Central Limit Theorems and Suppression of Anomalous Diffusion for Systems with Symmetry

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Abstract

We give general conditions for the central limit theorem and weak convergence to Brownian motion (the weak invariance principle / functional central limit theorem) to hold for observables of compact group extensions of nonuniformly expanding maps. In particular, our results include situations where the central limit theorem would fail, and anomalous behaviour would prevail, if the compact group were not present.

This has important consequences for systems with noncompact Euclidean symmetry and provides the rigorous proof for a conjecture made in our paper: A Huygens principle for diffusion and anomalous diffusion in spatially extended systems. *Proc. Natl. Acad. Sci. USA* **110** (2013) 8411–8416.

1 Introduction

It is by now well-understood that statistical limit laws such as the central limit theorem (CLT) and corresponding invariance principles (convergence to Brownian motion) hold for large classes of nonuniformly hyperbolic dynamical systems [14, 21, 28, 29, 30]. In this setting, summable decay of correlations is sufficient for the central limit theorem to hold. There are also numerous results for compact group extensions of such maps [9], in particular for equivariant observables which occur naturally in systems with symmetry [24, 10, 19, 20].

For systems modelled by Young towers with nonsummable decay of correlations [30], the central limit theorem generally fails for typical Hölder observables. In certain instances, there is convergence instead to a stable law with nonstandard normalisation n^s, $s > \frac{1}{2}$, see Gouëzel [14]. (The corresponding invariance principle, namely weak convergence to a stable Lévy process with superdiffusive growth rate t^s, is also valid [23].)

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Passing to compact group extensions of such systems, in a recent paper [13] we described a dichotomy whereby the superdiffusion persists or is suppressed in favour of normal diffusion. The dichotomy is characterised by the equivariance properties of the observables. The arguments in [13] are heuristic, backed up by numerical simulations. In this paper, we give a proof of the suppression statements. The work of [13] was motivated by the study of systems with noncompact Euclidean symmetry, see Subsection 1.3.

Our main theoretical result, Theorem 1.10, gives very general conditions under which the CLT and the weak invariance principle (WIP) hold for equivariant observables of compact group extensions of a nonuniformly expanding map $f : X \to X$. Essentially, the problem is reduced to proving that a certain derived observable (denoted V^* in the sequel) lies in L^2. The second main contribution of this paper is to verify this L^2 condition in the situation of [13].

Remark 1.1 There is a connection between our results and recent work of Peligrad & Wu [25] and Cohen & Conze [7]. They consider *rotated sums* of the form $\sum_{j=0}^{n-1} e^{ij\theta} v \circ f^j$ and prove central limit theorems under extremely mild conditions: it suffices [7] that f is exact and $v \in L^2$. This corresponds to the case of circle extensions with constant cocycle $h \equiv e^{i\theta}$ and observables $\phi(x,\psi) = e^{i\psi}v(x)$ in the notation of this paper. The constancy of h enables the use of Fourier-analytic techniques.

For our results we require much stronger assumptions on the dynamics and the function v, but we work throughout with general compact group extensions and do not require that h is constant.

First, we focus on the specific case of Pomeau-Manneville intermittency maps [26], before turning to a more general class of nonuniformly expanding maps in Subsection 1.2.

1.1 Intermittency maps

For ease of exposition, we consider the family of maps $f : X \to X$, where X is the interval $[0,1]$, studied by [18]. For $\gamma \geq 0$, let

$$f(x) = \begin{cases} x(1 + 2^\gamma x^\gamma), & x \in [0, \frac{1}{2}] \\ 2x - 1, & x \in (\frac{1}{2}, 1]. \end{cases} \tag{1.1}$$

We are interested in the statistical properties of compact group extensions of f for $\gamma \in [0,1)$.

The statistical properties of f itself are well understood. There is a unique absolutely continuous ergodic invariant probability measure μ. If $\gamma = 0$, then f is the doubling map with exponential decay of correlations. For $\gamma \in (0,1)$, it is known [17] that the correlation function $\rho(n) = \int_X v\, w \circ f^n \, d\mu - \int_X v \, d\mu \int_X w \, d\mu$ satisfies $|\rho(n)| \leq Cn^{-((1/\gamma)-1)}$ for v Hölder, $w \in L^\infty$, and moreover this is sharp.

In the case of summable decay of correlations, namely $\gamma \in [0, \frac{1}{2})$, the following central limit theorem holds for Hölder observables $v : X \to \mathbb{R}^d$. Suppose that $\int_X v \, d\mu = 0$ and define $v_n = \sum_{j=0}^{n-1} v \circ f^j$. Then $n^{-\frac{1}{2}} v_n \to_d Y$ where Y is normally distributed with mean 0 and variance σ^2 (typically positive). The case of nonsummable decay of correlations, $\gamma \in [\frac{1}{2}, 1)$, is quite different. For $\gamma = \frac{1}{2}$, there is still convergence to a normal distribution, but if $v(0) \neq 0$ then it is necessary to normalise by $(n \log n)^{\frac{1}{2}}$ instead of $n^{\frac{1}{2}}$. For $\gamma \in (\frac{1}{2}, 1)$ and $v(0) \neq 0$, the required normalisation is n^γ and $n^{-\gamma} v_n \to_d Y_\alpha$ where Y_α is a one-sided stable law of order $\alpha = 1/\gamma$. The results for $\gamma \in [\frac{1}{2}, 1)$ are due to Gouëzel [14], who also showed that the ordinary central limit theorem prevails for observables v with sufficiently large Hölder exponent when $v(0) = 0$.

To summarise, the CLT holds in the *strongly chaotic* case $\gamma \in [0, \frac{1}{2})$, but anomalous (superdiffusive) scaling rates hold typically in the *weakly chaotic* case $\gamma \in [\frac{1}{2}, 1)$. However, the anomalous diffusion is suppressed, and normal diffusion prevails, for smooth enough observables that vanish at the origin. In addition, the corresponding WIPs are valid: Dedecker & Merlevède [8] prove weak convergence to Brownian motion in the cases where [14] proves the CLT, and Melbourne & Zweimüller [23] prove weak convergence to a Lévy process in the cases where [14] obtains a stable law.

Compact group extensions of intermittency maps Next, we consider the generalisation of these results for compact group extensions and *equivariant* observables [24]. In certain situations [13] it turns out that the above results in the weakly chaotic case are reversed, namely that suppression of anomalous diffusion is generic and anomalous diffusion is the degenerate case. So far our claims in [13] on anomalous diffusion are conjectural, but we present here rigorous results on suppression.

Let G be a compact connected Lie group with Haar measure ν. Consider the group extension $f_h : X \times G \to X \times G$ given by $f_h(x, g) = (fx, gh(x))$ where $h : X \to G$ is a Hölder cocycle. The product measure $m = \mu \times \nu$ is an f_h-invariant probability measure, and is assumed throughout to be ergodic.

Remark 1.2 Ergodicity of m is typical in the following strong sense. The set of Hölder cocycles $h : X \to G$ for which m is not ergodic lies inside a closed subspace of infinite codimension in the space of all Hölder cocycles [11].

Let $\mathbb{R}^d$ be a representation of G; without loss G acts orthogonally on $\mathbb{R}^d$. We consider equivariant observables $\phi : X \times G \to \mathbb{R}^d$ of the form $\phi(x, g) = g \cdot v(x)$ where $v : X \to \mathbb{R}^d$ is Hölder. Let $\phi_n = \sum_{j=0}^{n-1} \phi \circ f_h^j$. Throughout, we suppose that $\int_{X \times G} \phi \, dm = 0$.

Theorem 1.3 (CLT) *Let $\gamma \in (0, \frac{1}{2})$. Assume that $v : X \to \mathbb{R}^d$, $h : X \to G$ are Hölder. Then $n^{-\frac{1}{2}} \phi_n \to_d N(0, \Sigma)$ as $n \to \infty$, where Σ is a $d \times d$ covariance matrix* *satisfying $g\Sigma = \Sigma g$ for all $g \in G$. That is,*

$$m((x,g) \in X \times G : n^{-\frac{1}{2}}\phi_n(x,g) \in I) \to \int_I \frac{1}{(2\pi)^{k/2}(\det\Sigma)^{1/2}} \exp\{-\frac{1}{2}y^T\Sigma^{-1}y\}\, dy,$$

as $n \to \infty$, for every open rectangle $I \subset \mathbb{R}^d$.

Remark 1.4 (WIP) In the situation of Theorem 1.3, we also obtain the following weak invariance principle. Define $W_n(t) = n^{-\frac{1}{2}}\phi_{nt}$ for $t = 0, \frac{1}{n}, \frac{2}{n}, \dots$ and linearly interpolate to obtain $W_n \in C([0,\infty), \mathbb{R}^d)$. Then W_n converges weakly to W in $C([0,\infty), \mathbb{R}^d)$, denoted $W_n \to_w W$, where W is d-dimensional Brownian motion with covariance matrix Σ.

Equivalently, for any $T > 0$, $k \geq 1$, and for any continuous function $\chi : C([0,T], \mathbb{R}^k) \to \mathbb{R}^k$, we have that $\chi(W_n) \to_d \chi(W)$ as ordinary $\mathbb{R}^k$-valued random variables (so $m(\chi(W_n) \in I) \to \mathbb{P}(\chi(W) \in I)$ for any open rectangle $I \subset \mathbb{R}^k$). Taking $T = 1$, $k = d$ and $\chi(p) = p(1)$ we recover Theorem 1.3, so the CLT is a special case of the WIP.

