result in [45] is formulated slightly differently in terms of a splitting $T_\Lambda M = E^s \oplus E^c \oplus E^u$, but their proof covers our situation (with the invariant splitting $T_\Lambda M = E^u \oplus E^{cs}$ there replaced by the symmetric situation $T_\Lambda M = E^s \oplus E^{cu}$). □

Theorem 6.2. *The stable foliation $\mathcal{W}^s$ is C^ϵ for some $\epsilon > 0$.*

Proof. Let $\{\gamma(x) : x \in U_0\}$ be the family of embeddings $\gamma(x) : \mathcal{D}^{d_s} \to W^s_x$ described in Proposition 2.8.

Let $x \in U_0$ and choose an embedded d_{cu}-dimensional disk $Y_0 \subset M$ containing x and transverse to W^s_x. By continuity of E^s, we can shrink Y_0 so that Y_0 is transverse to W^s_y at y for all $y \in Y_0$. Let $\psi : \mathcal{D}^{d_{cu}} \to Y_0$ be a smooth embedding. The proof of [7, Lemma 4.9] shows that the map $\chi : \mathcal{D}^{d_s} \times \mathcal{D}^{d_{cu}} \to U_0$ given by

$$\chi(u, v) = \gamma(\psi(v))(u)$$

is a topological chart for $\mathcal{W}^s$ at x. Note that χ maps horizontal lines $\{v = \text{const.}\}$ homeomorphically onto stable disks $W^s_{\psi(v)}$.

Moreover, we claim that χ maps vertical lines $\{u = \text{const.}\}$ onto smooth transversals Y_u to $\mathcal{W}^s$. To see this, we recall the notation $\gamma(y)(u) = Q(u, \varphi_y(u))$ from the proof of [7, Lemma 4.8]. Here $Q = Q_{x,0} : \mathbb{R}^d \to M$ is a diffeomorphism and $Q^{-1}(W^s_y)$ is given by the graph of $\varphi_y : \mathcal{D}^{d_s} \to \mathcal{D}^{d_{cu}}$. Hence

$$\begin{aligned} Y_u = \{\chi(u, v) : v \in \mathcal{D}^{d_{cu}}\} &= \{\gamma(\psi(v))(u) : v \in \mathcal{D}^{d_{cu}}\} \\ &= \{\gamma(y)(u) : y \in Y_0\} = Q\{(u, \varphi_y(u)) : y \in Y_0\}. \end{aligned}$$

The curves W^s_y foliate U_0, so the curves $Q^{-1}(W^s_y) = \{(u, \varphi_y(u))\}$ foliate $\mathcal{D}^{d_s} \times \mathcal{D}^{d_{cu}}$. Hence the set $\{(u, \varphi_y(u)) : y \in Y_0\}$ is precisely $\{u = \text{const.}\}$ and so $Y_u = Q(\{u = \text{const.}\})$ verifying the claim.

Moreover, via the diffeomorphism Q, the angles of Y_u with stable disks W^s_y are bounded away from zero. Hence for any $u \neq 0$, the stable holonomy $H_u : Y_0 \to Y_u$ satisfies $d(H_u(y), H_u(y')) \le Kd(y, y')^\epsilon$ for all $y, y' \in Y_0$ by Lemma 6.1. Also $H_u^{-1} : Y_u \to Y_0$ is a stable holonomy, so $d(H_u^{-1}(y), H_u^{-1}(y')) \le Kd(y, y')^\epsilon$ for all $y, y' \in Y_u$.

Now $\chi(u,v) = \gamma(\psi(v))(u) = H_u(\psi(v))$, so

$$\begin{aligned} d(\chi(u,v),\chi(u,v')) &= d(H_u(\psi(v)),H_u(\psi(v'))) \\ &\le K d(\psi(v),\psi(v'))^\epsilon \le K(\operatorname{Lip}\psi)^\epsilon \|v-v'\|^\epsilon. \end{aligned}$$

Also there is a constant $L_1 > 0$ such that

$$\begin{aligned} d(\chi(u,v'),\chi(u',v')) &= d(\gamma(\psi(v'))(u),\gamma(\psi(v'))(u')) \\ &\le \operatorname{Lip}\gamma(\psi)(v')\|u-u'\| \le L\|u-u'\| \le L_1\|u-u'\|^\epsilon. \end{aligned}$$

Altogether, letting $M = \max\{K(\operatorname{Lip}\psi)^\epsilon, L_1\}$,

$$d(\chi(u,v),\chi(u',v')) \le M\big(\|u-u'\|^\epsilon + \|v-v'\|^\epsilon\big) \le CM(\|u-u'\|^2 + \|v-v'\|^2)^{\epsilon/2}$$

where $C > 0$ is an upper bound for the homogeneous function $\dfrac{|x|^\epsilon + |y|^\epsilon}{(|x|^2+|y|^2)^{\epsilon/2}}$ over the set of $(x,y) \in \mathbb{R}^2$ such that $|x|^2 + |y|^2 = 1$. Hence χ is C^ϵ.

Next,

$$\begin{aligned} \|u-u'\| &\le \|(u,\varphi_{\psi(v)}(u)) - (u',\varphi_{\psi(v')}(u'))\| \\ &\le \operatorname{Lip}(Q^{-1}) d(Q(u,\varphi_{\psi(v)}(u)), Q(u',\varphi_{\psi(v')}(u'))) \\ &= \operatorname{Lip}(Q^{-1}) d(\gamma(\psi(v))(u),\gamma(\psi(v'))(u')) = \operatorname{Lip}(Q^{-1}) d(\chi(u,v),\chi(u',v')), \end{aligned}$$

and

$$\begin{aligned} \|v-v'\| \le \operatorname{Lip}(\psi^{-1}) d(\psi(v),\psi(v')) &= \operatorname{Lip}(\psi^{-1}) d(H_{u'}^{-1}\chi(u',v), H_{u'}^{-1}\chi(u',v')) \\ &\le K \operatorname{Lip}(\psi^{-1}) d(\chi(u',v),\chi(u',v'))^\epsilon. \end{aligned}$$

Moreover,

$$\begin{aligned} \|(u',\varphi_{\psi(v)}(u')) - (u',\varphi_{\psi(v')}(u'))\| &= \|\varphi_{\psi(v)}(u') - \varphi_{\psi(v')}(u')\| \\ &\le \|\varphi_{\psi(v)}(u') - \varphi_{\psi(v)}(u)\| + \|\varphi_{\psi(v)}(u) - \varphi_{\psi(v')}(u')\| \\ &\le L\|u-u'\| + \|\varphi_{\psi(v)}(u) - \varphi_{\psi(v')}(u')\| \\ &\le (L+1)\|(u,\varphi_{\psi(v)}(u)) - (u',\varphi_{\psi(v')}(u'))\|, \end{aligned}$$

so

$$d(\chi(u',v),\chi(u',v')) \le (L+1)\operatorname{Lip}Q\operatorname{Lip}(Q^{-1}) d(\chi(u,v),\chi(u',v')).$$

Combining these estimates, we obtain $\|(u,v)-(u',v')\| \le \text{const.}\, d(\chi(u,v),\chi(u',v'))^\epsilon$. Hence $\|\chi^{-1}(p) - \chi^{-1}(p')\| \le \text{const.}\, d(p,p')^\epsilon$ for $p,p' \in U_0$, so $\chi^{-1} \in C^\epsilon$. □

Theorem 6.3. *The stable holonomy $H : Y_0 \to Y_1$ is absolutely continuous. That is, $m_1 \ll H_* m_0$ where m_i is Lebesgue measure on Y_i, $i = 0,1$.*

Moreover, the Jacobian $JH : Y_0 \to \mathbb{R}$ given by

$$JH(x) = \frac{dm_1}{dH_* m_0}(Hx) = \lim_{r\to 0} \frac{m_1(H(B(x,r)))}{m_0(B(x,r))}, \quad x \in Y_0,$$

is bounded above and below and is C^ϵ for some $\epsilon > 0$.

