Symmetry Groups of Attractors *

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Abstract

For maps equivariant under the action of a finite group Γ on $\mathbb{R}^n$, the possible symmetries of fixed points are known and correspond to the isotropy subgroups. This paper investigates the possible symmetries of arbitrary, possibly chaotic, attractors and finds that the necessary conditions of Melbourne, Dellnitz and Golubitsky [15] are also sufficient, at least for continuous maps.

The result shows that the reflection hyperplanes are important in determining those groups which are admissible; more precisely a subgroup Σ of Γ is admissible as the symmetry group of an attractor if there exists a Δ with Σ/Δ cyclic such that Δ fixes a connected component of the complement of the set of reflection hyperplanes of reflections in Γ but not in Δ. For finite reflection groups this condition on Δ reduces to the condition that Δ is an isotropy subgroup. Our results are illustrated for finite subgroups of $\mathbf{O}(3)$.

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1 Introduction

An important observation in the theory of discrete equivariant dynamical systems is that there are representation-theoretic restrictions on the symmetry groups of fixed points. Indeed the symmetry group of the fixed point must be the isotropy subgroup of that point. This is a restriction since not every subgroup is an isotropy subgroup. There are also restrictions on the symmetry groups of periodic orbits. Let Δ be the common isotropy subgroup of the points in the periodic orbit. Then the symmetry group Σ of the periodic orbit must be a cyclic extension of Δ, that is Δ is normal in Σ and Σ/Δ is cyclic, see for example [15].

Even the symmetry groups of chaotic attractors are subject to restrictions, see Melbourne, Dellnitz and Golubitsky [15]. For example, mappings that are equivariant with respect to the standard action of $\mathbb{D}_m$ on $\mathbb{R}^2$ cannot have $\mathbb{D}_k$-symmetric attractors for $3 < k < m$. On the other hand, there is a great deal of numerical evidence (see for example [5], [7], [9], [14]) suggesting the existence of $\mathbb{D}_m$-symmetric attractors, and also of $\mathbb{D}_2$-symmetric attractors if m is even. In addition, King and Stewart [14] have proved that there exist attractors with cyclic symmetry for all cyclic subgroups of a finite group Γ.

In [15] a necessary condition was given for a subgroup Σ to arise as the symmetry group of an attractor of a continuous Γ-equivariant mapping. In this paper we demonstrate that when Γ is finite, this condition is optimal. The main result is Theorem 2.2 which gives a completely representation/group-theoretic characterization of the dynamical issues being discussed. The existence of connected Σ-symmetric attractors can be characterized representation-theoretically. Taking cyclic extensions of such subgroups Σ yields the symmetry groups of general (not necessarily connected) attractors. (This is analogous to the characterization of the symmetry groups of periodic orbits as cyclic extensions of isotropy subgroups.)

We discuss two special cases of Theorem 2.2. If Γ is a finite reflection group, then the subgroups Σ are precisely the cyclic extensions of the isotropy subgroups of Γ, see Corollary 3.3. It follows that for the afore-mentioned representations of $\mathbb{D}_m$ there do, as expected, exist attractors with $\mathbb{D}_m$ (and $\mathbb{D}_2$ for m even) symmetry. At the other extreme, if the representation contains no reflections, as is typically the case in applications, then there are *no* restrictions on the symmetry of an attractor.

The remainder of the paper is organized as follows. In Section 2 we state precisely our main results expressing necessary and sufficient conditions for existence of Σ-symmetric attractors. The conditions rely both on group theory and representation theory; as it stands, the representation-theoretic conditions are not easily computable. However, in Section 3 we reduce the representation-theoretic condition to one depending only on isotropy subgroups and reflections. This is more easily computable, and in addition leads to our result about finite reflection groups.

The first step in the proof of Theorem 2.2 is to construct connected symmetric graphs supporting the appropriate equivariant dynamics. This step is carried out in Section 4. Then the existence of connected Σ-symmetric attractors reduces to the embeddability of these graphs in $\mathbb{R}^n$. Also there is an extendability property of graphs corresponding to cyclic extensions of Σ and the existence of disconnected attractors when there is no connected attractor with the required symmetry. These notions of embeddability and extendability of symmetric graphs, and their implications for Theorem 2.2 are discussed in Section 5. Then the embedding and extension problems are solved in Section 6 for $n \geq 3$.

When $n = 1$ and $n = 2$, there arises the possibility of topological obstructions in the embedding problem. Hence these cases are treated separately in Section 7. It turns out that topological obstructions only occur when $n = 2$ and Γ is cyclic. Finally, in Section 8 we apply our results to the finite subgroups of $\mathbf{O}(3)$.

2 Admissible symmetries of attractors

Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, and $x \in \mathbb{R}^n$. As in [15], we say that the ω-limit set $A = \omega(x)$ of x under f is an *attractor* for f if the set A is stable, that is for any neighborhood U of A, there is a (smaller) neighborhood V of A such that $f^n(V) \subset U$ for $n \geq 0$.

Let $\Gamma \subset \mathbf{O}(n)$ be a finite group acting orthogonally on $\mathbb{R}^n$. Recall that if A is a nonempty subset of $\mathbb{R}^n$ then we define the *symmetry group* of A to be the subgroup of Γ,

$$\Sigma_A = \{\gamma \in \Gamma, \gamma A = A\}.$$

The set A is Σ-*symmetric* if

(i) $\Sigma_A = \Sigma$,

(ii) $\gamma A \cap A = \emptyset$ for $\gamma \in \Gamma - \Sigma$, and

(iii) A contains points of trivial isotropy.

Condition (iii) implies that A does not lie in a proper fixed-point subspace. By Proposition 1.1 in [5], condition (ii) is redundant if A is an attractor. We note that our definition of a Σ-symmetric set is more restrictive than the definition in [15].

Definition 2.1 A subgroup $\Sigma \subset \Gamma$ is *admissible* if there is a continuous Γ-equivariant mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ with a Σ-symmetric attractor A. The subgroup Σ is *strongly admissible* if f and A can be chosen so that A is connected.

Admissibility (or more precisely, strong admissibility) of a subgroup depends crucially on the geometry of those reflection hyperplanes in $\mathbb{R}^n$ that correspond to reflections contained in Γ but not in Σ. Recall that $\tau \in \Gamma$ is a reflection if $\mathrm{Fix}(\tau)$ has dimension $n-1$. Let

$$K_\Sigma = \{\tau \in \Gamma - \Sigma;\ \tau \text{ is a reflection}\},$$

and define

$$L_\Sigma = \bigcup_{\tau \in K_\Sigma} \mathrm{Fix}(\tau).$$

Theorem 2.2 *Suppose that Γ is a finite subgroup of $\mathbf{O}(n)$, $n \geq 3$.*

(a) A subgroup $\Sigma \subset \Gamma$ is strongly admissible if and only if Σ fixes a connected component of $\mathbb{R}^n - L_\Sigma$.

(b) A subgroup Σ is admissible if and only if it is a cyclic extension of a strongly admissible subgroup Δ, that is Δ is normal in Σ and Σ/Δ is cyclic.

Remark 2.3 (a) Necessary conditions for admissibility and strong admissibility were obtained in [15] for arbitrary compact Lie groups $\Gamma \subset \mathbf{O}(n)$, $n \geq 1$. Theorem 2.2 asserts that these necessary conditions are also sufficient when $n \geq 3$ and Γ is finite.
(b) In low dimensions ($n = 1, 2$) there are topological obstructions in addition to the group and representation-theoretic obstructions. The low-dimensional situations are treated on a case-by-case basis in Section 7. It transpires that

the end effect of the topological obstructions are quite mild. Part (b) of Theorem 2.2 is valid for all $n \geq 1$. Moreover, part (a) fails only when $n = 2$ and $\Gamma \subset \mathbf{O}(2)$ is cyclic. In this case, the strongly admissible subgroups are Γ and $\mathbf{1}$.
(c) Provided $n \geq 3$, if the representation of Γ contains no reflections then any subgroup of Γ is strongly admissible. More generally, any subgroup Σ that contains all the reflections present in Γ (so that $K_\Sigma = \emptyset$) is strongly admissible. In particular Γ is strongly admissible.
(d) Any isotropy subgroup of Γ is strongly admissible (see Theorem 3.2 below). Hence the trivial isotropy subgroup $\mathbf{1}$ is strongly admissible and any cyclic subgroup is admissible by part (b) of Theorem 2.2. In particular, we recover the result of [14] on cyclic subgroups.
(e) The attractors constructed in the proof of Theorem 2.2 possess several interesting properties:

1. The attractors have an open basin of attraction, that is there is a open neighborhood V of A such that $\omega(y) \subset A$ for all $y \in V$. Moreover, $\omega(y) = A$ for y lying in a residual subset of full (Lebesgue) measure in V.