Remark 1.5 (a) The convergence here is in distribution with respect to the probability measure m on $X \times G$. In fact, it follows from [31] that we obtain *strong* distributional convergence: convergence in distribution to $N(0, \Sigma)$ holds for any probability measure that is absolutely continuous with respect to m. The corresponding statement also holds for the WIP.

(b) By [19], we obtain strong distributional convergence also with respect to the measure $\mu \times \delta_{g_0}$ for $g_0 \in G$ fixed.

(c) The covariance matrix is typically nondegenerate. (Again, the degenerate situation $\det\Sigma = 0$ holds only on a closed subspace of infinite codimension in the space of Hölder functions $v : X \to \mathbb{R}^d$ [24].)

When $\gamma \in [\frac{1}{2}, 1)$ it is necessary to consider the values of the cocycle h and the observable v at the neutral fixed point 0. Let $\operatorname{Fix} g = \{w \in \mathbb{R}^d : gw = w\}$ for $g \in G$. We have the orthogonal splitting $\mathbb{R}^d = \operatorname{Fix} h(0) \oplus (\operatorname{Fix} h(0))^\perp$.

Theorem 1.6 *Let $\gamma \in [\frac{1}{2}, 1)$. Suppose that $v : X \to \mathbb{R}^d$ and $h : X \to G$ are η-Hölder, where $\eta > \gamma - \frac{1}{2}$.*

If $v(0) \in (\operatorname{Fix} h(0))^\perp$, then $n^{-\frac{1}{2}}\phi_n \to_d N(0, \Sigma)$ where Σ is a $d \times d$ covariance matrix satisfying $g\Sigma = \Sigma g$ for all $g \in G$.

Again the convergence is in the sense of strong distribution, Σ is typically nondegenerate, and the corresponding weak invariance principle holds.

The heuristic arguments in [13] generalise in the current context to yield the following conjecture:

Conjecture 1.7 If $\gamma \in (\frac{1}{2}, 1)$ and $v(0) \notin (\operatorname{Fix} h(0))^{\perp}$, then we conjecture that $n^{-\gamma}\phi_n$ converges in distribution to a d-dimensional stable law of order $\alpha = 1/\gamma$.

Similarly, if $\gamma = \frac{1}{2}$ and $v(0) \notin (\operatorname{Fix} h(0))^{\perp}$, then we conjecture that $(n \log n)^{-\frac{1}{2}}\phi_n$ converges in distribution to a d-dimensional normal distribution.

Remark 1.8 As in [10, Section 4(a)], our set up decomposes naturally into the cases where G acts trivially on $\mathbb{R}^d$ and where G acts fixed-point freely on $\mathbb{R}^d$ (so if $w \in \mathbb{R}^d$ and $g \cdot w = w$ for all $g \in G$, then $w = 0$). In the latter case, the condition $\int_{X \times G} \phi \, dm = 0$ is automatically satisfied [24].

1.2 Extensions of nonuniformly expanding maps

The intermittency maps (1.1) are examples of nonuniformly expanding maps. This is a large class of dynamical systems that can be modelled by Young towers [30] and whose statistical properties are well-understood. The main result of this paper, Theorem 1.10 below, gives very general conditions under which the CLT and WIP hold for equivariant observables of compact group extensions of such maps.

Let (X, d) be a locally compact separable bounded metric space with Borel probability measure μ_0 and let $f : X \to X$ be a nonsingular transformation for which μ_0 is ergodic[1]. Let $Y \subset X$ be a measurable subset with $\mu_0(Y) > 0$, and let α be an at most countable measurable partition of Y with $\mu_0(a) > 0$ for $a \in \alpha$. Suppose that there is an L^1 *return time* function $r : Y \to \mathbb{Z}^+$, constant on each $a \in \alpha$, and constants $\lambda > 1$, $\eta \in (0, 1]$, $C \geq 1$ such that for each $a \in \alpha$,

(1) $F = f^{r(a)} : a \to Y$ is a bijection with measurable inverse.

(2) $d(Fx, Fy) \geq \lambda d(x, y)$ for all $x, y \in a$.

(3) $d(f^{\ell}x, f^{\ell}y) \leq C d(Fx, Fy)$ for all $x, y \in a$, $0 \leq \ell < r(a)$.

(4) $g_a = \frac{d(\mu_0|a \circ F^{-1})}{d\mu_0|_Y}$ satisfies $|\log g_a(x) - \log g_a(y)| \leq C d(x, y)^{\eta}$ for all $x, y \in Y$.

Remark 1.9 For the intermittency maps (1.1) a natural choice is $Y = [\frac{1}{2}, 1]$. Conditions (1)–(4) are valid for all $\gamma \geq 0$ and the condition that r is integrable holds if and only if $\gamma \in [0, 1)$.

Such a dynamical system $f : X \to X$ is called *nonuniformly expanding*. There is a unique f-invariant probability measure μ on X equivalent to μ_0 (see for example [30, Theorem 1]).

As before, we consider compact group extensions $f_h : X \times G \to X \times G$, $f_h(x, g) = (fx, gh(x))$. Again, the invariant product measure $m = \mu \times \nu$ is assumed to be

[1] Recall that a not necessarily invariant measure μ_0 is *ergodic* if every f-invariant measurable subset $A \subset X$ satisfies $\mu(A) = 0$ or $\mu(X \setminus A) = 0$

ergodic. Let $\phi : X \times G \to \mathbb{R}^d$ be an observable of the form $\phi(x, g) = g \cdot v(x)$ where $v : X \to \mathbb{R}^d$ and G acts orthogonally on $\mathbb{R}^d$.

To study the statistical properties of the observable ϕ, we follow the standard approach of *inducing* where we pass from the nonuniformly expanding map $f : X \to X$ (and its group extension on $X \times G$) to the uniformly expanding map $F = f^r : Y \to Y$ (and its group extension on $Y \times G$). There is a trade-off between the improvement of f and the deterioration of the cocycle $h : X \to G$ and observable $\phi : X \times G \to \mathbb{R}^d$, stemming from the possibility that the return time function r may be large. Hence it is necessary to consider an induced cocycle $H : Y \to G$ and an induced observable $\Phi : Y \times G \to \mathbb{R}^d$ which incorporate this information (see Section 3).

To state our main result, it suffices to introduce induced versions of the function $v : X \to \mathbb{R}^d$. Define V, $V^* : Y \to \mathbb{R}^d$,

$$V(y) = \sum_{j=0}^{r(y)-1} h_j(y) v(f^j y), \quad V^*(y) = \max_{0 \le \ell < r(y)} \Big| \sum_{j=0}^{\ell} h_j(y) v(f^j y) \Big|,$$

where $h_j(y) = h(y) \cdots h(f^{j-1} y)$. Note that if h is measurable and $v \in L^\infty$, then $r \in L^p$ implies that V and V^* lie in L^p.

Theorem 1.10 *Suppose that $f : X \to X$ is nonuniformly expanding and that $r : Y \to \mathbb{Z}^+$ is constant on partition elements. Suppose further that $v : X \to \mathbb{R}^d$ and $h : X \to G$ are uniformly Hölder.*

If $r \in L^p$ for some $p > 1$ and $V \in L^2$, then the CLT holds for ϕ. If moreover $V^* \in L^2$, then the WIP holds for ϕ. In particular, this is the case if $r \in L^2$.

The additional conclusions in Remark 1.5 are again applicable.

For the intermittency maps (1.1), it is well-known that $r \in L^2$ if and only if $\gamma \in [0, \frac{1}{2})$. Hence Theorem 1.3 is an immediate consequence of Theorem 1.10. For $\gamma \in [\frac{1}{2}, 1)$ it is still the case that $r \in L^p$ for some $p > 1$, so given Theorem 1.10 it suffices to verify that $V^* \in L^2$ in order to prove Theorem 1.6.

1.3 Application to Euclidean group extensions

The work in this paper was motivated by questions related to the existence of anomalous diffusion in spatially extended systems that we raised in [13]. Isotropic spatially extended systems with d space dimensions, such as reaction-diffusion equations in $\mathbb{R}^d$, transform under the Euclidean group $\Gamma = \mathbf{SO}(d) \ltimes \mathbb{R}^d$ of rotations and translations of d-dimensional space.[2] The central thesis of [13] is that anomalous diffusion is suppressed in d-dimensions if and only if d is even.

As we now explain, Theorem 1.6 answers one of the main questions in [13], namely that anomalous diffusion is indeed suppressed in even dimensions.

[2] In [13] the Euclidean group is denoted by G, whereas the group is denoted here by Γ.