Proof. This essentially follows from [14, Theorems 8.6.1 and 8.6.13]. See also [44, Theorem 2.1] and [33, Section III.3]. The results there are formulated under a condition of the type $\sup_{x\in\Lambda}\|DZ_t|E^s_x\|\sup_{x\in\Lambda}\|DZ_{-t}|E^{cu}_{Z_tx}\|\le C\lambda^t$ which is more restrictive than the domination condition (2.2). However, it is standard that such results generalise to our setting. (See the remark in [44] after their theorem. Most of the required result is covered by [44] except that Hölder continuity of JH is not mentioned, only continuity.) □

7. One-dimensional quotient map $\bar f:\overline{X}\to\overline{X}$

In this section, we continue to suppose that Λ is a singular hyperbolic attracting set with $d_{cu}=2$. Let $f:X'\to X$ be the global Poincaré map defined in Section 3 with invariant stable foliation $\mathcal{W}^s(X)$. We now show how to obtain a one-dimensional piecewise $C^{1+\epsilon}$ uniformly expanding quotient map $\bar f:\overline{X}'\to\overline{X}$.

We begin by analysing the stable holonomies for f. Let γ_0, $\gamma_1\subset X$ be two u-curves such that for all $x\in\gamma_0$, the stable leaf $W^s_x(X)$ intersects each of γ_0 and γ_1 in precisely one point. The *(cross-sectional) stable holonomy* $h:\gamma_0\to\gamma_1$ is given by defining $h(x)$ to be the intersection point of $W^s_x(X)$ with γ_1.

Lemma 7.1. *The stable holonomy h is $C^{1+\epsilon}$ for some $\epsilon>0$.*

Proof. Recall that $X=\bigcup\Sigma_{y_j}$ where Σ_{y_j} is the cross-section associated to the flow box V_{y_j} for each j. Since the result is local, we can suppose that $\gamma_0,\gamma_1\subset\Sigma_{y_j}$ for some j and we can choose coordinates so that the local flow Z_t is linear.

Consider the 2-dimensional disks $Y_i=\bigcup_{t\in[-\delta_i,\delta_i]}Z_t(\gamma_i)=\gamma_i\times[-\delta_i,\delta_i], i=0,1$, for fixed $\delta_i>0$. These are smooth transversals to the stable foliation $\mathcal{W}^s$ of the flow. Provided δ_0 is small with respect to δ_1, we can then consider the holonomy $H:Y_0\to Y_1$ as in Section 6.

For $p=(v,0)\in\gamma_0\subset Y_0$ we write $H(v,0)=(H_1(v),\xi(v))$ with $H_1:\gamma_0\to\gamma_1$ and $\xi:\gamma_0\to[-\delta_1,\delta_1]$. Clearly $h=H_1$ by construction. Since $\mathcal{W}^s$ is flow invariant,

$$H(v,t)=(h(v),\xi(v)+t).$$

Let $\lambda_i=(\pi_i)_*m_i$ denote Lebesgue measure on $\gamma_i, i=0,1$, where $\pi_i:Y_i\to\gamma_i$ is the natural projection. By Theorem 6.3, $m_1\ll H_*m_0$. Since $\pi_1H=h\pi_0$,

$$\lambda_1=(\pi_1)_*m_1\ll(\pi_1H)_*m_0=(h\pi_0)_*m_0=h_*\lambda_0.$$

Hence h is absolutely continuous.

Taking balls $B(x,r)$ to be rectangles, we have for r sufficiently small

$$H(B(x,r))=\bigcup_{v'\in B(v,r)}\{h(v')\}\times(t+\xi(v')-r,t+\xi(v')+r).$$

By Fubini,

$$\frac{m_1(H(B(x,r)))}{m_0(B(x,r))}=\frac{2r\lambda_1(h(B(v)))}{2r\lambda_0(B(v,r))}=\frac{\lambda_1(h(B(v,r)))}{\lambda_0(B(v,r))},$$

showing that $JH(x) = Jh(v)$ for all $x = (v, t)$. By Theorem 6.3, Jh is Hölder. But $\dim \gamma_0 = \dim \gamma_1 = 1$, so $Jh = |Dh|$ and the result follows. □

Recall that X is a union of finitely many cross-sections Σ_{y_j}, and that f is smooth on a subset $X'' \subset X' \subset X$ which is obtained from X by removing finitely many stable leaves. Moreover, each $\Sigma_{y_j} \cap X''$ is a union of finitely many connected smooth strips S such that $f|_S : S \to f(S)$ is a diffeomorphism.

For each j, let $\gamma_j \subset \Sigma_{y_j}$ be a u-curve crossing Σ_{y_j}. Define $\overline{X} = \bigcup_j \operatorname{cl} \gamma_j$ and $\overline{X}' = X' \cap \overline{X}$. Given a smooth strip $S \subset \Sigma_{y_j}$, there exists k such that $f(S) \subset \Sigma_{y_k}$. Also $f(\gamma_j)$ is a u-curve by Theorem 4.3. Let $h : f(S \cap \gamma_j) \to \gamma_k$ be the associated stable holonomy and define $\bar{f}(x) = h(fx)$ for $x \in S \cap \gamma_j$. In this way we obtain a one-dimensional map $\bar{f} : \overline{X}' \to \overline{X}$.

Corollary 7.2. *The quotient map $\bar{f} : \overline{X}' \to \overline{X}$ is piecewise $C^{1+\epsilon}$ and consists of finitely many monotone $C^{1+\epsilon}$ branches. Choosing T_1 in Section 4 sufficiently large, we have $|D\bar{f}| \geq 2$ on $\overline{X}'$.*

Proof. Since f is smooth on smooth strips and the holonomies $h : f(S \cap \gamma_j) \to \gamma_k$ are $C^{1+\epsilon}$ by Lemma 7.1, it follows that $\bar{f}$ is piecewise $C^{1+\epsilon}$. The collection of intervals $S \cap \gamma_j$ is finite, so $\bar{f}$ has finitely many branches. By finiteness of the collection $\{\Sigma_{y_j}\}$, there is a constant $c > 0$ such that all the holonomies h considered above satisfy $|Dh| \geq c$. Hence taking λ_1 sufficiently small in Theorem 4.3, we can ensure that $|D\bar{f}|$ is as large as desired. □

8. Statistical properties for $\bar{f}$ and f

In this section, we investigate statistical properties for the $(d_s + 1)$-dimensional Poincaré map $f : X' \to X$ and the one-dimensional quotient map $\bar{f} : \overline{X}' \to \overline{X}$. Define $\pi : X \to \overline{X}$ Hölder by letting $\pi(x)$ be the point where $W^s_x(X)$ intersects $\overline{X}$. Then π defines a semiconjugacy between f and $\bar{f}$.

From now on, we write $f : X \to X$ and $\bar{f} : \overline{X} \to \overline{X}$ with the understanding that f and $\bar{f}$ are not defined everywhere (and are piecewise smooth where defined).