2. The attractors are topologically transitive: there is a point $x \in A$ with $\omega(x) = A$. In addition, periodic points are dense, and there is sensitive dependence on initial conditions in the following sense: there is an open neighborhood V of A and an $\epsilon > 0$ such that for any $y \in V$ and $\delta > 0$ there is a point z with $|z - y| < \delta$ and an integer $n > 0$ such that $f^n(y)$ and $f^n(z)$ are at least distance ϵ apart.

3. The connected attractors are topologically mixing: for any open subsets U and V in A, there is an integer N such that $f^{-n}(U) \cap V \neq \emptyset$ for all $n \geq N$. In [15] it is shown that topologically mixing attractors are automatically connected. Hence there exists a topologically mixing Σ-symmetric attractor if and only if Σ satisfies the condition for strong admissibility.

4. Each attractor consists of a finite union of one-dimensional manifolds on which there is a finite Lebesgue-equivalent f-invariant Σ-invariant ergodic measure. (Lebesgue-equivalent means that the sets of measure zero are the same for the ergodic measure and the Lebesgue measure induced on the one-dimensional sets). Moreover, the measure is a Sinai-Bowen-Ruelle measure. (See [16] for information about SBR measures.)

There are several related issues to those discussed in this paper that we have passed over. For example, we could ask what additional restrictions are imposed on the symmetry groups of (connected) attractors for differentiable maps, homeomorphisms, diffeomorphisms and flows. It is clear that there are further restrictions for invertible maps. However we conjecture that differentiability assumptions do not alter the results. Also we conjecture that the symmetry groups for attractors of flows are the same as those of connected attractors for homeomorphisms and diffeomorphisms.

We have only considered attractors that contain points with trivial isotropy and hence do not lie in a proper fixed-point subspace. In [15] it was indicated how the necessary conditions for admissibility translate naturally into necessary conditions for admissibility of symmetry groups of attractors in fixed-point spaces. However, there may be additional restrictions due to the presence of so-called *hidden symmetries*, see [11].

Finally, we have assumed that the underlying group of symmetries Γ is finite. It should be possible to obtain similar results when Γ is a compact Lie group. This and the other issues will be the subject of future work.

3 A computable criterion for strong admissibility, and finite reflection groups

In this section, we give a convenient rephrasing of the criterion in Theorem 2.2 that Σ fixes a connected component of $\mathbb{R}^n - L_\Sigma$.

Lemma 3.1 *Suppose that Σ fixes a connected component C of $\mathbb{R}^n - L_\Sigma$. Then $C \cap \mathrm{Fix}(\Sigma) \neq \emptyset$.*

Proof Choose $v_0 \in C$ and set $v = \sum_{\sigma\in\Sigma} \sigma v_0$. Then v lies in $\mathrm{Fix}(\Sigma)$. Since Σ fixes C, $\sigma v_0 \in C$ for each $\sigma \in \Sigma$. But C is a cone and hence $v \in C$. ■

If I is a subgroup of Γ, we define I_R to be the subgroup of I that is generated by reflections. Note that $I_R \subset I \cap \Gamma_R$ but that equality does not hold in general.

Theorem 3.2 *Let $\Gamma \subset \mathbf{O}(n)$ be a finite group with subgroup Σ. Then Σ fixes a connected component of $\mathbb{R}^n - L_\Sigma$ (equivalently Σ is strongly admissible) if* *and only if there is an isotropy subgroup $I \subset \Gamma$ such that*

$$I_R \subset \Sigma \subset I.$$

Proof Suppose that $I_R \subset \Sigma \subset I$ for I an isotropy subgroup. Let $x \in \mathbb{R}^n$ be a point with isotropy I. Since $I_R \subset \Sigma$, $x \in \mathbb{R}^n - L_\Sigma$. Let C be the connected component of $\mathbb{R}^n - L_\Sigma$ containing x. Since $\Sigma \subset I$, $x \in C \cap \mathrm{Fix}(\Sigma)$ and hence the connected component C intersects itself under the action of elements in Σ. But Σ acts on the connected components of $\mathbb{R}^n - L_\Sigma$ and hence fixes C.

Conversely, suppose that Σ fixes a connected component C of $\mathbb{R}^n - L_\Sigma$. If $L_\Sigma = \emptyset$, then $\Gamma_R \subset \Sigma$ and we can choose $I = \Gamma$. Otherwise, by the lemma, there is a point x in C that is fixed by Σ. Let I be the isotropy subgroup of x so that $\Sigma \subset I$. Since $x \notin L_\Sigma$, it follows that if $x \in \mathrm{Fix}(\tau)$, τ a reflection, then $\tau \in \Sigma$. Hence $I_R \subset \Sigma$. ■

Corollary 3.3 *Suppose that $\Gamma \subset \mathbf{O}(n)$ is a finite group generated by reflections. The strongly admissible subgroups of Γ are exactly the isotropy subgroups of Γ. Moreover, the admissible subgroups of Γ are precisely those subgroups that are cyclic extensions of isotropy subgroups.*

Proof It is a basic fact about finite reflection groups that all isotropy subgroups are generated by reflections, see for example [3]. The statement about strongly admissible subgroups follows immediately from this fact and the theorem. The statement about admissible subgroups now follows from Theorem 2.2(b). ■

4 Equivariant dynamics on graphs

In this section, we introduce the notion of a Σ-graph, and prove the existence of topologically transitive Σ-equivariant dynamics on certain Σ-graphs. In Subsection (a) below we recall a basic result about continuous equivariant extensions of mappings. Then in Subsection (b) we review some graph theory and in particular define what we mean by a Σ-symmetric graph. Our main results of this section appear in Subsection (c).

(a) An equivariant extension lemma

Lemma 4.1 *Suppose that* Γ *is a finite group acting on topological spaces* Y *and* Z*. Let* $X \subset Y$ *be a closed subset such that* $Y = \bigcup_{\gamma\in\Gamma} \gamma X$ *and let* $f : X \to Z$ *be a continuous mapping. Assume that if* $x, \gamma x \in X$ *for some* $\gamma \in \Gamma$*, then* $f(\gamma x) = \gamma f(x)$*. Then* f *can be uniquely extended to a continuous* Γ*-equivariant mapping* $g : Y \to Z$*.*

Proof Let $y \in Y$. Then $y = \gamma x$ for some $\gamma \in \Gamma$, $x \in X$. To obtain equivariance we must define $g(y) = \gamma f(x)$. This proves uniqueness of g. To show that g is well-defined suppose that $y = \gamma_1 x_1 = \gamma_2 x_2$ where $x_j \in X$, $\gamma_j \in \Gamma$ for $j = 1, 2$. We must prove that $\gamma_1 f(x_1) = \gamma_2 f(x_2)$. Let $\gamma = \gamma_2^{-1}\gamma_1$ so that $x_2 = \gamma x_1$. Then $x_1, \gamma x_1 \in X$ and

$$\gamma_1 f(x_1) = \gamma_2 \gamma f(x_1) = \gamma_2 f(\gamma x_1) = \gamma_2 f(x_2),$$

as required.

Next we show that g is Γ-equivariant. Suppose that $\gamma \in \Gamma$ and $y \in Y$. Then $y = \gamma' x$ where $\gamma' \in \Gamma$ and $x \in X$. Hence

$$g(\gamma y) = g(\gamma\gamma' x) = \gamma\gamma' f(x) = \gamma g(y).$$

Finally we show that g is continuous. The set Y is the union of finitely many copies γX of the closed set X and it is sufficient to show that $g|_{\gamma X}$ is continuous for each γ. But $g|_{\gamma X} = \gamma f \gamma^{-1}|_{\gamma X}$ which is continuous. ■

Remark 4.2 The assumption that Γ is finite is not crucial in Lemma 4.1. With slightly more work it can be shown that the conclusion is still valid if Γ is a compact Lie group.