We adopt the standard perspective for dynamical systems with a Lie group of continuous symmetries Γ, decomposing the full dynamics into the dynamics along the symmetry group Γ and the dynamics orthogonal to the group orbits (see [13] and references therein). Systems with symmetry are thus realised as a skew product

$$\dot{x} = f(x), \quad \dot{\gamma} = \gamma\xi(x),$$

on $X \times \Gamma$ where the dynamics on Γ is driven by the shape dynamics on a cross-section X transverse to the group directions. Here $\gamma\xi(x)$ denotes the action of $\gamma \in \Gamma$ on the Lie algebra element $\xi(x) \in T_x\Gamma$. A general question is:

> *Given the dynamics on X and the structure of the group Γ, what can be said about the dynamics on $X \times \Gamma$?*

In particularly, the dynamics of isotropic spatially extended systems reduce to dynamics on a skew product $X \times \Gamma$ where $\Gamma = \mathbf{SO}(d) \ltimes \mathbb{R}^d$ is the Euclidean group. In this situation, if X consists of an equilibrium or a periodic solution, it is easily seen [2] that the dynamics on $X \times \Gamma$ is typically bounded if and only if d is even, and has a linear drift when d is odd. This is a nonlinear version of the classical Huygens principle whereby sound waves propagate only in odd dimensions.

The main conjecture in [13] is that if the dynamics on X is weakly chaotic (so that one anticipates superdiffusive behaviour as discussed in Subsection 1.1) and Γ is the Euclidean group, then typically the dynamics on $X \times \Gamma$ is superdiffusive for d odd and diffusive for d even. Hence the Huygens dichotomy extends from regular dynamics to weakly chaotic dynamics, with bounded behaviour replaced by diffusion and linear drift replaced by superdiffusion. (In the strongly chaotic case, the dichotomy disappears and we obtain diffusion for d odd and even.)

When studying statistical properties of dynamical systems, there are standard techniques for passing between continuous and discrete time (see for example [15, 22, 23]). In particular, it follows from [23] that results on convergence to a Brownian motion or a Lévy process can be answered at the level of the Poincaré map. Hence from now on, we consider discrete time skew products of the form

$$f_\xi : X \times \Gamma \to X \times \Gamma, \qquad f_\xi(x, \gamma) = (fx, \gamma\xi(x)),$$

where $f : X \to X$ defines the dynamics on X and $\xi : X \to \Gamma$ is a measurable cocycle.

From now on we suppose that Γ is a *Euclidean-type group*, namely a semidirect product $\Gamma = G \ltimes \mathbb{R}^d$ where G is a connected closed subgroup of $\mathbf{SO}(d)$, the group of $d \times d$ orthogonal matrices. It is assumed that the group multiplication is given by $(g_1, p_1) \cdot (g_2, p_2) = (g_1g_2, p_1 + g_1p_2)$ where g_1p_2 is matrix multiplication. (The special case $G = \mathbf{SO}(d)$ yields the Euclidean group.)

Write $\gamma = (g, p)$ where $g \in G$, $p \in \mathbb{R}^d$. Similarly, write $\xi = (h, v)$ where $h : X \to G$, $v : X \to \mathbb{R}^d$. Then the group extension becomes

$$f_\xi(x, \gamma) = (fx, gh(x), p + gv(x)) = (f_h(x, g), p + \phi(x, g)),$$

where $\phi(x,g) = gv(x)$. Iterating, we obtain

$$f_\xi^n(x,\gamma) = (f_h^n(x,g), p + \phi_n(x,g)),$$

where $\phi_n = \sum_{j=0}^{n-1} \phi \circ f_h^j$. Hence, as noted in [24], the noncompact part of the dynamics is governed by the statistical properties of the equivariant observable $\phi : X \times G \to \mathbb{R}^d$.

Let $\mathbb{T}$ denote the maximal torus in G, with fixed point space $\mathrm{Fix}(\mathbb{T}) = \{v \in \mathbb{R}^d : gv = v \,\text{for all}\, g \in \mathbb{T}\}$. A full measure set of elements $g \in G$ generate a maximal torus [4] and so typically $h(0)$ generates a maximal torus. Hence if $\mathrm{Fix}(\mathbb{T}) = \{0\}$, then typically $\mathrm{Fix}(h(0)) = \{0\}$ so that the hypothesis $v(0) \in (\mathrm{Fix}(h(0))^\perp$ is automatically satisfied. For $f : X \to X$ an intermittent map as in Subsection 1.1 and $\xi = (h, v) : X \to \Gamma$ Hölder, our main results imply that superdiffusion is typically suppressed for such Euclidean-type groups.

On the other hand, if $\mathrm{Fix}(\mathbb{T}) \neq \{0\}$, then typically $v(0) \notin (\mathrm{Fix}(h(0))^\perp$ and superdiffusive behaviour is conjectured.

In the special case of the Euclidean group $\Gamma = \mathbf{SO}(d) \ltimes \mathbb{R}^d$ we have $\mathrm{Fix}(\mathbb{T}) = \{0\}$ if and only if d is even. In [13] we gave heuristic arguments, supported by numerics, for suppression of superdiffusion in even dimensions and existence of superdiffusion in odd dimensions. The claim that superdiffusion is typically suppressed in even dimensions is a consequence of Theorem 1.6. The claim about existence of superdiffusion in odd dimensions is a special case of Conjecture 1.7.

Remark 1.11 Suppose that d is even. The action of $G = \mathbf{SO}(d)$ on $\mathbb{R}^d$ is irreducible, so the property $g\Sigma = \Sigma g$ for $g \in G$ implies that $\Sigma = \sigma^2 I_d$ for some $\sigma \geq 0$. (Typically $\sigma > 0$.)

Similarly, for d odd the conjectured limits in Conjecture 1.7 are symmetric.

The structure of the remainder of the paper is as follows. In Section 2, we prove the CLT and WIP for group extensions of a class of uniformly expanding maps called Gibbs-Markov maps. In Section 3, we use the result in Section 2 to prove Theorem 1.10. In Section 4, we show that Theorem 1.6 follows from Theorem 1.10 by verifying that $V^* \in L^2$.

The argument in Section 3 relies on the method of inducing statistical limit laws, which is by now standard for the CLT. The corresponding result for the WIP is a special case of [23] (where the focus is on the superdiffusive case) but the method simplifies significantly in the situation of this paper. Hence we have included the required special case of [23] in Appendix A.

Notation We use "big O" and $\ll$ notation interchangeably, writing $a_n = O(b_n)$ or $a_n \ll b_n$ as $n \to \infty$ if there is a constant $C > 0$ such that $a_n \leq C b_n$ for all $n \geq 1$.

2 Central limit theorems for group extensions of Gibbs-Markov maps

Suppose that (Y, μ) is a probability space, and that α is a countable measurable partition of Y. Let $F : Y \to Y$ be an ergodic measure-preserving map. It is assumed that the partition α separates orbits of F and that $F|_a : a \to Y$ is a bijection for each $a \in \alpha$. If $a_0, \dots, a_{n-1} \in \alpha$, we define the n-cylinder $[a_0, \dots, a_{n-1}] = \cap_{i=0}^{n-1} F^{-i} a_i$. Fix $\theta \in (0, 1)$ and define $d_\theta(x, y) = \theta^{s(x,y)}$ where the *separation time* $s(x, y)$ is the greatest integer $n \geq 0$ such that x and y lie in the same n-cylinder.

An observable $V : Y \to \mathbb{R}^d$ is *Lipschitz* if $\|V\|_\theta = |V|_\infty + |V|_\theta < \infty$ where $|V|_\theta = \sup_{x \neq y} |V(x) - V(y)|/d_\theta(x, y)$. The space $F_\theta(Y, \mathbb{R}^d)$ of Lipschitz observables is a Banach space. More generally we say that an observable $V : Y \to \mathbb{R}^d$ is *locally Lipschitz*, $V \in F_\theta^{\rm loc}(Y, \mathbb{R}^d)$, if $V|_a \in F_\theta(Y, \mathbb{R}^d)$ for each $a \in \alpha$. Accordingly, we define $D_\theta V(a) = \sup_{x,y \in a : x \neq y} |V(x) - V(y)|/d_\theta(x, y)$.

Define the potential function $p = \log \frac{d\mu}{d\mu \circ F} : Y \to \mathbb{R}$ and assume that $p \in F_\theta^{\rm loc}(Y, \mathbb{R})$ and moreover that $\sup_a D_\theta p(a) < \infty$. In particular, $F : Y \to Y$ is *Gibbs-Markov* [1].

Let α_n denote the partition of Y into n-cylinders. Also let $q = e^p$ and $q_n = q\, q \circ F \cdots q \circ F^{n-1}$. Gibbs-Markov maps have the property that there exists a constant $D > 0$ such that for all $n \geq 1$, $a \in \alpha_n$ and $y, y' \in a$,

$$q_n(y) \leq D\mu(a), \quad \text{and} \quad |q_n(y) - q_n(y')| \leq D\mu(a) d_\theta(F^n y, F^n y'). \tag{2.1}$$

Next let G be a compact connected Lie group acting orthogonally on $\mathbb{R}^d$. Given a measurable cocycle $H : Y \to G$, we define the G-extension $F_H : Y \times G \to Y \times G$, $F_H(y, g) = (Fy, gH(y))$ with invariant measure $m = \mu \times \nu$ (recall that ν is Haar measure on G). The Euclidean metric on $\mathbb{R}^{d \times d}$ restricts to a pseudometric on G and we can speak of locally Lipschitz cocycles $H \in F_\theta^{\rm loc}(Y, G)$.