8.1. Spectral decomposition and physical measures.

Proposition 8.1. *There exists a finite number of ergodic absolutely continuous $\bar{f}$-invariant probability measures $\bar{\mu}_1, \ldots, \bar{\mu}_s$ whose basins cover a subset of $\overline{X}$ of full Lebesgue measure. For each j, the density $d\bar{\mu}_j / d\,\mathrm{Leb}$ lies in L^∞ and $\operatorname{Int}\operatorname{supp} \bar{\mu}_j \neq \emptyset$.*

Proof. By Corollary 7.2, $\bar{f}$ is a piecewise $C^{1+\epsilon}$ uniformly expanding one-dimensional map. Hence, most of the result is immediate from [28, Theorem 3.3]. We refer to [49, Lemma 3.1] for the fact that $\operatorname{Int}\operatorname{supp} \bar{\mu}_j \neq \emptyset$. □

Corollary 8.2. *There exists a finite number of ergodic f-invariant probability measures $\mu_1, \dots, \mu_s$ whose basins cover a subset of X of full Lebesgue measure. Moreover, $\pi_*\mu_j = \bar\mu_j$ for each j.*

Proof. This follows from the existence of the stable foliation $\mathcal{W}^s(X)$ (here, the fact that it is a topological foliation suffices) combined with Proposition 8.1. For details, see [10, Sections 6.1 and 6.2]. □

8.2. **Existence of an inducing scheme.** In this subsection, we suppose without loss that there is a unique absolutely continuous $\bar f$-invariant measure $\bar\mu$ in Proposition 8.1 (so $s = 1$).

Proposition 8.3. *There exists $k \ge 1$ such that* $\operatorname{supp}\bar\mu = \overline{X}_1 \cup \cdots \cup \overline{X}_k$ *where the sets $\overline{X}_j$ are permuted cyclically by $\bar f$, and $\bar f^k : \overline{X}_j \to \overline{X}_j$ is mixing for each j.*

Moreover, for any $\eta \in (0,1)$, there exist constants c, $C > 0$ such that

$$\Big|\int_{\overline{X}_j} v\, w\circ \bar f^{kn}\, d\bar\mu - \int_{\overline{X}_j} v\, d\bar\mu \int_{\overline{X}_j} w\, d\bar\mu\Big| \le C\|v\|_{C^\eta}|w|_1 e^{-cn} \quad \textit{for all } n\ge 1,\ j=1,\dots,k,$$

for all $v \in C^\eta(\overline{X})$ and $w \in L^1(\overline{X})$.

Proof. This is immediate from the quasicompactness of the transfer operator for $\bar f$ which is established in [28, Theorem 3.3]. Indeed the result in [28] is proved for the class of functions with finite η-variation (for all $\eta > 0$ sufficiently small). This includes observables that are C^η. □

For ease of exposition, we suppose for the remainder of this subsection that $k = 1$ and $\overline{X}_1 = \overline{X}$. Recall that a one-dimensional map $\overline{F} : \overline{Y} \to \overline{Y}$ is a full branch Gibbs-Markov map if there is an at most countable partition α of $\overline{Y}$ and constants $C > 0$, $\epsilon \in (0,1]$ such that for all $a \in \alpha$,

- $\overline{F}|_a : a \to \overline{Y}$ is a measurable bijection, and
- $\big|\log|D\overline{F}(y_1)| - \log|D\overline{F}(y_2)|\big| \le C|\overline{F}y_1 - \overline{F}y_2|^\epsilon$ for all $y_1, y_2 \in a$.

Lemma 8.4. *For all $\beta > 0$, there exists a positive measure subset $\overline{Y} \subset \overline{X}$ and a full branch Gibbs-Markov induced map $\overline{F} = \bar f^\rho : \overline{Y} \to \overline{Y}$, where $\rho : \overline{Y} \to \mathbb{Z}^+$ is constant on partition elements and satisfies* $\operatorname{Leb}(y \in \overline{Y} : \rho(y) > n) = O(n^{-\beta})$.

Proof. By Proposition 8.1, $\bar\mu$ is an ergodic absolutely continuous invariant probability measure on $\overline{X}$ with $d\bar\mu/d\operatorname{Leb} \in L^\infty$. The result follows from Theorem A.1 provided we verify that $\bar\mu$ is expanding and that conditions (C0)–(C3) hold. Let $\mathcal{S}$ denote the finite set consisting of singularities/discontinuities of $\bar f$. (In general $X \setminus \mathcal{S}$ is a proper subset of X' since $\mathcal{S}$ includes the discontinuities of the piecewise smooth map f.) Conditions (C0) and (C3) are redundant since $\bar f$ is one-dimensional. Conditions (C1) and (C2) become

(C1) $C^{-1}d(x,\mathcal{S})^q \le |D\bar f(x)| \le Cd(x,\mathcal{S})^{-q}$ for all $x \in \overline{X} \setminus \mathcal{S}$, (C2) $\big|\log|D\bar f(x)|-\log|D\bar f(x')|\big|\le C|x-x'|^\eta(|D\bar f(x)|^{-q}+|D\bar f(x)|^q)$ for all $x,x'\in \bar X\setminus\mathcal S$ with $|x-x'|<\operatorname{dist}(x,\mathcal S)/2$,

where $\eta\in(0,1)$ and $C,q>0$ are constants. Since $d\bar\mu/d\,\mathrm{Leb}\in L^\infty$ it is immediate from (C1) that $\log|(D\bar f)^{-1}|$ is integrable with respect to $\bar\mu$. Also $\int\log|(D\bar f)^{-1}|\,d\bar\mu\le \log\frac12<0$ by Corollary 7.2, so $\bar\mu$ is an expanding measure.

It remains to verify conditions (C1) and (C2). Note that they are trivially satisfied for functions $\bar f$ with $D\bar f$ Hölder and bounded below. Hence they are satisfied away from $\mathcal S$ and also near all discontinuity points in $\mathcal S$.

By Proposition 2.6, it remains to consider singularities $x_0\in X$ corresponding to Lorenz-like equilibria σ. The Poincaré map f can be written near x_0 as $f=h_1\circ g\circ h_2$ where g corresponds to the flow near σ and h_1, h_2 are the remaining parts of the Poincaré map. In particular $D\bar h_j$ is Hölder and bounded below for $j=1,2$.

Suppose first that the flow is $C^{1+\epsilon}$-linearizable for some $\epsilon>0$ in a neighborhood of σ. Incorporating the linearization into h_1 and h_2, we can suppose without loss that the flow is linear in a neighborhood of σ. Hence the flow is given by $x\mapsto e^{tA}x$ where $A=\lambda^u\oplus\lambda^s\oplus B$ with $-\lambda^u<\lambda^s<0<\lambda^u$ and $B=DG(\sigma)|E^s_\sigma$. A standard calculation shows that in suitable coordinates,

$$g(x,z)=(|x|^{-\lambda^s/\lambda^u},ze^{-\lambda_u^{-1}B\log|x|}).$$

In particular, $\bar g(x)=|x|^\omega$ where $\omega=-\lambda^s/\lambda^u\in(0,1)$.

Since $D\bar h_j$ is bounded above and below, it follows from the chain rule that

$$|D\bar f(x)|\cong|D\bar g(\bar h_2x)|=\omega|\bar h_2x|^{\omega-1}\cong|x-x_0|^{\omega-1},$$

so (C1) is satisfied. Next,

$$\begin{aligned}\big|\log|D\bar f(x)|-\log|D\bar f(x')|\big|&\le C_1\big(|x-x'|^\epsilon+\big|\log|D\bar g(\bar h_2x)|-\log|D\bar g(\bar h_2x')|\big|\\&\qquad+|\bar g(\bar h_2x)-\bar g(\bar h_2x')|^\epsilon\big)\\&=C_1\big(|x-x'|^\epsilon+(1-\omega)\big|\log|\bar h_2x|-\log|\bar h_2x'|\big|+\big||\bar h_2x|^\omega-|\bar h_2x'|^\omega\big|^\epsilon\big).\end{aligned}$$

Now

$$\begin{aligned}\big|\log|\bar h_2x|-\log|\bar h_2x'|\big|&\le C_2(|\bar h_2x-\bar h_2x'|/|\bar h_2x|)^{1-\omega}\le C_2'|x-x'|^{1-\omega}|x-x_0|^{\omega-1}\\&\le C_2''|x-x'|^{1-\omega}|D\bar f(x)|.\end{aligned}$$

Also without loss $|x-x_0|\le|x'-x_0|$, so

$$\begin{aligned}\big||\bar h_2x|^\omega-|\bar h_2x'|^\omega\big|&\le|\bar h_2x-\bar h_2x'|(|\bar h_2x|^{\omega-1}+|\bar h_2x'|^{\omega-1})\\&\le C_3|x-x'||x-x_0|^{\omega-1}\le C_3'|x-x'||D\bar f(x)|.\end{aligned}$$

Hence, there exists $\eta\in(0,1)$ such that

$$\begin{aligned}\big|\log|D\bar f(x)|-\log|D\bar f(x')|\big|&\le C_4|x-x'|^\eta(|D\bar f(x)|+1)\\&\le C_4|x-x'|^\eta(2|D\bar f(x)|+|D\bar f(x)|^{-1}),\end{aligned}$$

verifying (C2).