(b) Some graph theory

We shall now summarize the graph theory that we require. See [2] for further details. A *finite graph* G consists of a finite set of vertices and a finite set of edges that join pairs of vertices. A subset $J \subset G$ is a subgraph of G if J is a graph and the vertices and edges of J are vertices and edges of G. A *path* in G is a sequence of oriented edges where the initial vertex of each edge is the final vertex of the previous edge. A graph is *connected* if there is a path between any two vertices. If each pair of vertices in G is joined by an edge, the graph is *completely connected.* A completely connected oriented graph is one where each pair of distinct vertices is joined by two edges with opposite orientations.

Each edge of a graph G can be made into a metric space isometric to the unit interval. Then the length of a path in G can be defined in the obvious way, and the distance between any two points in the same connected component is defined to be the length of the shortest path between the points. If we define the distance between points in distinct connected components to be 1, then G becomes a compact metric space. Moreover the concepts of connectedness in the metric space and in the graph coincide.

The *degree* of a vertex is the number of edges emanating from that vertex. A connected graph is said to be *Eulerian* if each vertex has even degree. To avoid trivialities, we shall not consider the graph consisting of one vertex and no edges to be Eulerian. A completely connected oriented graph is an example of an Eulerian graph provided there are at least two vertices. We define the completely connected oriented graph on one vertex to be the Eulerian graph consisting of one vertex and one edge.

The sum of the degrees of the vertices of a graph is even (twice the number of edges) and it follows easily that an Eulerian graph cannot be disconnected by removing a single edge. It is well-known that Eulerian graphs are characterized by the property that there is a continuous path tracing through each edge precisely once such that the initial vertex and the final vertex are the same. More generally, there exists a path tracing through each edge of a graph precisely once if and only if the graph is connected and there are either two vertices or no vertices of odd degree. Any vertices of odd degree lie at the endpoints of the path.

Suppose that Σ is a finite group. A graph G is a Σ-*graph* if

(a) Σ acts isometrically on G,

(b) The set of edges (equivalently the set of vertices) of G is invariant under the action of Σ, and

(c) Σ acts fixed-point-free on the set of edges (but not necessarily vertices) of G, that is if E is an edge and $\sigma E = E$ for some $\sigma \in \Sigma$, then $\sigma = 1$.

A subgraph $J \subset G$ is a *fundamental subgraph* if

(a) $G = \bigcup_{\sigma \in \Sigma} \sigma J$, and

(b) If $E, \sigma E \in J$ for some $\sigma \in \Sigma$, then $\sigma = 1$.

Assumption (c) in the definition of Σ-graph is equivalent to assuming the existence of a fundamental subgraph.

(c) Dynamics

Theorem 4.3 *Suppose that G is an Eulerian Σ-graph. Then there exists a continuous Σ-equivariant mapping $f : G \to G$ such that*

(a) G is topologically mixing.

(b) Periodic points are dense in G and there is sensitive dependence.

(c) There is a unique f-invariant Σ-invariant Lebesgue-equivalent ergodic measure on G.

Proof Let m be the number of edges in G and suppose first that $m \geq 3$. Let J be a fundamental subgraph of G and suppose that E is an edge of J with vertices v and w. Since G is an Eulerian graph, $G - E$ is a connected graph with precisely two vertices of odd degree, namely v and w. It follows that there is a continuous path $f_E : E \to G$ such that $f_E(v) = v$, $f_E(w) = w$, and such that f_E passes precisely once through each edge of $G - E$.

Define $f_J : J \to G$ by $f_J(x) = f_E(x)$ if $x \in E$. Since distinct edges intersect only in vertices, and since the mappings f_E fix vertices, the mapping f_J is well-defined and continuous. Moreover, since J is a fundamental subgraph of G and f_J fixes vertices, the hypotheses of Lemma 4.1 are satisfied and f_J can be extended to a continuous Σ-equivariant mapping $f : G \to G$.

Denote the edges of the graph G by I_j. Reparametrizing the paths f_E if necessary, we may arrange that properties (i)–(iii) in the appendix are satisfied. For example, by defining the paths to be piecewise linear, we can arrange that $q = 1$ and $\theta = m - 1 > 1$ in (iii). Finally, compute that $f(I_j) = G - I_j$ and $f(G - I_j) = G$ (since $m \geq 3$) so that property (iv)$'$ in the appendix is satisfied with $p = 2$. The proposition now follows from Proposition A.1.

To deal with the case where $m < 3$, define a new graph G' where G and G' are equal as sets, but the vertices of G' consist of the vertices of G together with the midpoints of the edges of G. Then G' is still an Eulerian Σ-graph but has twice as many edges as G. Repeat if necessary until G' has at least 4 edges. Now define a mapping $f : G' \to G'$ as before. This gives the required mapping on G. ■

An example of an Eulerian Σ-graph is given by the following construction. Define $G(\Sigma)$ to be the completely connected oriented graph with vertices $\sigma \in \Sigma$. The action of Σ on the vertices of $G(\Sigma)$ by left multiplication induces (and uniquely determines) an orientation-preserving isometric action of Σ on $G(\Sigma)$: if $E_{\tau,\tau'}$ denotes the oriented edge joining the vertex τ to the vertex τ', and $\sigma \in \Sigma$, then $\sigma E_{\tau,\tau'} = E_{\sigma\tau,\sigma\tau'}$. Let $J_\sigma = E_{1,\sigma}$.

Proposition 4.4 *$G(\Sigma)$ is an Eulerian Σ-graph with fundamental subgraph $J = \bigcup_{\sigma\in\Sigma} J_\sigma$. Moreover, Σ acts fixed-point-free on the edges and vertices of $G(\Sigma)$.*

Proof $G(\Sigma)$ is a completely connected oriented graph and hence is Eulerian. It is clear that the action on the vertices and hence the edges is fixed-point-free, and it follows immediately that $G(\Sigma)$ is a Σ-graph.

It remains to show that J is a fundamental subgraph. Suppose that $E_{\tau,\tau'}$ is an edge. Then $\tau J_{\tau^{-1}\tau'} = E_{\tau,\tau'}$ and hypothesis (a) is satisfied. In addition, if $\sigma J_\tau = J_{\tau'}$ then $E_{\sigma,\sigma\tau} = E_{1,\tau'}$. Since the action of Σ preserves orientation of edges, $\sigma = 1$ thus verifying hypothesis (b). ■

We shall refer to $G(\Sigma)$ as the *complete Σ-graph.* In Figure 1(a) we illustrate the complete Σ-graph for $\Sigma = \mathbb{Z}_3$. In part (b) of the figure we demonstrate graphically the mapping f defined in the proof of Theorem 4.3.

5 Construction of attractors as embedded graphs

In this section we define the notions of embeddability and extendability of a Σ-graph, and prove that the existence of embeddable, extendable Eulerian Σ-graphs leads to Theorem 2.2.