Theorem 2.1 *Let $\theta \in (0, 1)$, $\epsilon \in (0, 1]$. Suppose that $F_H : Y \times G \to Y \times G$ is an ergodic G-extension of a Gibbs-Markov map $F : Y \to Y$ by a locally Lipschitz cocycle $H \in F_{\theta^{1/\epsilon}}^{\rm loc}(Y, G)$. Let $V \in L^2 \cap F_\theta^{\rm loc}(Y, \mathbb{R}^d)$, and define the equivariant observable $\Phi(y, g) = g \cdot V(y)$. Suppose that $\int_{Y \times G} \Phi \, dm = 0$ and define $\Phi_n = \sum_{j=0}^{n-1} \Phi \circ F_H^j$.*

Assume that

(i) $\sum_{a \in \alpha} \mu(a) |1_a V|_\infty < \infty$.

(ii) $\sum_{a \in \alpha} \mu(a) (D_{\theta^{1/\epsilon}} V(a))^\epsilon (1 + |1_a V|_\infty) < \infty$.

(iii) $\sum_{a \in \alpha} \mu(a) (D_{\theta^{1/\epsilon}} H(a))^\epsilon (1 + |1_a V|_\infty) < \infty$.

Then the limit $\Sigma = \lim_{n \to \infty} \frac{1}{n} \int_{Y \times G} \Phi_n \Phi_n^T \, dm$ exists, $g\Sigma = \Sigma g$ for all $g \in G$, and $\frac{1}{\sqrt{n}} \Phi_n \to_d N(0, \Sigma)$.

Moreover, if we define $W_n(t) = n^{-\frac{1}{2}} \Phi_{nt}$ for $t = 0, \frac{1}{n}, \frac{2}{n}, \dots$ and linearly interpolate to obtain $W_n \in C([0, \infty), \mathbb{R}^d)$, then $W_n \to_w W$ in $C([0, \infty), \mathbb{R}^d)$ where W is d-dimensional Brownian motion with covariance matrix Σ.

Remark 2.2 By Cauchy-Schwarz, the regularity hypotheses $V \in L^2$ and conditions (i)–(iii) are satisfied provided (1) $\sum \mu(a)|1_a V|_\infty^2 < \infty$, (2) $\sum_{a\in\alpha} \mu(a)(D_{\theta^{1/\epsilon}} V(a))^{2\epsilon} < \infty$, and (3) $\sum_{a\in\alpha} \mu(a)(D_{\theta^{1/\epsilon}} H(a))^{2\epsilon} < \infty$.

In the remainder of this section, we prove this result. Let L denote the transfer operator for $F_H : Y \times G \to Y \times G$ (so $\int_{Y\times G} Lv\, w\, dm = \int_{Y\times G} v\, w \circ F_H\, dm$). Similarly, let M denote the transfer operator for $F : Y \to Y$. Let M_H denote the twisted transfer operator, $M_H V = M(H^{-1} \cdot V)$. In the following result (and throughout the paper) $\Phi = g \cdot V$ is shorthand for $\Phi(y, g) = g \cdot V(y)$ and so on.

Proposition 2.3 *Let $V \in L^1(Y, \mathbb{R}^d)$. If $\Phi = g \cdot V$, then $L\Phi = g \cdot M_H V$.*

Proof Let $\langle\, ,\, \rangle$ denote a G-invariant inner product on $\mathbb{R}^d$. The operator $L : L^1(Y \times G, \mathbb{R}^d) \to L^1(Y \times G, \mathbb{R}^d)$ is defined by the relation $\int_{Y\times G} \langle L\Phi, \Psi\rangle\, dm = \int_{Y\times G} \langle \Phi, \Psi \circ F_H\rangle\, dm$ for all $\Psi \in L^\infty(Y \times G, \mathbb{R}^d)$. By the Peter-Weyl theorem and the orthogonality relations for compact groups [4], we can suppose without loss that $\Psi = g \cdot W$, $W \in L^\infty(Y, \mathbb{R}^d)$. Hence,

$$\begin{aligned}\int_{Y\times G} \langle L\Phi, \Psi\rangle\, dm &= \int_{Y\times G} \langle \Phi, \Psi \circ F_H\rangle\, dm = \int_{Y\times G} \langle g \cdot V, gH \cdot W \circ F\rangle\, dm \\ &= \int_Y \langle V, H \cdot W \circ F\rangle\, d\mu = \int_Y \langle H^{-1} \cdot V, W \circ F\rangle\, d\mu \\ &= \int_Y \langle M_H V, W\rangle\, d\mu = \int_{Y\times G} \langle g \cdot M_H V, \Psi\rangle\, dm.\end{aligned}$$

The result follows. ∎

The next strange-looking result is surprisingly useful.

Proposition 2.4 *Suppose that $x, a, b \ge 0$ and $x \le a$, $x \le b$. Then $x \le (1 + a)b^\epsilon$ for all $\epsilon \in (0, 1]$. If in addition $a \ge 1$, then $x \le ab^\epsilon$ for all $\epsilon \in (0, 1]$.*

Proof If $b \le 1$, then $x \le b \le b^\epsilon$. Hence certainly $x \le (1 + a)b^\epsilon$. If $b \ge 1$, then $x \le a \le 1 + a \le (1 + a)b^\epsilon$. The last sentence follows from obvious modifications. ∎

Lemma 2.5 *Let $\theta \in (0, 1)$, $\epsilon \in (0, 1]$. Suppose that $V \in F^{\mathrm{loc}}_{\theta^{1/\epsilon}}(Y, \mathbb{R}^d)$ and $H \in F^{\mathrm{loc}}_{\theta^{1/\epsilon}}(Y, G)$.*

(a) If $\sum_{a\in\alpha} \mu(a)(D_{\theta^{1/\epsilon}} H(a))^\epsilon < \infty$, then the essential spectral radius of $M_H : F_\theta(Y, \mathbb{R}^d) \to F_\theta(Y, \mathbb{R}^d)$ is at most θ.

(b) Suppose that (i) $\sum_{a\in\alpha} \mu(a)|1_a V|_\infty < \infty$, (ii) $\sum_{a\in\alpha} \mu(a)(D_{\theta^{1/\epsilon}} V(a))^\epsilon(1 + |1_a V|_\infty) < \infty$, and (iii) $\sum_{a\in\alpha} \mu(a)(D_{\theta^{1/\epsilon}} H(a))^\epsilon |1_a V|_\infty < \infty$. Then $M_H V \in F_\theta(Y, \mathbb{R}^d)$.

Proof We prove part (b) first. Now $(M_H V)(y) = \sum_{a\in\alpha} q(y_a)H(y_a)^{-1}V(y_a)$ where y_a denotes the unique preimage of y under F in a. Using (2.1), $|(M_H V)(y)| \le \sum_{a\in\alpha} q(y_a)|V(y_a)| \le D\sum_{a\in\alpha}\mu(a)|1_a V|_\infty$. By (i), $|M_H V|_\infty < \infty$.

Similarly, $|(M_H V)(y) - (M_H V)(y')| \le I + II + III$, where

$$I = \sum_{a\in\alpha} |q(y_a) - q(y_a')||V(y_a)|, \qquad II = \sum_{a\in\alpha} q(y_a')|H(y_a) - H(y_a')||V(y_a)|,$$

$$III = \sum_{a\in\alpha} q(y_a')|V(y_a) - V(y_a')|.$$

By (2.1) and property (i),

$$|I| \le D\sum_{a\in\alpha}\mu(a)d_\theta(y,y')|1_a V|_\infty = Dd_\theta(y,y')\sum_{a\in\alpha}\mu(a)|1_a V|_\infty \ll d_\theta(y,y').$$

Next, the estimates $|H(y_a) - H(y_a')| \le 2$ and $|H(y_a) - H(y_a')| \le D_{\theta^{1/\epsilon}}H(a)d_{\theta^{1/\epsilon}}(y_a,y_a')$ together imply by Proposition 2.4 that $|H(y_a) - H(y_a')| \le 2(D_{\theta^{1/\epsilon}}H(a))^\epsilon d_{\theta^{1/\epsilon}}(y_a,y_a')^\epsilon$. Moreover, $d^\epsilon_{\theta^{1/\epsilon}} = d_\theta$. By (2.1) and property (iii),

$$\begin{aligned} |II| &\le 2D\sum_{a\in\alpha}\mu(a)(D_{\theta^{1/\epsilon}}H(a))^\epsilon d_\theta(y_a,y_a')|1_a V|_\infty \\ &= 2D\theta d_\theta(y,y')\sum_{a\in\alpha}\mu(a)(D_{\theta^{1/\epsilon}}H(a))^\epsilon|1_a V|_\infty \ll d_\theta(y,y'). \end{aligned}$$

Similarly, we have $|V(y_a) - V(y_a')| \le 2|1_a V|_\infty$ and $|V(y_a) - V(y_a')| \le D_{\theta^{1/\epsilon}}V(a)d_{\theta^{1/\epsilon}}(y_a,y_a')$ which together imply by Proposition 2.4 that $|V(y_a)-V(y_a')| \le (1+2|1_a V|_\infty)(D_{\theta^{1/\epsilon}}V(a))^\epsilon d_{\theta^{1/\epsilon}}(y_a,y_a')^\epsilon$. By (2.1) and property (ii),

$$|III| \le D\sum_{a\in\alpha}\mu(a)(1+2|1_a V|_\infty)(D_{\theta^{1/\epsilon}}V(a))^\epsilon d_\theta(y_a,y_a') \ll d_\theta(y,y').$$

Hence $\|M_H V\|_\theta = |M_H V|_\infty + |M_H V|_\theta < \infty$ as required.