To complete the proof, we remove the assumption that the flow near σ is $C^{1+\epsilon}$-linearizable. By the center manifold theorem (eg. [26, Theorem 5.1]), locally we can choose a flow-invariant $C^{1+\epsilon}$ two-dimensional manifold W tangent to E^{cu}_σ (for some $\epsilon > 0$). Note that the quotient of $g|W$ coincides with $\bar g$. By a result of Newhouse [42] (stated previously but without proof in [25]), the flow restricted to W (being two-dimensional) can be $C^{1+\epsilon'}$ linearized for some $\epsilon' > 0$. The proof now proceeds as before. □

Remark 8.5. Since we have exponential decay of correlations in Proposition 8.3, there is the hope of obtaining an induced Gibbs-Markov map as in Lemma 8.4 but with exponential tails for ρ. (We note that Theorem A.1(2) which would give stretched exponential tails does not apply because the density $d\bar\mu/d\,\mathrm{Leb}$ is not bounded below.) In certain situations, it is possible to construct an inducing scheme with exponential tails by using different methods, controlling the tail of hyperbolic times and relating this with the tail of inducing times more directly [22, 5, 11]. One repercussion of the existence of such an inducing scheme would be that the error rate in the vector-valued ASIP would be improved to $n^{\frac14+\epsilon}$ for $\epsilon > 0$ arbitrarily small [23].

However, our construction here with superpolynomial tails holds in complete generality and suffices for our results on singular hyperbolic flows in Section 9, so we do not pursue this further.

Proposition 8.6. *There is a constant $C > 0$ such that*

$$\textstyle\sum_{\ell=0}^{\rho(y)-1}|\tau(f^\ell y)-\tau(f^\ell y')| \le C|\bar F y-\bar F y'|^\epsilon \quad \textit{for all } y,y'\in a,\ a\in\alpha.$$

Proof. It follows from the proof of Lemma 8.4 that the roof function $\tau : X \to \mathbb{R}^+$ satisfies $\tau(x) = -\lambda_u^{-1}\log|\bar h_2 x| + t(x)$ where $\bar h_2$ and t are $C^{1+\epsilon}$. Hence

$$|\tau(x)-\tau(x')| \le \lambda_u^{-1}|\bar h_2 x-\bar h_2 x'|/|\bar h_2 x| + |Dt|_\infty|x-x'| \le C_1|x-x'|d(x,\mathcal{S})^{-1},$$

and the result follows from (A.1). □

8.3. **Statistical limit laws for the Poincaré map.** By Corollary 8.2, there is a unique ergodic f-invariant probability measure μ on X corresponding to $\bar\mu$, with $\pi_*\mu = \bar\mu$.

Theorem 8.7. *Fix $\eta \in (0,1)$ and let $v \in C^\eta(X)$ with $\int_X v\,d\mu = 0$. Write $v_n = \sum_{j=0}^{n-1} v\circ f^j$. Then the limit $\sigma^2 = \lim_{n\to\infty} n^{-1}\int_\Lambda v_n^2\,d\mu$ exists. Suppose that $\sigma^2 > 0$. Then the following limit laws hold.*

ASIP [16]**:** *Let $\epsilon > 0$. There exists a probability space Ω supporting a sequence of random variables $\{S_n,\, n\ge 1\}$ with the same joint distributions as $\{v_n,\, n\ge1\}$, and a sequence $\{Z_n,\, n\ge1\}$ of i.i.d. random variables with distribution $N(0,\sigma^2)$, such that*

$$\textstyle\sup_{1\le k\le n}\big|S_k-\sum_{j=1}^k Z_j\big| = O(n^\epsilon) \textit{ a.e. as } n\to\infty.$$

Berry-Esseen [21]**:** *There exists $C>0$ such that*

$$\big|\mu\{x\in X: n^{-1/2}v_n(x)\le a\}-\mathbb{P}\{N(0,\sigma^2)\le a\}\big|\le Cn^{-1/2}\quad \textit{for all } a\in\mathbb{R},\ n\ge 1.$$

local limit theorem [21]**:** *Suppose that v is aperiodic (so it is not possible to write $v=c+g-g\circ f+\lambda q$ where $c\in\mathbb{R}$, $\lambda>0$, $g:X\to\mathbb{R}$ measurable and $q:X\to\mathbb{Z}$). Then for any bounded interval $J\subset\mathbb{R}$,*

$$\lim_{n\to\infty} n^{1/2}\mu(x\in X: v_n(x)\in J)=(2\pi\sigma^2)^{-1/2}|J|.$$

For C^η vector-valued observables $v:X\to\mathbb{R}^d$ with $\int_X v\,d\mu=0$, the limit $\Sigma=\lim_{n\to\infty} n^{-1}\int_\Lambda v_n v_n^T\,d\mu\in\mathbb{R}^{d\times d}$ exists and we obtain

vector-valued ASIP [37, 29]**:** *There exists $\lambda\in(0,\frac12)$ and a probability space Ω supporting a sequence of random variables $\{S_n,\, n\ge 1\}$ with the same joint distributions as $\{v_n,\, n\ge 1\}$, and a sequence $\{Z_n,\, n\ge 1\}$ of i.i.d. random variables with distribution $N(0,\Sigma)$, such that*

$$\sup_{1\le k\le n}\big|S_k-\textstyle\sum_{j=1}^k Z_j\big|=O(n^\lambda)\ \textit{a.e. as } n\to\infty.$$

Proof. The strategy is to model $\overline{F}:\overline{Y}\to\overline{Y}$ and $F:Y\to Y$ by "one-sided" and "two-sided" Young towers $\bar\Delta$ and Δ, and to construct an observable $\bar v:\bar\Delta\to\mathbb{R}$ to which the various results in the references can be applied. The desired statistical properties for v are deduced from those for $\bar v$.

Using $\overline{F}:\overline{Y}\to\overline{Y}$ and $\rho:\overline{Y}\to\mathbb{Z}^+$ as given in Lemma 8.4, we define the one-sided Young tower map $\bar f_\Delta:\bar\Delta\to\bar\Delta$,

$$\bar\Delta=\{(y,\ell)\in\overline{Y}\times\mathbb{Z}^+:0\le\ell\le\rho(y)-1\},\quad \bar f_\Delta(y,\ell)=\begin{cases}(y,\ell+1) & \ell\le\rho(y)-2\\ (\overline{F}y,0) & \ell=\rho(y)-1\end{cases}.$$

Let $\bar\mu_Y$ denote the unique absolutely continuous invariant probability measure for the Gibbs-Markov map $\overline{F}:\overline{Y}\to\overline{Y}$. Then $\bar\mu_\Delta=\bar\mu_Y\times\text{counting}/\int_{\overline{Y}}\rho\,d\bar\mu_Y$ is an ergodic $\bar f_\Delta$-invariant probability measure on $\bar\Delta$.