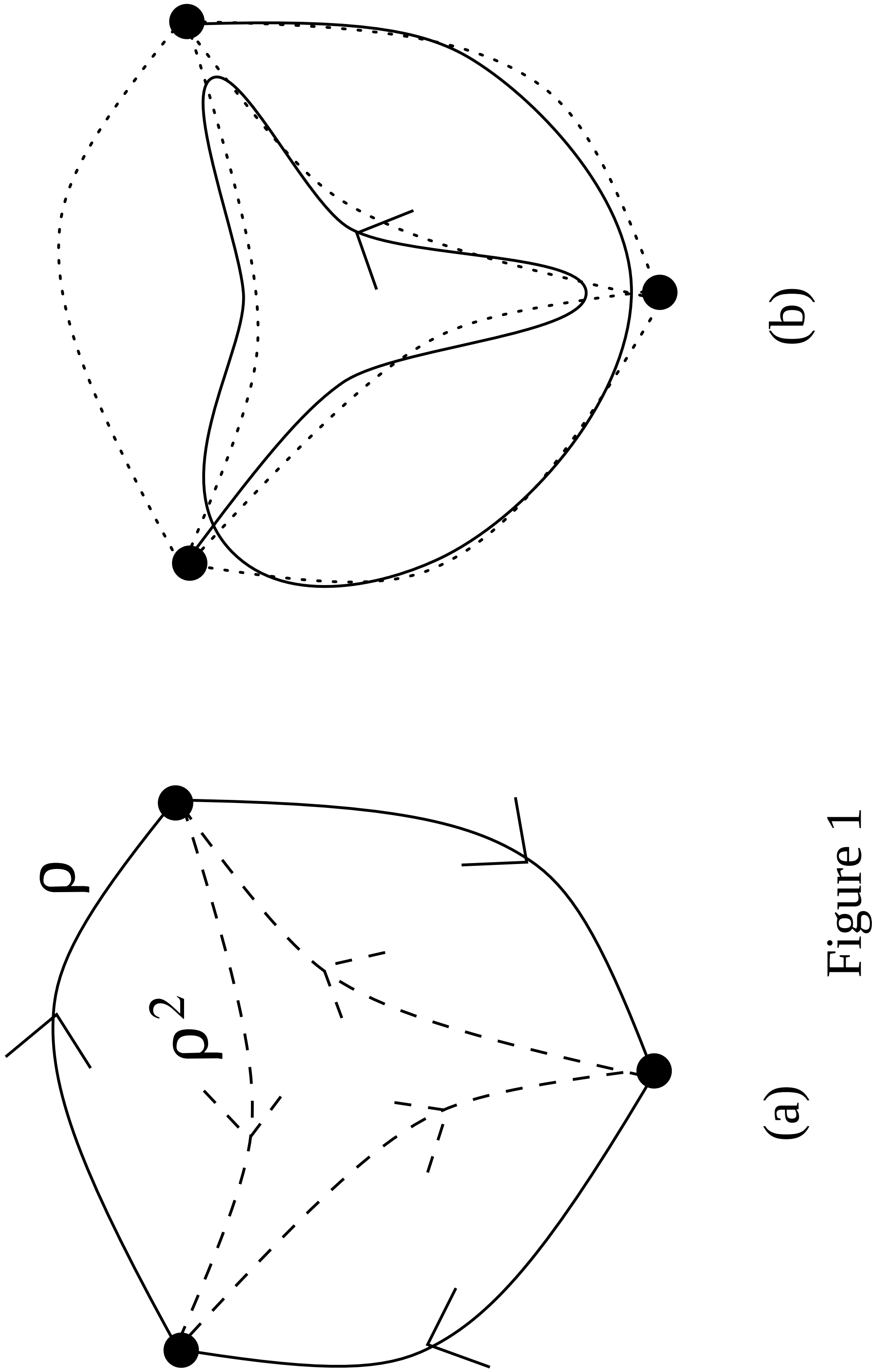


Figure 1: The complete $\mathbb{Z}_3$-graph is shown in (a). The edges marked ρ and ρ^2 form a fundamental subgraph. The map f defined in the proof of Theorem 4.3 is illustrated in part (b). The edge marked ρ is mapped onto the whole graph minus the edge ρ in such a way that there is a uniform expansion everywhere.

An *(equivariant) embedding* of G in $\mathbb{R}^n$ is a continuous one-to-one map $i : G \to \mathbb{R}^n$ (here we are viewing G as a metric space rather than a graph) such that

1. i is Σ-equivariant.

2. $\gamma i(G) \cap i(G) = \emptyset$ for all $\gamma \in \Gamma - \Sigma$.

Since G is compact, i is a homeomorphism onto its image. If there is an embedding of G in $\mathbb{R}^n$, we say that G is *embeddable.*

Proposition 5.1 *Suppose that $\Gamma \subset \mathbf{O}(n)$ is a finite group with subgroup Σ. If $i : G \to \mathbb{R}^n$ is an embedding of a Σ-graph G, then $i(G)$ is a Σ-symmetric subset of $\mathbb{R}^n$.*

Proof Since G is Σ-invariant, it follows from property 1 that $i(G)$ is Σ-invariant. Moreover there are points in G (those that are not vertices) that have trivial isotropy. Since i is one-to-one, the corresponding points in $i(G)$ have trivial isotropy. We conclude from property 2 that $i(G)$ is Σ-symmetric. ■

Let $N(\Sigma)$ denote the normalizer of Σ and let $\rho \in N(\Sigma) - \Sigma$. If $\sigma \in \Sigma$ let σ_ρ denote the element $\rho^{-1}\sigma\rho \in \Sigma$. Suppose that G is a Σ-graph. Then ρG can be made into a Σ-graph as follows: the edges of ρG are given by ρE where E is an edge of G and $\sigma(\rho E)$ is defined to be $\rho\sigma_\rho E$.

Definition 5.2 Suppose $\rho \in N(\Sigma) - \Sigma$. A Σ-graph G is *ρ-extendable* if there exists a Σ-equivariant isometry $h : G \to \rho G$. The isometry h is called an *extension.* If G is ρ-extendable for all $\rho \in N(\Sigma) - \Sigma$, then G is *extendable.*

Remark 5.3 If ρ lies in the centralizer of Σ, then any Σ-graph G is ρ-extendable. Simply define $h : G \to \rho G$ by $h(x) = \rho x$ and use the fact that $\sigma_\rho = \sigma$ for all $\sigma \in \Sigma$.

Theorem 5.4 *Suppose that G is an Eulerian, embeddable, extendable Σ-graph. Then Σ is strongly admissible, and any cyclic extension of Σ is admissible.*

Proof By Theorem 4.3 there is a continuous topologically transitive (even mixing) Σ-equivariant mapping $f : G \to G$. Let $i : G \to \mathbb{R}^n$ be an embedding, and set $A = i(G)$. Then A is a Σ-symmetric subset of $\mathbb{R}^n$ by Proposition 5.1. The mapping $\tilde{f} = i \circ f \circ i^{-1} : A \to A$ is topologically conjugate to f and hence A is an ω-limit set. We claim that it is possible to extend $\tilde{f}$ continuously and equivariantly to a closed Σ-invariant neighborhood U of A in such a way that A is an attractor for the extended mapping. Strong admissibility follows easily from the claim: shrink U if necessary so that U is Σ-symmetric ($\gamma U \cap U = \emptyset$ for all $\gamma \in \Gamma - \Sigma$) and define $U' = \bigcup_{\gamma \in \Gamma} \gamma U$. By Lemma 4.1 $\tilde{f}$ extends to a Γ-equivariant mapping on U'. Now apply the Tietze-Glaeser extension theorem [4] to obtain the required mapping on $\mathbb{R}^n$.

In order to verify the claim, we first consider the case when A is homeomorphic to a circle. Then A has a Σ-invariant tubular neighborhood U that is homeomorphic to the cross product $A \times E$ where E is the $(n-1)$-dimensional disk. Clearly $\tilde{f}(a, e) = (\tilde{f}(a), 0)$ satisfies the requirements in the claim. (It is more natural to take the less degenerate extension $\tilde{f}(a, e) = (\tilde{f}(a), e/2)$ but this does not generalize so easily to the case when A is not a circle.)

If A is not a circle, there are degeneracies at the set of vertices A^0. However we can find a Σ-invariant neighborhood U of the form $U = U_1 \cup U_2$ where $U_1 = (A - A^0) \times E$ and $U_2 = \bigcup_{v \in A_0} E_v$, for E_v a union of half disks. Define $\tilde{f}$ on U_1 as before. Since $\tilde{f}$ fixes points in A^0 we can define $\tilde{f}(v, e) = (v, 0)$ on U_2 and obtain the required continuous mapping on U. This completes the proof of the claim and hence the proof that Σ is strongly admissible. (If we try to use the less degenerate extension mentioned before, then there is a difficulty not at the vertices but at the points that map directly onto the vertices.)

Next, suppose that Δ is a cyclic extension of Σ. We must show that Δ is admissible. Choose $\rho \in \Delta - \Sigma$ such that the cyclic group Δ/Σ is generated by the coset $\rho\Sigma$. Let k be the order of Δ/Σ and define

$$G' = G \cup \rho G \cup \cdots \cup \rho^{k-1} G.$$

Then G' is a (generally disconnected) Δ-graph. Let $h : G \to \rho G$ be an extension, and define $g : G \to G'$ by $g = h \circ f$. By Lemma 4.1, g extends uniquely to a continuous Δ-equivariant map $g : G' \to G'$. Provided that we can verify properties (i)–(iv) in the appendix, Proposition A.1 implies that G' is topologically transitive under g. The embedding of G then extends to an embedding of G' leading to admissibility of Δ by identical arguments to those used in the proof of strong admissibility of Σ above.