Next we prove part (a). We claim that $\|M_H^n V\|_\theta \le C(|V|_\infty + \theta^n|V|_\theta)$. Since the unit ball of $F_\theta(Y,\mathbb{R}^d)$ is compact in L^∞, the result then follows from [16].

It remains to prove the claim. This is done by combining an argument in [5, Corollary 4.3(a)] with the method used for term II in part (b). Let $V \in F_\theta(Y,\mathbb{R}^d)$. First, it is standard that $|M_H V|_\infty \le |V|_\infty$ and so $|M_H^n V|_\infty \le |V|_\infty$. Also, $(M_H^n V)(y) = \sum_{a\in\alpha_n} q_n(y_a)H_n(y_a)^{-1}V(y_a)$ where y_a denotes the unique preimage of y under F^n in a and $H_n = H\, H\circ F\cdots H\circ F^{n-1}$. Hence $|(M_H^n V)(y) - (M_H^n V)(y')| \le I + II + III$ where

$$I = \sum_{a\in\alpha_n} |q_n(y_a) - q_n(y_a')||V|_\infty, \qquad II = \sum_{a\in\alpha_n} q_n(y_a')||H_n(y_a) - H_n(y_a')||V|_\infty,$$

$$III = \sum_{a\in\alpha_n} q_n(y_a')|V(y_a)-V(y_a')|.$$

Now $I \le D\sum_{a\in\alpha_n}\mu(a)d_\theta(y,y')|V|_\infty = Dd_\theta(y,y')|V|_\infty$ and $III \le D\sum_{a\in\alpha_n}\mu(a)|V|_\theta d_\theta(y_a,y_a') = D\theta^n|V|_\theta d_\theta(y,y')$. Also,

$$II \le D\sum_{a\in\alpha_n}\mu(a)|H_n(y_a)-H_n(y_a')||V|_\infty,$$

and

$$\begin{aligned}|H_n(y_a)-H_n(y_a')| &\le \sum_{j=0}^{n-1}|H(F^jy_a)-H(F^jy_a')| \le 2\sum_{j=0}^{n-1}(D_{\theta^{1/\epsilon}}H(F^ja))^\epsilon d_\theta(F^jy_a,F^jy_a') \\ &= 2\sum_{j=0}^{n-1}(D_{\theta^{1/\epsilon}}H(F^ja))^\epsilon\theta^{n-j}d_\theta(y,y').\end{aligned}$$

Hence $II \le 2D\sum_{a\in\alpha_n}\mu(a)\sum_{j=0}^{n-1}(D_{\theta^{1/\epsilon}}H(F^ja))^\epsilon\theta^{n-j}|V|_\infty d_\theta(y,y')$. Now

$$\begin{aligned}&\sum_{a\in\alpha_n}\mu(a)\sum_{j=0}^{n-1}(D_{\theta^{1/\epsilon}}H(F^ja))^\epsilon\theta^{n-j} = \sum_{j=0}^{n-1}\sum_{b\in\alpha_{n-j}}\sum_{a\in\alpha_n,F^ja=b}\mu(a)(D_{\theta^{1/\epsilon}}H(F^ja))^\epsilon\theta^{n-j} \\ &= \sum_{j=0}^{n-1}\theta^{n-j}\sum_{b\in\alpha_{n-j}}(D_{\theta^{1/\epsilon}}H(b))^\epsilon\sum_{a\in\alpha_n,F^ja=b}\mu(a) = \sum_{j=0}^{n-1}\theta^{n-j}\sum_{b\in\alpha_{n-j}}(D_{\theta^{1/\epsilon}}H(b))^\epsilon\mu(b) \\ &\le \sum_{j=0}^{n-1}\theta^{n-j}\sum_{a\in\alpha}(D_{\theta^{1/\epsilon}}H(a))^\epsilon\mu(a) \le (1-\theta)^{-1}\sum_{a\in\alpha}(D_{\theta^{1/\epsilon}}H(a))^\epsilon\mu(a),\end{aligned}$$

so $II \ll |V|_\infty d_\theta(y,y')$. The claim follows by combining these estimates. ∎

Proof of Theorem 2.1 The proof largely follows [10, 20]. Suppose first that $M_H : F_\theta(Y,\mathbb{R}^d)\to F_\theta(Y,\mathbb{R}^d)$ has no eigenvalues on the unit circle. By Lemma 2.5(a), there exists $\tau<1$ such that the spectrum of M_H lies strictly inside the ball of radius τ. In particular, there is a constant $C>0$ such that $\|M_H^n\|\le C\tau^n$.

By Lemma 2.5(b), $W = M_HV \in F_\theta(Y,\mathbb{R}^d)$. Define $\chi = \sum_{j=1}^\infty M_H^jV = \sum_{j=0}^\infty M_H^jW$. Our assumptions guarantee that this series is absolutely convergent in $F_\theta(Y,\mathbb{R}^d)$ and hence $\chi\in F_\theta(Y,\mathbb{R}^d)$. Write $V = \hat V + H\cdot(\chi\circ F)-\chi$. Then $\hat V\in L^2$ (since $\chi\in F_\theta(Y,\mathbb{R}^d)$ and $V\in L^2$). Moreover,

$$M_HV = M_H\hat V + \chi - M_H\chi = M_H\hat V + \sum_{j=1}^\infty M_H^jV - \sum_{j=2}^\infty M_H^jV = M_H\hat V + M_HV,$$

and so $M_H\hat V = 0$. At the level of $Y\times G$, we have

$$\Phi = \hat\Phi + (g\cdot\chi)\circ F_H - g\cdot\chi,$$

where $\hat{\Phi} = g \cdot \hat{V}$ is an L^2 observable and $g \cdot \chi$ lies in L^∞. By Proposition 2.3, $L\hat{\Phi} = 0$. It follows that the sequence $\{\hat{\Phi} \circ F_H^n;\, n \geq 1\}$ defines a reverse martingale sequence. Hence we have decomposed Φ into a (reverse) L^2 martingale $\hat{\Phi}$ and an L^∞ coboundary, as in Gordin [12]. By ergodicity, it follows as usual that we obtain the CLT and WIP for one-dimensional projections, and hence in $\mathbb{R}^d$ by the Cramer-Wold device.

It remains to remove the assumption about eigenvalues for M_H on the unit circle. Suppose that there are k such eigenvalues $e^{i\omega_\ell}$, $\omega_\ell \in [0, 2\pi)$, $\ell = 1, \ldots, k$ (including multiplicities). Generalised eigenfunctions yield polynomial growth rates under iteration by M_H; this is impossible since M_H is a contraction in L^∞. Hence we can write $V = V_0 + \sum_{\ell=1}^k V_\ell$ where $\|M_H^n V_0\|_\theta \leq C\tau^n \|V_0\|_\theta$ and $M_H V_\ell = e^{i\omega_\ell} V_\ell$. Correspondingly $\Phi = \Phi_0 + \sum_{\ell=1}^k \Phi_\ell$ where $\Phi_\ell = g \cdot V_\ell$, $\ell = 0, \ldots, k$. In particular, we obtain the CLT and WIP for Φ_0 by the above argument, while $L\Phi_\ell = e^{i\omega_\ell}\Phi_\ell$, $\ell = 1, \ldots k$.

By ergodicity, the eigenvalue 1 for L corresponds to constant eigenfunctions (these only occur if the representation $\mathbb{R}^d$ of G includes trivial representations). Restricting to observables of mean zero removes these eigenfunctions, and then 1 is not an eigenvalue. It follows that $\omega_\ell \in (0, 2\pi)$, $\ell = 1, \ldots, k$.

Next, a simple argument (see [20]) shows that $\Phi_\ell \circ F_H = e^{-i\omega_\ell}\Phi_\ell$ for $\ell = 1, \ldots, k$, so that $|\sum_{j=1}^n \Phi_\ell \circ F_H^j|_\infty \leq 2|e^{i\omega_\ell} - 1|^{-1}|\Phi_\ell|_\infty$ which is bounded in n. Hence the result for Φ follows from the result for Φ_0. ∎

3 Central limit theorems for group extensions of nonuniformly expanding maps

Let $f : X \to X$ be a nonuniformly expanding map of a metric space (X, d), with probability measure μ_0, partition α, integrable return time $r : Y \to \mathbb{Z}^+$, and return map $F = f^r : Y \to Y$, satisfying conditions (1)–(4) as described in Section 1.2. There is a unique F-invariant probability measure μ_Y absolutely continuous with respect to $\mu_0|_Y$. (So from now, the probability measure on Y denoted by μ in Section 2 is denoted μ_Y.) It is easily verified that the map $F : Y \to Y$ is Gibbs-Markov on the probability space (Y, μ_Y) with partition α and $\theta = \lambda^{-\eta}$. A standard elementary argument shows that there is a constant $C > 0$ such that $d(x, y) \leq C d_\theta(x, y)^{1/\eta}$ for all $x, y \in Y$.

We continue to let μ denote the f-invariant probability measure on X as described after Remark 1.9. The construction of μ starting from μ_Y is given explicitly at the beginning of the proof of Theorem 1.10 at the end of this section.

As usual, we suppose that $h : X \to G$ is a measurable cocycle into a compact connected Lie group G with Haar measure ν, and we define the group extension $f_h : X \times G \to X \times G$, $f_h(x, g) = (fx, gh(x))$. The invariant product measure $m = \mu \times \nu$ is assumed to be ergodic.