Next, define $Y=\pi^{-1}\overline{Y}\subset X$ to be the union of stable leaves $W^s_y(X)$ where $y\in\overline{Y}$. In the proof of Corollary 8.2, we used an argument from [10] which constructs μ on X starting from $\bar\mu$ on $\overline{X}$. The same argument constructs an ergodic F-invariant probability measure μ_Y on Y starting from $\bar\mu_Y$. Define $\rho:Y\to\mathbb{Z}^+$ and $F:Y\to Y$ by setting $\rho(y)=\rho(\pi y)$ and $F(y)=f^{\rho(y)}y$. Using these (instead of $\rho:\overline{Y}\to\mathbb{Z}^+$ and $\overline{F}:\overline{Y}\to\overline{Y}$) we obtain a two-sided Young tower map $f_\Delta:\Delta\to\Delta$ with ergodic f_Δ-invariant probability measure $\mu_\Delta=\mu_Y\times\text{counting}/\int_Y\rho\,d\mu_Y$. The projection $\pi:X\to\overline{X}$ extends to a semiconjugacy $\pi:\Delta\to\bar\Delta$ given by $\pi(y,\ell)=(\pi y,\ell)$, and $\pi_*\mu_\Delta=\bar\mu_\Delta$. Moreover, the projection

$$\pi_\Delta:\Delta\to X,\qquad \pi_\Delta(y,\ell)=f^\ell y,$$

is a semiconjugacy from f_Δ to f and $\pi_{\Delta\,*}\mu_\Delta=\mu$.

The separation time $s(y, y')$ of points $y, y' \in \overline{Y}$ is the least integer $n \geq 0$ such that $\overline{F}^n y$ and $\overline{F}^n y'$ lie in distinct elements of the partition α. This extends to $\bar{\Delta}$ by setting $s((y, \ell), (y', \ell')) = s(y, y')$ when $\ell = \ell'$ and zero otherwise, and then to Δ by setting $s(p, p') = s(\pi p, \pi p')$.

For each $\theta \in (0, 1)$, define the symbolic metric d_θ on $\bar{\Delta}$ given by $d_\theta(p, p') = \theta^{s(p,p')}$. Given $w : \bar{\Delta} \to \mathbb{R}$, we define

$$\|w\|_\theta = |w|_\infty + \sup_{p \neq p'} |w(p) - w(p')|/d_\theta(p, p').$$

Let $\lambda_1 \in (0, 1)$ be as in Propositions 4.1 and 4.2, and set $\theta = \lambda_1^{\eta/2}$. Let $v \in C^\eta(X, \mathbb{R}^d)$ with $\int_X v \, d\mu = 0$. We claim that there exists $\chi \in L^\infty(\Delta, \mathbb{R}^d)$ and $\bar{v} \in L^\infty(\bar{\Delta}, \mathbb{R}^d)$ with $\|\bar{v}\|_\theta < \infty$ such that

$$v \circ \pi_\Delta = \bar{v} \circ \pi + \chi \circ f_\Delta - \chi. \tag{8.1}$$

Suppose that the claim is true. Since $\bar{\Delta}$ is a one-sided Young tower [52] with superpolynomial tails (in fact $\beta > 2$ suffices here) and $\|\bar{v}\|_\theta < \infty$, it follows that $\bar{v}$ satisfies all of the desired statistical properties by the mentioned references. These are inherited (since π is measure-preserving) by $\bar{v} \circ \pi : \Delta \to \mathbb{R}^d$. Since $\chi \in L^\infty$, the properties are inherited by $v \circ \pi_\Delta : \Delta \to \mathbb{R}^d$ and thereby v (since π_Δ is measure-preserving).

It remains to verify the claim. Define $\chi : \Delta \to \mathbb{R}^d$,

$$\chi(p) = \sum_{j=0}^{\infty} \big(v \circ f^j \circ \pi_\Delta(\pi p) - v \circ f^j \circ \pi_\Delta(p) \big).$$

For $p = (y, \ell)$, using Proposition 4.1(a), we have

$$\begin{aligned} |\chi(p)| &\leq \sum_{j=0}^{\infty} |v|_{C^\eta} \|f^j \circ \pi_\Delta(\pi p) - f^j \circ \pi_\Delta(p)\|^\eta = |v|_{C^\eta} \sum_{j=0}^{\infty} \|f^{j+\ell}(\pi y) - f^{j+\ell}(y)\|^\eta \\ &\leq |v|_{C^\eta} \sum_{j=0}^{\infty} \lambda_1^{\eta j} \|\pi y - y\|^\eta < \infty. \end{aligned}$$

Hence $\chi \in L^\infty(\Delta)$.

Let $\hat{v} = v \circ \pi_\Delta - \chi \circ f_\Delta + \chi$. It follows from the definitions that $\hat{v} : \Delta \to \mathbb{R}^d$ is constant along fibres $\pi^{-1}\bar{p}$ for $\bar{p} \in \bar{\Delta}$. Indeed,

$$\hat{v}(p) = \sum_{j=0}^{\infty} v \circ f^j \circ \pi_\Delta(\pi p) - \sum_{j=0}^{\infty} v \circ f^j \circ \pi_\Delta(\pi f_\Delta p).$$

Hence we can write $\hat{v} = \bar{v} \circ \pi$ where $\bar{v} : \bar{\Delta} \to \mathbb{R}^d$ satisfies (8.1).

Clearly, $|\bar{v}|_\infty \leq |v|_\infty + 2|\chi|_\infty < \infty$.

Let $p = (y, \ell)$, $p' = (y', \ell') \in \Delta$. If $\ell \neq \ell'$, then $|\bar{v}(p) - \bar{v}(p')| \leq 2|\bar{v}|_\infty = 2|\bar{v}|_\infty d_\theta(p, p')$. When $\ell = \ell'$, set $N = [s(p, p')/2]$. Then

$$|\hat{v}(p) - \hat{v}(p')| \leq A_N(p) + A_N(p') + B_N(p, p') + B_{N-1}(f_\Delta p, f_\Delta p'),$$

where

$$A_N(q) = \sum_{j=N}^{\infty} |v \circ f^j \circ \pi_\Delta(\pi q) - v \circ f^{j-1} \circ \pi_\Delta(\pi f_\Delta q)|,$$

$$B_N(q, q') = \sum_{j=0}^{N-1} |v \circ f^j \circ \pi_\Delta(\pi q) - v \circ f^j \circ \pi_\Delta(\pi q')|.$$

The calculation for χ gives $A_N(q) = O(\lambda_1^{\eta N}) = O(\theta^{s(p,p')})$ for $q = p, p'$. Next,

$$B_N(p, p') = \sum_{j=0}^{N-1} |v \circ f^{j+\ell}(\pi y) - v \circ f^{j+\ell}(\pi y')|.$$

Write $n = s(p, p')$. By Proposition 4.2,

$$\begin{aligned} \operatorname{diam} X &\geq \|f^n \circ f^\ell(\pi y) - f^n \circ f^\ell(\pi y')\| = \|f^{n-j} \circ f^j(f^\ell \pi y) - f^n \circ f^j(f^\ell \pi y')\| \\ &\geq \lambda_1^{-(n-j)} \|f^j(f^\ell \pi y) - f^j(f^\ell \pi y')\|, \end{aligned}$$

for all $j \leq n$. Hence

$$\|f^j(f^\ell \pi y) - f^j(f^\ell \pi y')\| = O(\lambda_1^{s(y,y')-j}), \tag{8.2}$$

and so $B_N(p, p') \leq C \sum_{j=0}^{N-1} \lambda_1^{\eta(s(y,y')-j)} = O(\theta^{s(p,p')})$. Similarly, $B_{N-1}(f_\Delta p, f_\Delta p') = O(\theta^{s(p,p')})$.