Let I_j denote the edges of G'. It follows from the proof of Theorem 4.3 and the fact that h is an isometry that properties (i)-(iii) are satisfied (with $q = 1$). Moreover, provided G has at least 3 edges, we compute that for any edge I_j in G, $g^2(I_j) = \rho^2 G$. By equivariance, we have

$$\bigcup_{i=2}^{k+1} g^i(I_j) = G',$$

for all edges I_j in G' and property (iv) is satisfied. Finally, if G has less than 3 edges, we can modify the mapping f just as we did in the proof of Theorem 4.3. ■

6 Existence of embeddable, extendable Eulerian Σ-graphs

Suppose that $n \geq 3$ and Γ is a finite subgroup of $\mathbf{O}(n)$ with subgroup Σ. In this section we prove that if Σ fixes a connected component of $\mathbb{R}^n - L_\Sigma$ then there exists an embeddable, extendable Eulerian Σ-graph. This combined with Theorem 5.4 proves Theorem 2.2. We must consider two cases depending on whether or not Σ contains reflections. In Subsection (a) we show that the complete Σ-graph is extendable, and is embeddable if and only if Σ contains no reflections. Then in Subsection (b) we construct the required graphs when Σ contains reflections.

(a) Σ contains no reflections

Theorem 6.1 *Let Γ be a finite subgroup of $\mathbf{O}(n)$ with subgroup Σ.*

(a) The complete Σ-graph $G(\Sigma)$ is extendable.

(b) Suppose that Σ fixes a connected component of $\mathbb{R}^n - L_\Sigma$ and that $n \geq 3$. Then $G(\Sigma)$ is embeddable if and only if Σ contains no reflections.

Proof (a) Let $E_{\tau,\tau'}$ denote the edge joining the vertices τ and τ' in $G(\Sigma)$ and define $h(E_{\tau,\tau'}) = \rho E_{\tau_\rho,\tau'_\rho}$. (Recall that $\tau_\rho = \rho^{-1}\tau\rho$.) We must show that

$h(\sigma E_{\tau,\tau'}) = \sigma h(E_{\tau,\tau'})$. Now $\sigma h(E_{\tau,\tau'}) = \sigma\rho E_{\tau_\rho,\tau'_\rho}$ which by definition of the action of Σ on $\rho G(\Sigma)$ is given by $\rho\sigma_\rho E_{\tau_\rho,\tau'_\rho}$. Now compute that

$$\begin{aligned} \rho\sigma_\rho E_{\tau_\rho,\tau'_\rho} &= \rho E_{(\sigma\tau)_\rho,(\sigma\tau')_\rho} \\ &= h(E_{\sigma\tau,\sigma\tau'}) \\ &= h(\sigma E_{\tau,\tau'}) \end{aligned}$$

as required.

(b) First we show that Σ must contain no reflections if $G(\Sigma)$ is to be embedded. Suppose that $i : G(\Sigma) \to \mathbb{R}^n$ is an embedding and let $\tau \in \Sigma$ be a reflection. Then the vertices 1 and τ in $G(\Sigma)$ map into distinct connected components of $\mathbb{R}^n - \mathrm{Fix}(\tau)$. Hence the image of the edge joining 1 to τ intersects $\mathrm{Fix}(\tau)$ which contradicts the fact that all points in $G(\Sigma)$ and hence in $i(G(\Sigma))$ have trivial isotropy.

Finally, we show that $G(\Sigma)$ is embeddable provided Σ contains no reflections. Suppose that C is a connected component of $\mathbb{R}^n - L_\Sigma$ fixed by Σ. Since Γ is finite, there is an open dense subset of $\mathbb{R}^n$ consisting of points with trivial isotropy. Let C' consist of those points in C with trivial isotropy. Then C' is open and dense in C and moreover C' is connected since Σ contains no reflections. We proceed to embed $G(\Sigma)$ into C'.

Choose a point $x \in C'$ with trivial isotropy. Then $\sigma x \in C'$ for $\sigma \in \Sigma$, and for $\sigma \neq 1$ there is a continuous one-to-one path $i_\sigma : J_\sigma \to C'$ with $i_\sigma(1) = x$ and $i_\sigma(\sigma) = \sigma x$. By Proposition 4.4, $J = \bigcup_{\sigma\in\Sigma} J_\sigma$ is a fundamental subgraph of $G(\Sigma)$ and it follows from Lemma 4.1 that there is a unique continuous Σ-equivariant mapping $i : G(\Sigma) \to \mathbb{R}^n$ such that $i|_{J_\sigma} = i_\sigma$. The image $i(G(\Sigma))$ consists of the paths τi_σ where $\sigma, \tau \in \Sigma$. Let Σx denote the image of the set of vertices of $G(\Sigma)$ under the map i. Then i is one-to-one if and only if the paths τi_σ intersect only at points in Σx. Moreover, i satisfies in addition the second condition in the definition of embedding if and only if paths γi_σ intersect only at points in Γx for $\sigma \in \Sigma$, $\gamma \in \Gamma$.

To show that such a choice of paths i_σ is possible, we pass to the orbit space $\mathbb{R}^n/\Gamma$. The set Γx becomes a point represented by x and the set $i(J)$ becomes a union of loops at x. Our requirements reduce to demanding that these loops have no intersections other than at the point x. Any other intersections are nontransverse (since $n \geq 3$ and Γ is finite) and the paths i_σ can be perturbed so that the loops in $\mathbb{R}^n/\Gamma$ intersect only at x as required. ■

(b) Σ contains reflections

When Σ contains reflections, it is easy to modify $G(\Sigma)$ so as to make it embeddable. Unfortunately, it is not clear that the resulting Σ-graph is extendable. For this reason, we embark on an entirely different construction of a Σ-graph.

First, consider the case when Σ is generated by reflections. Let C be a connected component of $\mathbb{R}^n - L_\Sigma$ fixed by Σ and choose $D \subset C$ a fundamental domain for the action of Σ on C (for information on fundamental domains, see [10]). Let $\overline{D}$ denote the closure of D in the relative topology on C (so $\overline{D}$ is the the closure of D in $\mathbb{R}^n$ intersected with C). Similarly, let ∂D denote the boundary of D in the relative topology.

We may write ∂D as a union of reflection hyperplanes $\mathrm{Fix}(\tau_i)$ for reflections τ_i in Σ (where we only consider those hyperplanes satisfying $\dim(\mathrm{Fix}(\tau_i) \cap \partial D) = n-1$). For each i we can choose a point x_i in ∂D such that the only nontrivial element of Γ that fixes x_i is τ_i. We can choose the x_i so that no two of these points lie on the same Γ-orbit.

Let J be the complete (nonoriented) graph with vertices x_i and define $G = \bigcup_{\sigma \in \Sigma} \sigma J$. Then G is a Σ-graph with fundamental subgraph J and is clearly Eulerian. Moreover, provided $n \geq 3$, G is embeddable by the same arguments used to prove embeddability of $G(\Sigma)$ in Theorem 6.1.

Next we show that G is extendable. Since Σ fixes C it is clear that $\mathrm{Fix}(\tau) \cap C \neq \emptyset$ for all reflections $\tau \in \Sigma$. If $\rho \in N(\Sigma) - \Sigma$, then Σ also fixes the connected component ρC. In particular, all the reflection hyperplanes that make up ∂D intersect ρC. It follows that there is a fundamental domain $D' \subset \rho C$ whose boundary is made up of the same reflection hyperplanes that comprise the boundary of D. Let $A = i(G)$ denote the embedded graph and set $y_i = \rho A \cap \mathrm{Fix}(\tau_i)$. If E is an edge of J joining x_i to x_j then we define $h(E)$ to be the edge of ρG joining y_i to y_j. In this way we obtain an isometry $h : J \to \rho G$ satisfying the hypotheses of Lemma 4.1. Hence we obtain the required extension $h : G \to \rho G$.