The proof of Theorem 1.10 proceeds by considering the return map for f_h to the set $Y \times G$ and reducing to the set up in Section 2. The return time $r : Y \times G \to \mathbb{Z}^+$

is simply $r(y,g)=r(y)$. Define the *induced cocycle* $H:Y\to G$, $H=h_r=h(h\circ f)\cdots(h\circ f^{r-1})$. Then the return map $F_H=(f_h)^r:Y\times G\to Y\times G$ is given by $F_H(y,g)=(Fy,gH(y))$, again with ergodic measure $m_Y=\mu_Y\times\nu$.

Let $\phi:X\times G\to\mathbb{R}^d$ be an observable of the form $\phi(x,g)=g\cdot v(x)$ where $v:X\to\mathbb{R}^d$ and G acts orthogonally on $\mathbb{R}^d$. We define the *induced observable* $\Phi(y,g)=\sum_{j=0}^{r(y)-1}\phi\circ f_h^j(y,g)$. Then $\Phi(y,g)=g\cdot V(y)$ where $V(y)=\sum_{j=0}^{r(y)-1}h_j(y)v(f^jy)$. Let $Z_n=\{y\in Y:r(y)=n\}$.

Proposition 3.1 *Let $p\ge 1$. Suppose that v and h are C^η for some $\eta\in(0,1]$, with Hölder constants $|v|_\eta$ and $|h|_\eta$. Let $\theta\in[\lambda^{-\eta},1)$. Then*

(a) $|1_{Z_n}V|_\infty\le|v|_\infty n$.

(b) If $r\in L^p$, then $V\in L^p$ and moreover $\sum_{a\in\alpha}\mu_Y(a)|1_aV|_\infty^p<\infty$.

(c) $D_\theta V(Z_n)\ll(|v|_\eta+|v|_\infty|h|_\eta)n^2$.

(d) $D_\theta H(Z_n)\ll|h|_\eta n$.

Proof Part (a) is immediate. Note that $\int_Y r^p\,d\mu_Y=\sum_{n=1}^\infty\mu_Y(Z_n)n^p$ implying part (b).

Next, let $y,y'\in Z_n$. Then

$$\begin{aligned}|H(y)-H(y')|&\le\sum_{j=0}^{n-1}|h(f^jy)-h(f^jy')|\le|h|_\eta\sum_{j=0}^{n-1}d(f^jy,f^jy')^\eta\\&\ll n|h|_\eta d(Fy,Fy')^\eta\ll n|h|_\eta d_\theta(y,y'),\end{aligned}$$

and part (d) follows.

Finally, for $y,y'\in Z_n$, $|V(y)-V(y')|=|\sum_{j=0}^{n-1}h_j(y)v(f^jy)-\sum_{j=0}^{n-1}h_j(y')v(f^jy')|\ll|\sum_{j=0}^{n-1}h_j(y)-h_j(y')||v|_\infty+n|v|_\eta d_\theta(y,y')$. Moreover, $|\sum_{j=0}^{n-1}h_j(y)-h_j(y')|\le\sum_{j=0}^{n-1}\sum_{k=0}^{j-1}|h(f^kx)-h(f^ky')|\ll|h|_\eta n^2d_\theta(y,y')$. Part (c) follows. ∎

Lemma 3.2 *Under the hypotheses of Theorem 1.10, the induced observable $\Phi=g\cdot V:Y\times G\to\mathbb{R}^d$ satisfies the CLT and WIP.*

Proof We verify the hypotheses of Theorem 2.1. By assumption, $V\in L^2$. By Proposition 3.1(b), condition (i) holds.

Let $\epsilon\in(0,1)$, $\epsilon\le(p-1)/2$. Increase $\theta\in(0,1)$ if necessary so that $V\in F^{\rm loc}_{\theta^{1/\epsilon}}(Y,\mathbb{R}^d)$ and $H\in F^{\rm loc}_{\theta^{1/\epsilon}}(Y,G)$. By Proposition 3.1(c),

$$\sum_{a\in\alpha}\mu_Y(a)(D_{\theta^{1/\epsilon}}V(a))^\epsilon(1+|1_aV|_\infty)\ll\sum_{n\ge1}\mu_Y(Z_n)n^{2\epsilon+1}\le\sum_{n\ge1}\mu_Y(Z_n)n^p=\int_Y r^p\,d\mu_Y<\infty,$$

verifying condition (ii) of Theorem 2.1. Similarly, condition (iii) follows from Proposition 3.1(d). ∎

The next result, which is proved in the Appendix, is a special case of [23] showing that, under a mild condition, to prove the WIP it suffices to prove the WIP for an induced map.

Theorem 3.3 *Suppose that $q : \Omega \to \Omega$ is an ergodic measure-preserving transformation of a probability space (Ω, m) and $\phi : \Omega \to \mathbb{R}^d$ is an integrable observable of mean zero. Let $\Lambda \subset \Omega$ have positive measure and set $m_\Lambda = (m|\Lambda)/m(\Lambda)$. Let $r : \Lambda \to \mathbb{Z}^+$ be the first return time to Λ, namely $r(y) = \inf\{n \geq 1 : q^n y \in \Lambda\}$. Suppose that r is integrable and set $\bar{r} = \int_\Lambda r \, dm_\Lambda$.*

Define the first return map $Q = q^r : \Lambda \to \Lambda$ and the induced observable $\Phi = \sum_{j=0}^{r-1} \phi \circ q^j : \Lambda \to \mathbb{R}^d$. Define the Birkhoff sums $\phi_n = \sum_{j=0}^{n-1} \phi \circ q^j$, $\Phi_n = \sum_{j=0}^{n-1} \Phi \circ Q^j$. Also define $\Psi = \max_{0 \leq \ell < r} |\phi_\ell| : \Lambda \to \mathbb{R}$.

Let $w_n(t) = n^{-\frac{1}{2}} \phi_{nt}$ and $W_n(t) = n^{-\frac{1}{2}} \Phi_{nt}$ for $t = 0, \frac{1}{n}, \frac{2}{n}, \ldots$ and linearly interpolate to obtain processes $w_n, W_n \in C([0,\infty), \mathbb{R}^d)$.

Assume that

(a) $W_n \to_w W$ in $C([0,\infty), \mathbb{R}^d)$ on (Λ, m_Λ) where W is a d-dimensional Brownian motion with covariance matrix Σ, and

(b) $n^{-\frac{1}{2}} \max_{j=0,\ldots,n} \Psi \circ Q^j \to 0$ in probability on (Λ, m_Λ).

Then $w_n \to_w \widetilde{W}$ in $C([0,\infty), \mathbb{R}^d)$ on (Ω, m) where $\widetilde{W} = (\bar{r})^{-\frac{1}{2}} W$ is a d-dimensional Brownian motion with covariance matrix $\widetilde{\Sigma} = (\bar{r})^{-1} \Sigma$.

If condition (b) fails, then we still have the CLT: $n^{-\frac{1}{2}} \phi_n \to_d N(0, \widetilde{\Sigma})$.

Remark 3.4 Condition (b) provides control during individual excursions in Ω from Λ. By Corollary A.2 in the Appendix, it suffices that $\Psi \in L^2$ (which is certainly the case if $\phi \in L^\infty$ and $r \in L^2$).

The only property of Brownian motion that is used in the proof is that the sample paths are continuous (relaxing this condition is the main point of [23]). Also, the argument goes through if the normalisation factor $n^{\frac{1}{2}}$ is replaced by a general regularly varying function.

Analogous methods for obtaining the CLT by inducing can be found for example in [6, 15, 22]. If the CLT is the main goal, then these approaches may be preferable to Theorem 3.3.

Proof of Theorem 1.10 Since $r : Y \times G \to \mathbb{Z}^+$ is not necessarily the *first* return time to $Y \times G$ for $f_h : X \times G \to X \times G$, we cannot directly apply Theorem 3.3. This is circumvented by using a tower construction to build an extension of $X \times G$ for which $r : Y \times G \to \mathbb{Z}^+$ is the first return time.

First we recall the definition of the tower for $f : X \to X$. (In doing so, we specify how μ is constructed from μ_Y.) Define the tower map $f_\Delta : \Delta \to \Delta$ by $\Delta = \{(y,\ell) \in Y \times \mathbb{Z} : 0 \le \ell < r(y)\}$ and $f_\Delta(y,\ell) = \begin{cases} (y,\ell+1), & \ell \le r(y)-2 \\ (Fy,0), & \ell = r(y)-1 \end{cases}$. The probability measure $\mu_\Delta = \mu_Y \times \text{counting}/\int_Y r\, d\mu_Y$ is f_Δ-invariant. The projection $p : \Delta \to X$, $p(y,\ell) = f^\ell y$, defines a semiconjugacy between f_Δ and f, and μ is defined to be $\mu = p_*\mu_\Delta$.