Hence we have shown that $|\hat{v}(p) - \hat{v}(p')| = O(\theta^{s(p,p')})$ and so $\|\bar{v}\|_\theta < \infty$ as claimed. □

Remark 8.8. The ASIP and vector-valued ASIP have numerous consequences summarised in [41, p. 233]. These include the central limit theorem (CLT); the functional CLT, also known as the weak invariance principle; the (functional, vector-valued) law of the iterated logarithm (LIL); upper and lower class refinements of the LIL and Chung's LIL.

Remark 8.9. The nondegeneracy assumption $\sigma^2 > 0$ fails only on a closed subspace of infinite codimension in the space of C^η observables. Indeed if $\sigma^2 = 0$ and $x \in X$ is a periodic point, then there exists $N \geq 1$ such that $\sum_{j=0}^{N-1} v(f^j x) = 0$ for all $v \in C^\eta(X)$ with mean zero. (See [8, Theorem B] for such a result in a more difficult context.) Similar comments apply to the covariance matrix Σ in the vector-valued ASIP. Taking one-dimensional projections, we obtain that the nondegeneracy assumption $\det \Sigma > 0$ fails only on a closed subspace of infinite codimension.

9. Statistical properties of singular hyperbolic attractors

In this section, we investigate statistical properties of the flow Z_t on a codimension two singular hyperbolic attracting set. We begin by modifying the Poincaré section so that the roof function τ becomes constant along stable leaves.

Let $\overline{X}$ be the union of u-curves in Section 7 and define $X_+ = \bigcup_{x \in \overline{X}} W^s_x$. Then X_+ is a Hölder-embedded cross-section and we obtain a new Poincaré map $f_+ : X_+ \to X_+$ with return time function $\tau_+ : X_+ \to \mathbb{R}^+$. We also define the quotient map $\bar{f}_+ = h \circ f_+ : \overline{X} \to \overline{X}$ where h is the stable holonomy in X_+.

Proposition 9.1. *τ_+ is constant along stable leaves in $\mathcal{W}^s$ and $\bar{f}_+ = \bar{f}$.*

Proof. For fixed $x \in \overline{X}$, set $T_0 = \tau(x)$. The stable foliation $\mathcal{W}^s$ is invariant under the time T_0-map Z_{T_0} so $Z_{T_0}(W^s_x) = W^s_{T_0 x} \subset X_+$. Hence $\tau_+(x) = T_0$ for each $x \in W^s_x$.

Next, recall that $W^s_{fx}(X)$ is the intersection of $\bigcup_{|t|<\epsilon_0} Z_t W^s_x$ with X for suitably chosen ϵ_0. Then $\bar{f}x$ is the unique intersection point of $\bigcup_{|t|<\epsilon_0} Z_t W^s_x$ with $\overline{X}$. But $f_+ x = Z_t f x$ for some small t so $\bar{f}_+ x$ also lies in the intersection of $\bigcup_{|t|<\epsilon_0} Z_t W^s_x$ with $\overline{X}$. Hence $\bar{f}_+ x = \bar{f} x$. □

In this section, we work with the new Poincaré map and roof function which we relabel $f : X \to X$ and $\tau : X \to \mathbb{R}^+$. In doing so we lose the smoothness properties of f and τ — they are now only piecewise Hölder. However we gain the property that τ is constant along the stable foliation in X. Since $\bar{f} : \overline{X} \to \overline{X}$ is unchanged; we still have that $\bar{f}$ is piecewise $C^{1+\epsilon}$ and the results on $\bar{f}$ in Section 7 and the physical measures and statistical properties in Section 8 remain valid.

Define the suspension

$$X^\tau = \{(x,u) \in X \times \mathbb{R} : 0 \le u \le \tau(x)\}/\sim \quad \text{where } (x,\tau(x)) \sim (fx, 0),$$

and the suspension flow $(x,u) \mapsto (x, u+t)$ (computed modulo identifications).

Theorem 9.2. *There exists a finite number of ergodic Z_t-invariant probability measures $\mu_{M,1}, \dots, \mu_{M,s}$ whose basins cover a subset of U_0 of full Lebesgue measure.*

Proof. For each μ_j in Corollary 8.2, we obtain an ergodic flow-invariant probability measure $\mu^\tau_j = \mu_j \times \text{Lebesgue}/\int_X \tau \, d\mu_j$ on X^τ. The projection $\pi^\tau : X^\tau \to M$, $\pi^\tau(x,u) = Z_u x$ defines a semiconjugacy from X^τ to M and $\mu_{M,j} = \pi^\tau_* \mu^\tau_j$ is an ergodic Z_t-invariant probability measure on M. By [10, Section 7], these form a finite family of physical measures $\mu_{M,j}$ for the flow Z_t whose basins cover a subset of U_0 of full Lebesgue measure. □

Suppose without loss that there is a unique physical measure $\mu_M = \pi^\tau_* \mu^\tau$ where $\mu^\tau = \mu \times \text{Lebesgue}/\int_X \tau \, d\mu$ (in the notation above). Recall that $\bar{\mu}$, and hence μ, is mixing up to a finite cycle of length $k \ge 1$. By shrinking the cross-section X we may suppose without loss that the measure μ on X is mixing.

Define the *induced* roof function

$$\varphi : \overline{Y} \to \mathbb{R}^+, \qquad \varphi(y) = \textstyle\sum_{\ell=0}^{\rho(y)-1} \tau(\bar{f}^\ell y).$$

Proposition 9.3. *$\mu_Y(\varphi > t) = O(t^{-\beta})$ for any $\beta > 0$.*

Proof. A standard general calculation (see for example [13, Proposition A.1]) shows that

$$\mu_Y(\varphi > t) \le \mu_Y(\rho > k) + \bar{\rho}\mu(\tau > t/k),$$

for all $t > 0$, $k \ge 1$, where $\bar{\rho} = \int_Y \rho \, d\mu_Y$. In particular, since ρ has superpolynomial tails and τ has at most logarithmic singularities, there is a constant $c > 0$ such that $\mu_Y(\varphi > t) = O(k^{-2\beta} + e^{-ct/k})$. Now take $k = [t^{1/2}]$. □

Recall that $\bar{F} : \overline{Y} \to \overline{Y}$ is a Gibbs-Markov map with partition α and separation time $s(y, y')$.

Proposition 9.4. *There exists $\theta \in (0,1)$ and $C > 0$ such that*

$$|\varphi(y) - \varphi(y')| \le C\theta^{s(y,y')} \quad \textit{for all } y, y' \in a,\ a \in \alpha.$$

Proof. We can write $\tau = \tau_0 + \tau_1$ where τ_0 is as in previous sections and in particular satisfies the estimate Proposition 8.6, and τ_1 is C^ϵ. Setting $\theta = 2^{-\epsilon}$ and using uniform expansion of $\bar{f}$,

$$|\tau_1(\bar{f}^\ell y) - \tau_1(\bar{f}^\ell y')| \le |\tau_1|_{C^\epsilon} |\bar{f}^\ell y - \bar{f}^\ell y'|^\epsilon \le C_1 \theta^{\rho(y)-\ell} |\bar{F}y - \bar{F}y'|^\epsilon.$$

Combining this with the estimate for τ_0, we obtain that $\sum_{\ell=0}^{\rho(y)-1} |\tau(\bar{f}^\ell y) - \tau(\bar{f}^\ell y')| \le C_1' |\bar{F}y - \bar{F}y'|^\epsilon$. By (8.2), $|\bar{F}y - \bar{F}y'| = O(2^{-s(y,y')})$ and the result follows. □

9.1. **Statistical limit laws for the flow.** If $\Lambda = \operatorname{supp} \mu_M$ contains no equilibria, then Λ is a nontrivial hyperbolic basic set for an Axiom A flow and the CLT for Hölder observables follows from [46, 38]. Moreover, [17] obtains a version of the (scalar) ASIP that implies the functional CLT and functional LIL.