Now we turn to the case where Σ contains reflections but is not generated by reflections. Let Σ_R denote the subgroup of Σ that is generated by reflections. We may define an embeddable, extendable Eulerian Σ_R-graph G_R with fundamental subgraph J_R using the above construction.

Elements of Σ permute the fundamental domains for the action of Σ_R. Let Σ_0 denote the subgroup of Σ that preserves D. Then J_R and σJ_R are disjoint graphs in $\overline{D}$ for $\sigma \in \Sigma_0$. Since elements of Σ_0 permute the subspaces

$\mathrm{Fix}(\tau_i)$, σJ_R has vertices $x_{\sigma(i)} \in \mathrm{Fix}(\tau_i)$. For each $\sigma \in \Sigma_0$, introduce an edge in D with vertices at $x_i \in J_R$ and $x_{\sigma(i)} \in \sigma J_R$. Let J be the graph consisting of J_R together with these additional edges and define $G = \bigcup_{\sigma\in\Sigma} \sigma J$. Then G is an embeddable Eulerian Σ-graph. The embedding can be chosen so that J is embedded in $\overline{D}$.

It remains to define an isometry $h : J \to \rho G$ satisfying the hypotheses of Lemma 4.1, thus verifying that G is extendable. Take $h|J_R$ as before. We have used the fact that $\rho\Sigma_R \cdot J_R$ intersects each reflection hyperplane in ρC in a single point. The same is true of $\rho\Sigma_R\sigma \cdot J_R$ for each $\sigma \in \Sigma_0$. Define $y_{\sigma(i)}$ to be the vertex of $\rho\Sigma_R\sigma \cdot J_R$ lying in $\mathrm{Fix}(\tau_i)$. If E is the edge of J connecting x_i to $x_{\sigma(i)}$, we define $h(E)$ to be the edge of J'_σ joining y_i to $y_{\sigma(i)}$. The resulting isometry $h : J \to \rho G$ gives the required extension.

7 Admissibility in one and two dimensions

In this section we consider analogous results when $n = 1$ and $n = 2$ to those described in Theorem 2.2 for $n \geq 3$. For these remaining values of n, the conditions for admissibility can be verified on a case by case basis. The case $\Gamma \subset \mathbf{O}(2)$ cyclic is different from the others.

We begin with the case $n = 1$. Now $\mathbf{O}(1) = \{\pm 1\}$ so that we need only consider the groups $\Gamma = \mathbf{1}$ and $\Gamma = \mathbb{Z}_2$.

Proposition 7.1 *If $\Gamma \subset \mathbf{O}(1)$ then all subgroups of Γ are strongly admissible.*

Proof Strong admissibility of the subgroup $\mathbf{1}$ is easily verified both for $\Gamma = \mathbf{1}$ and $\Gamma = \mathbb{Z}_2$. Strictly speaking, the mapping $f(x) = 3(x - x^3)/2$ will suffice in both cases, since there is an attracting fixed point at $\sqrt{1/3}$ that has no symmetry. If we want to have a 'nontrivial' attractor we can use the logistic mapping or variants thereof.

For the subgroup $\Sigma = \mathbb{Z}_2$ of $\Gamma = \mathbb{Z}_2$ we are forced to construct a nontrivial attractor. (We cannot take $A = \{0\}$ since then A contains no points with trivial isotropy.) The (unnormalized) Tchebycheff polynomial $f(x) = 4x^3 - 3x$ is $\mathbb{Z}_2$-equivariant, and is known to have a $\mathbb{Z}_2$-symmetric invariant set $A = [-1, 1] \subset \mathbb{R}$. Moreover, $f|_A$ is topologically conjugate to the mapping $g = 3\theta$ on S^1, see Exercise 1 in Section 1.8 of Devaney [8]. The desired properties of A follow easily. ■

The finite subgroups of $\mathbf{O}(2)$ are given up to conjugacy by

$$\mathbb{Z}_m, \quad \mathbb{D}_m, \; m \geq 1,$$

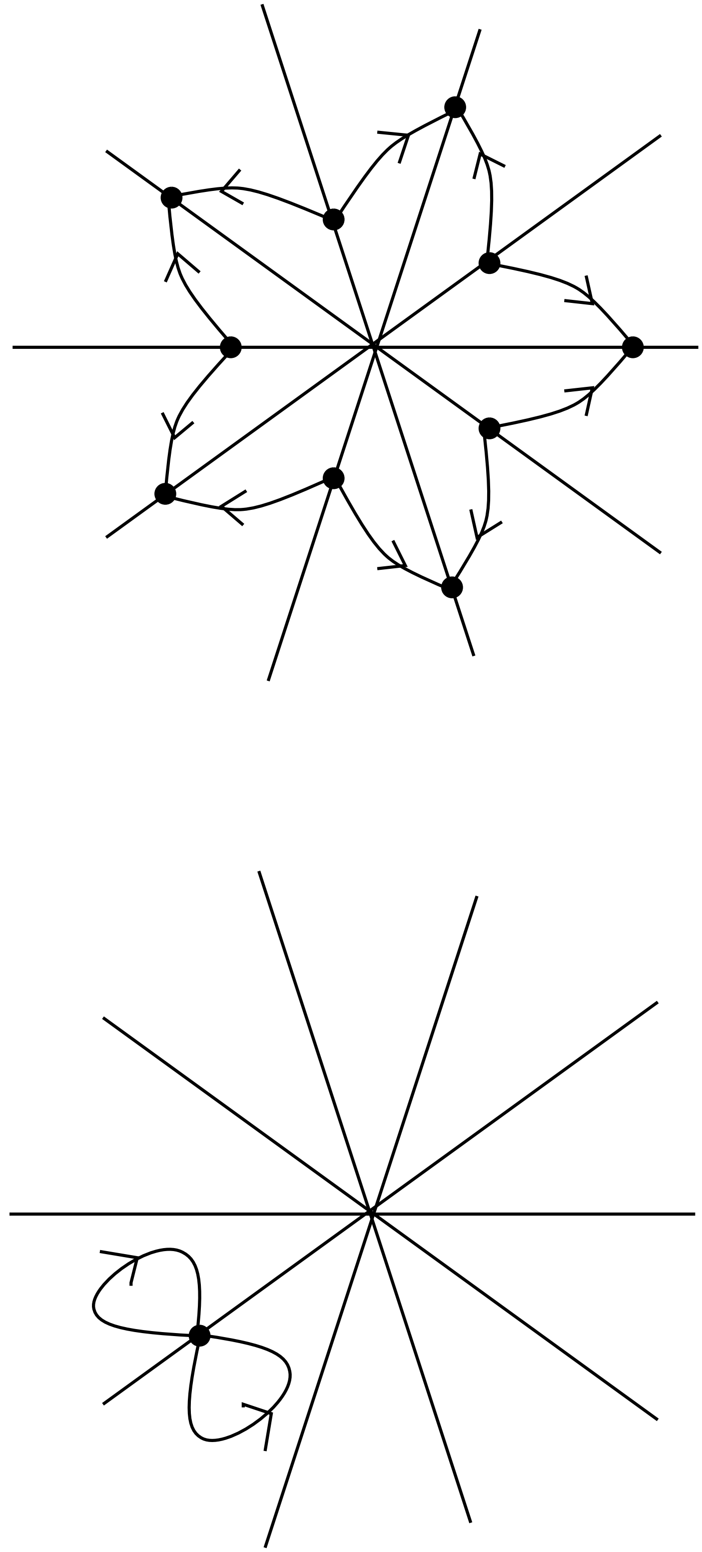
where $\mathbb{Z}_m$ is generated by rotation through angle $2\pi/m$ and $\mathbb{D}_m$ is generated by this element together with a reflection.

Theorem 7.2 *(a) Suppose that $\mathbb{Z}_m \subset \mathbf{O}(2)$, $m \geq 1$. The strongly admissible subgroups are $\mathbb{Z}_m$ and $\mathbf{1}$. The remaining subgroups are $\mathbb{Z}_k$ where k divides m, $1 < k < m$, and are admissible.*
(b) Suppose that $\mathbb{D}_m \subset \mathbf{O}(2)$, $m \geq 1$. The strongly admissible subgroups are $\mathbb{D}_m$, $\mathbb{D}_1$ and $\mathbf{1}$. The subgroups $\mathbb{Z}_k$, $k > 1$, (k divides m) are admissible as is $\mathbb{D}_2$ when m is even (so that $\mathbb{D}_2$ is a subgroup of $\mathbb{D}_m$). The remaining subgroups are $\mathbb{D}_k$, $2 < k < m$, (k divides m) and are inadmissible.