Similarly, we define the tower map $f_{\Delta\times G} : \Delta \times G \to \Delta \times G$ by $\Delta \times G = \{(y,g,\ell) \in Y \times G \times \mathbb{Z} : 0 \le \ell < r(y)\}$ and $f_{\Delta\times G}(y,g,\ell) = \begin{cases} (y,g,\ell+1), & \ell \le r(y)-2 \\ (F_H(y,g),0), & \ell = r(y)-1 \end{cases}$. The probability measure $m_\Delta = \mu_\Delta \times \nu$ is $f_{\Delta\times G}$-invariant. The projection $\pi : \Delta \times G \to X \times G$, $\pi(y,g,\ell) = T_h^\ell(y,g)$, defines a semiconjugacy between $f_{\Delta\times G}$ and f_h. Moreover, $m = \mu \times \nu$ satisfies $m = \pi_* m_\Delta$.

Starting with the original observable $\phi = g \cdot v : X \times G \to \mathbb{R}^d$, we define $\hat v = v \circ p : \Delta \to \mathbb{R}^d$ and $\hat\phi = \phi \circ \pi = g \cdot \hat v : \Delta \times G \to \mathbb{R}^d$. Since $m = \pi_* m_\Delta$, it follows that $\{\hat\phi \circ f_{\Delta\times G} : j \ge 0\} =_d \{\phi \circ f_h : j \ge 0\}$. Hence to prove the CLT/WIP for ϕ on $X \times G$, it suffices to prove the CLT/WIP for $\hat\phi$ on $\Delta \times G$.

Since $r : Y \times G \to \mathbb{Z}^+$ is the first return time for $f_{\Delta\times G} : \Delta\times G \to \Delta \times G$, with first return map $F_H : Y \times G \to Y \times G$, we are now in a position to apply Theorem 3.3. Take $q = f_{\Delta\times G}$, $Q = F_H$, $\Omega = \Delta \times G$, $\Lambda = Y \times G$. Also, we have $\hat\phi(x,g) = g \cdot \hat v(x)$ and $\Phi(y,g) = \sum_{j=0}^{r(y)-1} \hat\phi \circ q^j$. It follows from the definitions that

$$\Phi = g \cdot V, \quad V(y) = \sum_{j=0}^{r(y)-1} h_j(y) v(f^j y).$$

Assumption Theorem 3.3(a) is immediate from Lemma 3.2. The CLT for $\hat\phi$ follows by the last statement of Theorem 3.3.

Next, $\hat\phi_\ell(y,g,0) = \phi_\ell(y,g)$ and

$$|\phi_\ell(y,g)| = \Big| g \cdot \sum_{j=0}^{\ell} h_j(y) v(f^j y)\Big| = \Big|\sum_{j=0}^{\ell} h_j(y) v(f^j y)\Big|,$$

so $\Psi(y,g) = V^*(y)$. Hence the assumption that $V^* \in L^2$ in Theorem 1.10 implies that $\Psi \in L^2$ and so assumption Theorem 3.3(b) is satisfied. ∎

4 Central limit theorems for group extensions of intermittency maps

In the case of the intermittency maps (1.1), it is well-known that there is a constant $c = c_\gamma > 0$ such that $\mu(r > n) \sim cn^{-1/\gamma}$. In particular, $r \in L^2$ if and only if $\gamma < \frac{1}{2}$. Hence, Theorem 1.10 applies immediately when $\gamma \in [0, \frac{1}{2})$.

For $\gamma \in [\frac{1}{2}, 1)$, we have $r \in L^p$ where $p \in (1,2)$. The CLT and WIP still hold provided we can verify that $V^* \in L^2$. Here we require further more specific information about the maps (1.1). Let $Z_n = \{y \in Y : r(y) = n\}$. Then it is well known that in fact $\mu(Z_n) \ll n^{-(1+1/\gamma)}$. Furthermore, $\operatorname{diam}(f^k Z_n) \ll (n-k)^{-(1+1/\gamma)}$ and $|f^k y| \ll (n-k)^{-1/\gamma}$ for $y \in Z_n$, $k = 1, \dots, n$. (See for example [18, 14, 27].)

Theorem 4.1 *Suppose that f is one of the maps (1.1). Suppose that $v, h \in C^\eta$, $\eta \in (0,1]$. Suppose further that $\eta > \gamma - \frac{1}{2}$. If $v(0) \in (\operatorname{Fix} h(0))^\perp$, then $V^* \in L^2$ and hence ϕ satisfies the CLT and WIP.*

Proof We may suppose without loss that $\eta \in (\gamma - \frac{1}{2}, \gamma)$.

Writing $v = (v - v(0)) + v(0)$, we may consider the cases $v(0) = 0$ and $v \equiv v(0)$ separately. The case $v(0) = 0$ is identical to the argument in [14] and is repeated here for completeness. For $y \in Z_n$,

$$|V^*(y)| \le \sum_{j=0}^{n-1} |h_j(y) v(f^j y)| \le \sum_{j=0}^{n-1} |v|_\eta |f^j y|^\eta \ll \sum_{j=1}^{n-1} (n-j)^{-\eta/\gamma} \ll n^{1-\eta/\gamma}.$$

Hence,

$$\int_Y |V^*(y)|^2 \, d\mu \ll \sum_n n^{2-2\eta/\gamma} n^{-(1+1/\gamma)} < \infty. \tag{4.1}$$

It remains to consider the case $v \equiv v(0)$. Since G acts orthogonally and $v(0) \in (\operatorname{Fix} h(0))^\perp$,

$$\sup_{\ell \ge 0} \Big| \sum_{j=0}^{\ell} [h(0)]^j v(0) \Big| < \infty. \tag{4.2}$$

Set $\bar{h}(y) = h(y) h(0)^{-1}$ and $A_k = h(0)^k \, (\bar{h} \circ f^k) \, h(0)^{-k}$. In the noncommutative products below, we write $\prod_{k=0}^{j-1} a_k = a_0 a_1 \cdots a_{j-1}$. Then for $j \ge 2$,

$$\begin{aligned} h_j(y) &= \prod_{k=0}^{j-1} h(f^k y) = \prod_{k=0}^{j-1} [\bar{h}(f^k y) h(0)] = \Big[\prod_{k=0}^{j-1} A_k(y)\Big] h(0)^j \\ &= \sum_{k=1}^{j-1} \Big[\prod_{i=0}^{k-1} A_i(y)\Big] (A_k(y) - I) h(0)^j + A_0(y) h(0)^j. \end{aligned}$$

Moreover, for $y \in Z_n$,

$$|A_k(y) - I| = |\bar{h}(f^k y) - \bar{h}(0)| = O(|f^k y|^\eta) = O((n-k)^{-\eta/\gamma}).$$

Hence by (4.2), for $\ell \le n$,

$$
\begin{aligned}
\Big|\sum_{j=0}^{\ell} h_j(y)v(0)\Big| &= \Big|\sum_{j=2}^{\ell}\sum_{k=1}^{j-1}\Big[\prod_{i=0}^{k-1} A_i(y)\Big](A_k(y)-I)h(0)^j v(0)\Big| + O(1) \\
&= \Big|\sum_{k=1}^{\ell-1}\Big[\prod_{i=0}^{k-1} A_i(y)\Big](A_k(y)-I)\sum_{j>k} h(0)^j v(0)\Big| + O(1) \\
&\ll \sum_{k=1}^{\ell-1}|A_k(y)-I| + 1 \ll \sum_{k=1}^{\ell-1}(n-k)^{-\eta/\gamma} + 1 \\
&\le \sum_{k=1}^{n-1}(n-k)^{-\eta/\gamma} + 1 \ll n^{1-\eta/\gamma}.
\end{aligned}
$$

It follows that

$$
|V^*(y)| = \max_{0\le \ell < n-1}\Big|\sum_{j=0}^{\ell} h_j(y)v(0)\Big| \ll n^{1-\eta/\gamma},
$$

establishing the required estimate just as in (4.1). ∎

In particular, Theorem 1.6 holds for v, h sufficiently Hölder.

Remark 4.2 The resummation argument in the proof of Theorem 4.1 is required in order to fully exploit (4.2). The more direct estimate $|V^*(y)| \le \sum_{j=0}^{n}|h_j(y) - h_j(0)||v(0)| + |\sum_{j=0}^{n} h(0)^j v(0)| \ll n^{2-\eta/\gamma}$ establishes that $V \in L^2$ provided $\eta > 2\gamma - \frac{1}{2}$. However, even for h Lipschitz ($\eta = 1$) this approach succeeds only for $\gamma < \frac{3}{4}$.

Our original version of this resummation argument led to the same result but under the unnecessarily stringent restriction $\eta > \gamma$. The improved (and simplified) argument was pointed out to us by Sébastien Gouëzel.

A Inducing the weak invariance principle

Theorem 3.3 is a special case of [23, Theorem 2.2]. Since the proof is greatly simplified, and since the published version of [23] refers to this appendix, we provide the full details here.

As in the proof of Theorem 2.1, by the Cramer-Wold device we may suppose without loss that $d = 1$.

It is convenient to work throughout with the Skorokhod spaces $\mathcal{D}[0,T]$ and $\mathcal{D}[0,\infty)$ of real-valued cadlag functions (right-continuous $g(t^+) = g(t)$ with left-hand limits $g(t^-)$) on the respective interval, with the sup-norm topology in the case of $\mathcal{D}[0,T]$ and the topology of uniform convergence on compact subsets in the case of $\mathcal{D}[0,\infty)$. (We could equally work with the spaces of continuous functions (replacing certain piecewise constant functions by the piecewise linear continuous interpolants throughout.)