When Λ contains equilibria, the CLT and its functional version still holds by [27] at least for geometric Lorenz attractors. As pointed out in [13], a simpler argument than in [27] applies in general situations where the roof function is unbounded and includes the entire class of singular hyperbolic attractors analysed in this paper. We refer to the introduction of [13] for a more comprehensive list of statistical limit laws, with precise statements, that can be obtained in this way.

9.2. **Mixing and superpolynomial mixing for the flow.**

Theorem 9.5. *There is a C^2-open and C^∞-dense set of singular hyperbolic flows such that each nontrivial attractor Λ is mixing with superpolynomial decay of correlations: for any $\beta > 0$,*

$$\Big| \int_\Lambda v\, w \circ Z_t \, d\mu_M - \int_\Lambda v \, d\mu_M \int_\Lambda w \, d\mu_M \Big| \le C t^{-\beta} \quad \textit{for all } t > 0,$$

for all $v, w : M \to \mathbb{R}$ such that one of v or w is C^∞ and the other is Hölder. Here C is a constant depending on v, w and β.

Proof. If $\Lambda = \operatorname{supp} \mu_M$ contains no equilibria, then Λ is uniformly hyperbolic and the result is due to [18, 20]. The general case follows essentially from [34, 35].

More precisely, we have seen that the semiflow and flow is modelled as a suspension over a Young tower with superpolynomial tails. Using the induced roof function $\varphi : Y \to \mathbb{R}^+$, we obtain a suspension Y^φ over the uniformly hyperbolic map $F : Y \to Y$ where the roof function $\varphi : Y \to \mathbb{R}^+$ has superpolynomial tails.

We are now in a position to apply [12, Theorem 3.1] (see also [36, Theorem 4.1]). Conditions (3.1) and (3.2) in [12] follow from Propositions 4.1 and 4.2. Moreover, φ is constant along stable leaves by Proposition 9.1 and projects to a well-defined roof function $\varphi : \overline{Y} \to \mathbb{R}^+$ satisfying the estimate in Proposition 9.4 which is condition (3.3) in [12]. Hence the suspension flow on Y^φ is a skew product Gibbs-Markov flow in the terminology of [12]. Hence superpolynomial mixing follows from [12, Theorem 3.1] subject to a nondegeneracy condition (absence of approximate eigenfunctions).

Finally, it is shown in [20] that absence of approximate eigenfunctions is C^2-open and C^∞-dense (cf. [35, Remark 2.5] or [36, Subsection 5.2]). □

We have already seen that statistical limit laws such as the CLT hold for all singular hyperbolic flows. In the situation of Theorem 9.5, we can obtain such results also for the time-one map of a singular hyperbolic flow.

Corollary 9.6. *Assume that $Z_t : \Lambda \to \Lambda$ has superpolynomial decay of correlations as in Theorem 9.5. Let $v : M \to \mathbb{R}$ be C^∞ (or at least sufficiently smooth) with mean zero. Then the ASIP holds for the time-one map Z_1 for all C^∞ observables $v : M \to \mathbb{R}$.*

In particular, the limit $\sigma^2 = \lim_{n\to\infty} n^{-1} \int_\Lambda (\sum_{j=0}^{n-1} v \circ Z_j)^2 \, d\mu_M$ exists, and after passing to an enriched probability space, there exists a sequence $A_0, A_1, \dots$ of i.i.d. normal random variables with mean zero and variance σ^2 such that

$$\sum_{j=0}^{n-1} v \circ Z_j = \sum_{j=0}^{n-1} A_j + O(n^{1/4}(\log n)^{1/2}(\log\log n)^{1/4}), \quad \text{a.e.}$$

Moreover, if $\sigma^2 = 0$, then for every periodic point $q \in \Lambda$, there exists $T > 0$ (independent of v) such that $\int_0^T v(Z_t q) \, dt = 0$.

Proof. This is proved in the same way as [8, Theorems B and C]. □

In the case of the classical Lorenz attractor, it was shown in [32] and [8] that mixing and superpolynomial mixing is automatic. The proof exploits the *locally eventually onto (l.e.o.)* property as well as smoothness properties of the stable foliation. We now show that the mixing argument in [32] does not require the stable foliation to be smooth. In the general situation of this paper, we assume hypotheses that are more complicated to state but which are implied by l.e.o. for the classical Lorenz attractor.

We require that Λ contains at least one equilibrium. Let $q \in \mathcal{S}$ be the corresponding singularity for $\bar{f} : \overline{X} \to \overline{X}$. (Again, $\bar{f}$ is not defined at q.) Assume that the set of preimages of q under iterates of $\bar{f}$ is dense in $\overline{X}$. (This condition is always satisfied for geometric Lorenz attractors.) By Lemma 8.4 and Remark A.3, we can construct an induced Gibbs-Markov map $\overline{F} = \bar{f}^\sigma : \overline{Y} \to \overline{Y}$ where the inducing set $\overline{Y}$ contains q. Let $K = \bigcup_{\ell \ge 0} \bar{f}^\ell \overline{Y}$; this is an open and dense full measure subset of $\overline{X}$. Our final assumption is that $\bar{f}^p q_+ = \lim_{y \to q+} \bar{f}^p y \in K$ for some $p \ge 1$. (This would work equally well with q_+ replaced by q_-.)

Theorem 9.7. *Under the above assumptions, Λ is automatically mixing (and even Bernoulli).*

Proof. We sketch the proof following [32]. By [47], it suffices to show that the quotient suspension semiflow $\bar{f}^\tau_t : \overline{X}^\tau \to \overline{X}^\tau$ is weak mixing. Equivalently, the cohomological equation $u \circ \bar{f} = e^{ib\tau} u$ has no measurable solutions $u : \overline{X} \to S^1$ for all $b \ne 0$. (Here S^1 denotes the unit circle in $\mathbb{C}$.)

Suppose for contradiction that there exists $u : \overline{X} \to S^1$ measurable and $b \ne 0$ such that $u \circ \bar{f} = e^{ib\tau} u$. A Livšic regularity theorem of [15], exploiting the fact that $\overline{F}$ is Gibbs-Markov and that the roof function τ is Hölder with at most logarithmic growth (Lemma 3.2) ensures that u has a version that is continuous on K.

Also, $q \in \overline{Y} \subset K$. Choose $p \ge 1$ with $\bar{f}^p q_+ \in K$. Then $u \circ \bar{f}^p = e^{ib\tau_p} u$ where $\tau_p = \sum_{j=0}^{p-1} \tau \circ \bar{f}^j$. By Remark 3.3, $\tau_p(y) \ge \tau(y) \to \infty$ as $y \to q_+$, whereas $u(y) \to u(q)$ and $u(\bar{f}^p y) \to u(\bar{f}^p q_+)$. Since $b \ne 0$, this contradicts the equality $u \circ \bar{f}^p = e^{ib\tau_p} u$. □

Remark 9.8. If we assume in addition that the stable foliation $\mathcal{W}^s$ for the flow is $C^{1+\epsilon}$, then we can deduce exponential decay of correlations following [6].

However, without smoothness of $\mathcal{W}^s$, the roof function τ (on the modified cross-section) is only Hölder and the cancellation argument of [18] fails. In fact, we are unable even to prove superpolynomial mixing for fixed flows (without perturbing as in Theorem 9.5). It should be possible to use the techniques in [8] to prove that the stable and unstable foliations (defined appropriately) for the flow are not jointly integrable – this is a stronger property than mixing. However, we do not see how to use this to prove superpolynomial mixing when τ is only Hölder.