Remark 7.3 (a) Much of this theorem is contained in [14] and [15]. The new information is the admissibility of $\mathbb{D}_m$ and of $\mathbb{D}_2$ for m even, the strong admissibility of various subgroups, and the fact that certain subgroups of $\mathbb{Z}_m$ are not strongly admissible.
(b) Strictly speaking, we should distinguish the two copies of $\mathbb{D}_k$ that are conjugate in $\mathbf{O}(2)$ but not conjugate in $\mathbb{D}_m$ when m/k is even. (The copies of $\mathbb{D}_k$ are generated by reflectional symmetries of the regular m-gon corresponding to axes joining opposite vertices or joining midpoints of opposite edges.) We have not done this as it turns out that the analysis for each copy of $\mathbb{D}_k$ is identical. For subgroups of $\mathbf{O}(3)$ the issue of nonconjugate subgroups plays an important role, see Section 8.

Proof We begin with part (b). In [15], it was shown that the only subgroups of $\mathbb{D}_m$ that can be strongly admissible are $\mathbb{D}_m$, $\mathbb{D}_1$ and $\mathbf{1}$. Moreover, the only subgroups other than these that can be admissible are $\mathbb{Z}_k$ where k divides m and $\mathbb{D}_2$ for m even.

It remains to show that these subgroups are indeed admissible and strongly admissible. Strong admissibility of $\mathbf{1}$ is trivial, and admissibility of the cyclic subgroups follows from Remark 5.3. We show how to embed Eulerian $\mathbb{D}_1$ and $\mathbb{D}_m$-graphs in $\mathbb{R}^2$ in Figure 2 thus establishing strong admissibility of these subgroups. Finally, when m is even $\mathbb{D}_2$ is a subgroup of $\mathbb{D}_m$ and is the direct sum of $\mathbb{D}_1$ and $\mathbb{Z}_2$. By Remark 5.3, $\mathbb{D}_2$ is admissible.



(a) (b)

Figure 2

Figure 2: The figures show how Eulerian $\mathbb{D}_1$ and $\mathbb{D}_m$-graphs can be embedded in $\mathbb{R}^2$ acted on by $\mathbb{D}_m$. In this example $m = 5$. All the reflection axes are illustrated. Note that all points on the graphs are of trivial isotropy apart from the vertices.

Strong admissibility of **1** and admissibility of the cyclic subgroups in part (a) is identical to part (b). In addition we can construct a $\mathbb{Z}_m$-symmetric connected attractor using the same graph that we used for $\mathbb{D}_m$ in part (b).

It remains to show that $\mathbb{Z}_k$ is not strongly admissible for $1 < k < m$. Suppose for contradiction that A is a connected attractor with $\mathbb{Z}_k$ symmetry. Let $x \in A$ and let U be the connected component of the basin of A that contains x. The U is open and contains A, so that there is a simple closed curve S in U that contains the group orbit σx, $\sigma \in \mathbb{Z}_k$.

Let $\rho \in \mathbb{Z}_m - \mathbb{Z}_k$. We claim that $S \cap \rho S \neq \emptyset$. It then follows that the basins of the attractors A and ρA intersect, hence that $A \cap \rho A \neq \emptyset$. By Proposition 1.1 of [5], $A = \rho A$ and A is $\mathbb{Z}_m$-symmetric. This is the required contradiction.

To prove the claim, observe that $\mathbb{R}^2 - S$ consists of two connected components, one of which, C_1 say, contains the origin. Let $s_1 \in S$ be a point where the function $|s|$ attains its minimum. Then $\rho s_1 \in \overline{C_1}$. It follows that $\rho S \not\subset C_2$. A similar argument shows that $\rho S \not\subset C_1$ and hence $\rho S \cap S \neq \emptyset$ verifying the claim. ■

8 Examples

In this section, we list the (strongly) admissible subgroups for each finite subgroup of $\mathbf{O}(3)$. We assume familiarity with the notation in [6], [12] or [13]. Our results are tabulated in Tables 1 and 2. For each subgroup we list those subgroups that are strongly admissible, nonstrongly admissible and inadmissible. (Nonstrongly admissible means admissible but not strongly admissible.)

The subgroups of $\mathbf{O}(3)$ fall into classes I, II and III. We list the conjugacy classes of finite subgroups in each class

Class I $\mathbb{I}$, $\mathbb{O}$, $\mathbb{T}$, $\mathbb{D}_m$, $\mathbb{Z}_m$, $m \geq 1$,

Class II $\mathbb{I} \oplus \mathbb{Z}_2^c$, $\mathbb{O} \oplus \mathbb{Z}_2^c$, $\mathbb{T} \oplus \mathbb{Z}_2^c$, $\mathbb{D}_m \oplus \mathbb{Z}_2^c$, $\mathbb{Z}_m \oplus \mathbb{Z}_2^c$,

Class III $\mathbb{O}^-$, $\mathbb{D}_{2m}^d$, $\mathbb{D}_m^z$, $\mathbb{Z}_{2m}^-$, $m \geq 1$.

It should be noted that not all of these conjugacy classes are distinct. In particular, $\mathbb{D}_1$ is conjugate to $\mathbb{Z}_2$, $\mathbb{D}_1 \oplus \mathbb{Z}_2^c$ is conjugate to $\mathbb{Z}_2 \oplus \mathbb{Z}_2^c$, $\mathbb{D}_1^z$ is conjugate to $\mathbb{Z}_2^-$ and $\mathbb{D}_2^z$ is conjugate to $\mathbb{D}_2^d$ inside of $\mathbf{O}(3)$. If a finite subgroup

$\mathbb{O}^-$	strongly admissible nonstrongly admissible inadmissible	$\mathbb{O}^-$, $\mathbb{D}_3^z$, $\mathbb{D}_2^z$, $\mathbb{D}_1^z$, $\mathbf{1}$ $\mathbb{D}_4^d$, $\mathbb{Z}_4^-$, $\mathbb{Z}_3$, $\mathbb{Z}_2$ $\mathbb{T}$, $\mathbb{D}_2$
$\mathbb{I} \oplus \mathbb{Z}_2^c$	strongly admissible nonstrongly admissible inadmissible	$\mathbb{I} \oplus \mathbb{Z}_2^c$, $\mathbb{D}_5^z$, $\mathbb{D}_3^z$, $\mathbb{D}_2^z$, $\mathbb{Z}_2^-$, $\mathbf{1}$ $\mathbb{D}_5 \oplus \mathbb{Z}_2^c$, $\mathbb{D}_3 \oplus \mathbb{Z}_2^c$, $\mathbb{D}_2 \oplus \mathbb{Z}_2^c$, $\mathbb{Z}_5 \oplus \mathbb{Z}_2^c$, $\mathbb{Z}_3 \oplus \mathbb{Z}_2^c$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2^c$, $\mathbb{Z}_2^c$, $\mathbb{Z}_5$, $\mathbb{Z}_3$, $\mathbb{Z}_2$ $\mathbb{T} \oplus \mathbb{Z}_2^c$, $\mathbb{I}$, $\mathbb{T}$, $\mathbb{D}_5$, $\mathbb{D}_3$, $\mathbb{D}_2$
$\mathbb{O} \oplus \mathbb{Z}_2^c$	strongly admissible nonstrongly admissible inadmissible	$\mathbb{O} \oplus \mathbb{Z}_2^c$, $\mathbb{D}_4^z$, $\mathbb{D}_3^z$, $\mathbb{D}_2^z(e)$, $\mathbb{Z}_2^-(e)$, $\mathbb{Z}_2^-(f)$, $\mathbf{1}$ $\mathbb{D}_2^d$, $\mathbb{D}_2^z(f)$, $\mathbb{Z}_4^-$, $\mathbb{D}_4 \oplus \mathbb{Z}_2^c$, $\mathbb{D}_3 \oplus \mathbb{Z}_2^c$, $\mathbb{D}_2(e) \oplus \mathbb{Z}_2^c$, $\mathbb{Z}_4 \oplus \mathbb{Z}_2^c$, $\mathbb{Z}_3 \oplus \mathbb{Z}_2^c$, $\mathbb{Z}_2(e) \oplus \mathbb{Z}_2^c$, $\mathbb{Z}_2(f) \oplus \mathbb{Z}_2^c$, $\mathbb{Z}_2^c$, $\mathbb{Z}_4$, $\mathbb{Z}_3$, $\mathbb{Z}_2(e)$, $\mathbb{Z}_2(f)$ $\mathbb{O}^-$, $\mathbb{T} \oplus \mathbb{Z}_2^c$, $\mathbb{O}$, $\mathbb{T}$, $\mathbb{D}_4^d$, $\mathbb{D}_2(f) \oplus \mathbb{Z}_2^c$, $\mathbb{D}_4$, $\mathbb{D}_3$, $\mathbb{D}_2(e)$, $\mathbb{D}_2(f)$
$\mathbb{T} \oplus \mathbb{Z}_2^c$	strongly admissible nonstrongly admissible inadmissible	$\mathbb{T} \oplus \mathbb{Z}_2^c$, $\mathbb{D}_2^z$, $\mathbb{Z}_2^-$, $\mathbb{D}_2 \oplus \mathbb{Z}_2^c$, $\mathbb{Z}_3$, $\mathbf{1}$ $\mathbb{Z}_3 \oplus \mathbb{Z}_2^c$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2^c$, $\mathbb{Z}_2^c$, $\mathbb{Z}_2$ $\mathbb{T}$, $\mathbb{D}_2$
$\mathbb{I}$	strongly admissible	$\mathbb{I}$, $\mathbb{T}$, $\mathbb{D}_5$, $\mathbb{D}_3$, $\mathbb{D}_2$, $\mathbb{Z}_5$, $\mathbb{Z}_3$, $\mathbb{Z}_2$, $\mathbf{1}$
$\mathbb{O}$	strongly admissible	$\mathbb{O}$, $\mathbb{T}$, $\mathbb{D}_4$, $\mathbb{D}_3$, $\mathbb{D}_2(e)$, $\mathbb{D}_2(f)$, $\mathbb{Z}_4$, $\mathbb{Z}_3$, $\mathbb{Z}_2(e)$, $\mathbb{Z}_2(f)$, $\mathbf{1}$
$\mathbb{T}$	strongly admissible	$\mathbb{T}$, $\mathbb{D}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_2$, $\mathbf{1}$