Let (Ω, m, q), (Λ, m_Λ, Q) and $r : \Lambda \to \mathbb{Z}^+$ be as in Theorem 3.3. Recall the relation $Q = q^r$ and the notation $\bar{r} = \int_\Lambda r \, dm_\Lambda$. Define the Birkhoff sums $\phi_n = \sum_{j=0}^{n-1} \phi \circ q^j$, $\Phi_n = \sum_{j=0}^{n-1} \Phi \circ Q^j$, $r_n = \sum_{j=0}^{n-1} r \circ Q^j$. Also define $\Psi = \max_{0 \le \ell < r} |\phi_\ell| : \Lambda \to \mathbb{R}$ and the cadlag processes w_n, W_n, setting

$$w_n(t) = n^{-\frac{1}{2}} \phi_{[nt]}, \quad W_n(t) = n^{-\frac{1}{2}} \Phi_{[nt]}.$$

Let $N_k = \sum_{\ell=1}^{k} 1_\Lambda \circ q^\ell = \max\{n \ge 0 : r_n \le k\}$ denote the lap numbers of Λ. The visits to Λ, as counted by the lap numbers N_k, separate the consecutive *excursions from* Λ. Then we can write

$$\phi_k = \Phi_{N_k} + R_k \quad \text{on } \Lambda$$

with remainder term $R_k = \sum_{\ell=r_{N_k}}^{k-1} \phi \circ q^\ell = \phi_{k - r_{N_k}} \circ Q^{N_k}$ encoding the contribution of the incomplete last excursion (if any). Next, decompose the rescaled process $w_n(t) = n^{-\frac{1}{2}} \phi_{[nt]}$ accordingly, writing

$$w_n(t) = U_n(t) + V_n(t) \quad \text{on } \Lambda$$

where

$$U_n(t) = n^{-\frac{1}{2}} \Phi_{N_{[nt]}}, \ \text{ and } \ V_n(t) = n^{-\frac{1}{2}} R_{[nt]}.$$

The excursions correspond to the intervals $[t_{n,j}, t_{n,j+1})$, $j \ge 0$, where $t_{n,j} : \Lambda \to [0, \infty)$ is given by $t_{n,j} = r_j / n$. Note that

$$t \in [t_{n,N_{[nt]}}, t_{n,N_{[nt]}+1}) \quad \text{for } t > 0 \text{ and } n \ge 1. \tag{A.1}$$

Some almost sure results We record some consequences of the ergodic theorem. But first an elementary observation, the proof of which we omit.

Proposition A.1 *Let $s > 0$ and let $(c_n)_{n \ge 1}$ be a sequence in $\mathbb{R}$ such that $n^{-s} c_n \to c$. Define a sequence of functions $C_n : [0, \infty) \to \mathbb{R}$ by letting $C_n(t) = n^{-s} c_{[nt]} - t^s c$. Then, for any $T > 0$, $(C_n)_{n \ge 1}$ converges to 0 uniformly on $[0, T]$.* ∎

Corollary A.2 *If $\Psi \in L^2$, then condition (b) of Theorem 3.3 is satisfied.*

Proof By the ergodic theorem, $n^{-1} \sum_{j=0}^{n-1} \Psi^2 \circ Q^j \to \int_\Lambda \Psi^2 \, dm_\Lambda$ almost everywhere on Λ and hence $n^{-1} \Psi^2 \circ Q^n \to 0$ almost everywhere. Now take square roots and apply Proposition A.1 with $s = \frac{1}{2}$. ∎

Lemma A.3 *The lap numbers N_k satisfy $k^{-1} N_k \to 1/\bar{r}$ a.e. on Ω as $k \to \infty$. Moreover, for any $T > 0$,*

$$\sup_{t \in [0,T]} |k^{-1} N_{[kt]} - t/\bar{r}| \to 0 \quad \textit{a.e. on } \Omega \quad \textit{as } k \to \infty.$$

Proof Recall that $m(\Lambda) = 1/\bar{r}$ (Kac' formula). Hence the first statement is immediate from the ergodic theorem. The second then follows by Proposition A.1 with $s = 1$. ∎

Convergence of U_n. We require a standard but technical result.

Proposition A.4 *Suppose that A_n, B_n are sequences in $\mathcal{D}[0,T]$ and that $A_n \to_w A$, $B_n \to_w B$ in the sup-norm topology. Suppose further that A, B are continuous and that B is nonrandom. Then $(A_n, B_n) \to_w (A, B)$ in $\mathcal{D}[0,T] \times \mathcal{D}[0,T]$ with the sup-norm topology.*

Proof The issue here is that the sup-norm topology is not separable. But since A, B are continuous, convergence in the sup-norm topology is equivalent to convergence in the Skorokhod topology which is separable, and then the result is standard. ∎

Lemma A.5 *$U_n \to_w \widetilde{W}$ in $\mathcal{D}[0,\infty)$ on (Λ, m_Λ).*

Proof For $n \geq 1$ and $t \in [0,\infty)$ we define non-negative random variables on (Λ, m_Λ) by letting $u_n(t) = n^{-1} N_{[nt]}$. Since $[u_n(t)\, n] = N_{[nt]}$, we have

$$U_n(t) = W_n(u_n(t)) \quad \text{on } \Lambda \text{ for } n \geq 1 \text{ and } t \geq 0.$$

We regard U_n, W_n, W, u_n as random elements of $\mathcal{D} = \mathcal{D}[0,\infty)$. Let u denote the constant random element of $\mathcal{D}$ given by $u(t) \equiv t/\bar{r}$, $t \geq 0$.

By Lemma A.3, for almost every $y \in \Lambda$ we have $u_n(.)(y) \to u(.)(y)$ uniformly on compact subsets of $[0,\infty)$. Hence, $u_n \to u$ a.e. in $\mathcal{D}$. By condition (a) of Theorem 3.3, we also have $W_n \to_w W$ in $\mathcal{D}$. Since W and u are continuous and u is nonrandom, it follows from Proposition A.4 that

$$(W_n, u_n) \to_w (W, u) \quad \text{in } \mathcal{D} \times \mathcal{D}.$$

The composition map $\mathcal{D} \times \mathcal{D} \to \mathcal{D}$, $(g, v) \mapsto g \circ v$, is well-defined and is easily seen to be continuous in the sup-norm topology. Hence it follows from the continuous mapping theorem that $U_n = W_n \circ u_n \to_w W \circ u = \widetilde{W}$ as required. ∎

Convergence of w_n.

Lemma A.6 *$\|w_n - U_n\|_\infty \leq n^{-1/2} \max_{0 \leq j \leq [Tn]} \Psi \circ Q^j$ a.e. on Λ.*

Proof We decompose $[0,T]$ according to the consecutive excursions, letting $T_j = t_{n,j} \wedge T$, $j \leq N_{[Tn]} + 1$. Then $\|w_n - U_n\|_\infty \leq \max_{1 \leq j \leq N_{[Tn]}+1} \sup_{t \in [T_{j-1}, T_j]} |w_n(t) - U_n(t)|$.

But $U_n(t) = w_n(T_{j-1})$ for $t \in [T_{j-1}, T_j)$ and $U_n(T_j) = w_n(T_j)$ so

$$\begin{aligned}\sup_{t\in[T_{j-1},T_j]} |w_n(t) - U_n(t)| &\le \sup_{t\in[T_{j-1},T_j]} |w_n(t) - w_n(T_{j-1})| \\ &\le n^{-1/2} \max_{0\le\ell<r\circ Q^{j-1}} |\phi_\ell \circ Q^{j-1}| = n^{-1/2}\Psi \circ Q^{j-1}.\end{aligned}$$

Since $N_{[Tn]} \le [Tn]$, this yields the required result. ∎

Proof of Theorem 3.3 First we prove the WIP assuming conditions (a) and (b). Fix any $T > 0$. It suffices to prove that $w_n \to_w \widetilde{W}$ in $\mathcal{D}[0,T]$ on (Ω, m). Moreover, since m_Λ viewed as a probability measure on Ω is absolutely continuous with respect to m, it suffices by [31, Corollary 3] to prove that $w_n \to_w \widetilde{W}$ in $\mathcal{D}[0,T]$ on (Λ, m_Λ).

By assumption (b) of Theorem 3.3 and Lemma A.6, $\|w_n - U_n\|_\infty \to 0$ in probability on (Λ, m_λ). Also, by Lemma A.5, $U_n \to_w \widetilde{W}$ in $\mathcal{D}[0,T]$ on (Λ, m_Λ). Hence, by [3, Theorem 3.1], $w_n \to_w \widetilde{W}$ in $\mathcal{D}[0,T]$ on (Λ, m_Λ) as required.

Finally, we prove the CLT assuming only condition (a). By the above, it suffices to prove that $w_n(1) - U_n(1) \to 0$ in probability on (Λ, m_λ). A simple argument for this is given in [15, Appendix A]; we sketch the main steps. First, we can pass to the natural extension so that q is invertible. Then $w_n(1) - U_n(1) = n^{-1/2} H \circ q^n$ where $H : \Omega \to \mathbb{R}$ is the measurable function given by $H(x) = \sum_{j=1}^{s(x)} \phi(q^{-j}x)$ and $s(x) \ge 0$ is least such that $q^{-s(x)}x \in \Lambda$. It follows from invariance of m that $n^{-1/2} H \circ q^n \to 0$ in probability on (Ω, m) and hence on (Λ, m_λ). ∎

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