Appendix A. Theorem of Alves *et al.* [3]

In this appendix, we recall a result of Alves *et al.* [3] that is required in Section 8.1. Although the argument in [3] is essentially correct, there are certain problems with the formulation of the hypotheses. First, the hypotheses (C2) and (C3) in [3] are stated too strongly, since the right-hand side of their conditions are zero for points $x \ne y$ equidistant from $\mathcal{S}$, whereas the left-hand side is generally nonzero. Second, the hypotheses are not stated strongly enough for the first half of the proof of [3, Lemma 5.1], since the estimate for $d(x,\mathcal{S})^{-\alpha}$ is false in general. We state below a corrected version of the hypotheses in [3]. The conclusion in Theorem A.1 is identical to that in [3, Theorem C], and the proof is largely unchanged.

Throughout, (M,d) is a compact Riemannian manifold and $f:M\to M$ is a local C^{1+} diffeomorphism with singularity set $\mathcal{S}$. We suppose that there are constants $\eta\in(0,1)$ and C, $q>0$ such that

(C0) $\operatorname{Leb}(x: d(x,\mathcal{S})\le\epsilon)\le C\epsilon^\eta$ for all $\epsilon\ge 0$.

(C1) $C^{-1}d(x,\mathcal{S})^q\le\|Df(x)v\|\le Cd(x,\mathcal{S})^{-q}$, for all $x\in M\setminus\mathcal{S}$, $v\in T_xM$ with $\|v\|=1$.

(C2) $\big|\log\|Df(x)^{-1}\|-\log\|Df(y)^{-1}\|\big|\le Cd(x,y)^\eta(\|Df(x)^{-1}\|^q+\|Df(x)^{-1}\|^{-q})$ for all $x,y\in M\setminus\mathcal{S}$ with $d(x,y)<d(x,\mathcal{S})/2$.

(C3) $\big|\log|\det Df(x)|-\log|\det Df(y)|\big|\le Cd(x,y)^\eta d(x,\mathcal{S})^{-q}$ for all $x,y\in M\setminus\mathcal{S}$ with $d(x,y)<d(x,\mathcal{S})/2$.

Recall [3, Definition 1.2] that a measure μ is expanding if $\log\|(Df)^{-1}\|$ is integrable with respect to μ and $\int_M\log\|(Df)^{-1}\|\,d\mu<0$.

Let $\operatorname{Cov}(v,w)=\int_M v\,w\,d\mu-\int_M v\,d\mu\int_M w\,d\mu$.

Theorem A.1 ([3, Theorem C]). *Let $f:M\to M$ be a C^{1+} local diffeomorphism satisfying (C0)–(C3), and let $\alpha\in(0,1)$. Let μ be an ergodic expanding absolutely continuous invariant probability measure with $d\mu/d\operatorname{Leb}\in L^p$ for some $p>1$.*

(1) Suppose that there exists $\beta>1$ and $C>0$ such that $|\operatorname{Cov}(v,w\circ f^n)|\le C\|v\|_{C^\alpha}|w|_\infty\,n^{-\beta}$ for all $v\in C^\alpha$, $w\in L^\infty$, $n\ge1$.

Then there is a full branch Gibbs-Markov induced map $F=f^\rho:Y\to Y$, where $\rho:Y\to\mathbb{Z}^+$ is constant on partition elements and satisfies $\operatorname{Leb}(y\in Y:\rho(y)>n)=O(n^{-(\beta-1)})$. *Moreover, there are constants C, $\epsilon>0$ such that*

$$\textstyle\sum_{\ell=0}^{\rho(y)-1}d(f^\ell y,f^\ell y')^\eta d(x,\mathcal{S})^{-q}\le Cd(Fy,Fy')^\epsilon, \tag{A.1}$$

for all y,y' lying in the same partition element.

(2) Suppose that $d\mu/d\operatorname{Leb}$ is bounded below on its support and that there exist $\gamma\in(0,1]$, C, $c>0$ such that $|\operatorname{Cov}(v,w\circ f^n)|\le C\|v\|_{C^\alpha}|w|_\infty\,e^{-cn^\gamma}$ for all $v\in C^\alpha$, $w\in L^\infty$, $n\ge1$.

Then the conclusion in (1) holds and moreover for any $\gamma'\in(0,\gamma/(3\gamma+6))$ there exists $c'>0$ such that $\operatorname{Leb}(y\in Y:\rho(y)>n)=O\big(e^{-c'n^{\gamma'}}\big)$.

Remark A.2. The estimate (A.1) is a crucial component of the proofs in [3, 4]. (See the calculation at the end of the proof of [4, Lemma 4.1].) We make it explicit here since it is used in the proof of Proposition 8.6.

Remark A.3. Let $x\in M$ be any point with dense preimages in M. By [4, Remarks 1.4], the inducing set Y can be chosen to be an open ball containing x.

In the remainder of this appendix, we indicate the modifications to the argument in [3] required to obtain the corrected version of Theorem A.1.

We begin by noting that a consequence of (C1) and (C2) is that

$$\big|\log\|Df(x)^{-1}\| - \log\|Df(y)^{-1}\|\big| \le Cd(x,y)^\eta d(x,\mathcal{S})^{-q^2} \quad \text{for all } x,y\in M\setminus\mathcal{S}.$$

Combined with (C3), this means that the C^{1+} version of the C^2 set up in [2, 4] is satisfied. It is well-known, and routine, that the theory of hyperbolic times and the resulting constructions in [2, 4] work just as well in the C^{1+} setting. Hence as in [3], it suffices to verify the hypotheses of [4, Theorem 2]. This all proceeds exactly as in [3] except for the estimate of $\phi_{1,k}$ in [3, Lemma 5.1]. Recall from [3] that $\phi_1 = \log\|(Df)^{-1}\|$ and that $\phi_{1,k} = \phi_1 1_{\{|\phi_1|\le k\}}$. (The definition in [3] has $\phi_{1,k} = \phi_1 1_{\{\phi_1\le k\}}$, but it is clear from the proof of [3, Lemma 4.3] that this is what was meant.)

Proposition A.4. *For any $\alpha>0$, there exists $\eta'\in(0,1)$, $C>0$ such that $\|\phi_{1,k}\|_{C^{\eta'}} \le Ce^{\alpha k}$.*

Proof. We can suppose without loss that $\alpha<2q$.

Let $x,y\in M$. It is immediate that $|\phi_{1,k}(x)|\le k$ and that $|\phi_{1,k}(x)-\phi_{1,k}(y)|\le 2k$. Also, by (C2), assuming without loss that $\phi_{1,k}(x)\le\phi_{1,k}(y)$,

$$|\phi_{1,k}(x)-\phi_{1,k}(y)| \le C_1 d(x,y)^\eta(e^{q\phi_{1,k}(y)}+e^{-q\phi_{1,k}(x)}) \le C_1' d(x,y)^\eta e^{qk}.$$

The inequality $\min\{1,a\}\le a^\epsilon$ holds for all $a\ge0$, $\epsilon\in[0,1]$. Hence taking $\epsilon=\frac12\alpha/q$ and $\eta'=\epsilon\eta$ we obtain that

$$|\phi_{1,k}(x)-\phi_{1,k}(y)| \le C_2 k\min\{1,d(x,y)^\eta e^{qk}\} \le C_2 k d(x,y)^{\eta'} e^{\frac12\alpha k} \le C_2' d(x,y)^{\eta'} e^{\alpha k}.$$

We have shown that $\|\phi_{1,k}\|_{C^{\eta'}} = O(k+e^{\alpha k}) = O(e^{\alpha k})$ as required. □

The remainder of the proof of Theorem A.1 proceeds exactly as in [3]. (We note that in [3] it is asserted that $\eta'=\alpha$, but this is not required in the proof.)

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