Table 1: Admissible subgroups of the exceptional groups in $\mathbf{O}(3)$

Γ of $\mathbf{O}(3)$ contains these subgroups, it must be determined whether or not they are conjugate in Γ. In addition, there are isomorphic but nonconjugate subgroups of $\mathbb{O}$ and $\mathbb{O} \oplus \mathbb{Z}_2^c$. These and the resulting notation are described later.

In the remainder of this section, we give the verification of the entries in Tables 1 and 2. We follow the strategy described in Section 3. In order to carry out this program we need certain information about the subgroups of $\mathbf{O}(3)$. This is provided in Propositions 8.1, 8.2 and 8.3.

Proposition 8.1 *The finite subgroups of* $\mathbf{O}(3)$ *that are generated by reflections are*

$$\mathbb{I} \oplus \mathbb{Z}_2^c,\ \mathbb{O} \oplus \mathbb{Z}_2^c,\ \mathbb{O}^-,\ \mathbb{D}_m^z,\ \mathbb{D}_{2m}^d, m \textit{ odd},\ \mathbb{D}_m \oplus \mathbb{Z}_2^c,\ m \textit{ even},\ \mathbb{Z}_2^-, \mathbf{1}.$$

$\mathbb{D}^d_{2m}, m$ odd	strongly admissible nonstrongly admissible inadmissible	$\mathbb{D}^d_{2m}$, $\mathbb{D}^z_m$, $\mathbb{D}^d_2$, $\mathbb{D}^z_1$, $\mathbb{Z}^-_2$, $\mathbf{1}$ $\mathbb{Z}^-_{2k}$, $\mathbb{Z}_k$ $\mathbb{D}^d_{2k}\,(k \neq m)$, $\mathbb{D}^z_k\,(k \neq m)$, $\mathbb{D}_k$
$\mathbb{D}^d_{2m}, m$ even	strongly admissible nonstrongly admissible inadmissible	$\mathbb{D}^d_{2m}$, $\mathbb{D}^z_m$, $\mathbb{D}_1$, $\mathbb{D}^z_1$, $\mathbf{1}$ $\mathbb{D}^z_2$, $\mathbb{D}_2$, $\mathbb{Z}^-_{2k}$, $\mathbb{Z}_k$ $\mathbb{D}^d_{2k}\,(k \neq m)$, $\mathbb{D}^z_k\,(k \neq 2, m)$, $\mathbb{D}_k\,(k \neq 2)$
$\mathbb{D}^z_m$	strongly admissible nonstrongly admissible inadmissible	$\mathbb{D}^z_m$, $\mathbb{Z}^-_2$, $\mathbf{1}$ $\mathbb{D}^z_2$, $\mathbb{Z}_k$ $\mathbb{D}^z_k\,(k \neq 2, m)$
$\mathbb{Z}^-_{2m}$	strongly admissible	$\mathbb{Z}^-_{2k}$, $\mathbb{Z}_k$
$\mathbb{D}_m \oplus \mathbb{Z}^c_2, m$ odd	strongly admissible nonstrongly admissible inadmissible	$\mathbb{D}_m \oplus \mathbb{Z}^c_2$, $\mathbb{D}^z_m$, $\mathbb{D}^z_1$, $\mathbb{D}_1$, $\mathbf{1}$ $\mathbb{D}_m \oplus \mathbb{Z}^c_2$, $\mathbb{D}_1 \oplus \mathbb{Z}^c_2$, $\mathbb{D}^z_m$, $\mathbb{D}^z_1$, $\mathbb{D}_1$, $\mathbb{Z}_k \oplus \mathbb{Z}^c_2$, $\mathbb{Z}_k$ $\mathbb{D}_k \oplus \mathbb{Z}^c_2$, $\mathbb{D}^z_k\,(k \neq m)$, $\mathbb{D}_k$
$\mathbb{D}_m \oplus \mathbb{Z}^c_2, m$ even	strongly admissible nonstrongly admissible inadmissible	$\mathbb{D}_m \oplus \mathbb{Z}^c_2$, $\mathbb{D}^z_m$, $\mathbb{D}^z_1$, $\mathbb{D}^d_2$, $\mathbb{Z}^-_2$, $\mathbf{1}$ $\mathbb{D}_2 \oplus \mathbb{Z}^c_2$, $\mathbb{D}_1 \oplus \mathbb{Z}^c_2$, $\mathbb{D}^z_2$, $\mathbb{D}_1$, $\mathbb{Z}^-_{2k}$, $\mathbb{Z}_k \oplus \mathbb{Z}^c_2$ $\mathbb{D}_k \oplus \mathbb{Z}^c_2$, $\mathbb{D}^z_k\,(k \neq 2, m)$, $\mathbb{D}^d_{2k}$, $\mathbb{D}_k$
$\mathbb{Z}_m \oplus \mathbb{Z}^c_2, m$ odd	strongly admissible	$\mathbb{Z}_k \oplus \mathbb{Z}^c_2$, $\mathbb{Z}_k$
$\mathbb{Z}_m \oplus \mathbb{Z}^c_2, m$ even	strongly admissible nonstrongly admissible	$\mathbb{Z}^-_{2k}, k$ odd, $\mathbb{Z}_k \oplus \mathbb{Z}^c_2, k$ even, $\mathbb{Z}_k$ $\mathbb{Z}^-_{2k}, k$ even, $\mathbb{Z}_k \oplus \mathbb{Z}^c_2, k$ odd
$\mathbb{D}_m$	strongly admissible	$\mathbb{D}_k$, $\mathbb{Z}_k$
$\mathbb{Z}_m$	strongly admissible	$\mathbb{Z}_k$

Table 2: Admissible subgroups of the planar groups in $\mathbf{O}(3)$ (k divides m, $k > 1$)

Proposition 8.2 *(a) The subgroups $\mathbb{D}_m^z$, $m \geq 2$, $\mathbb{Z}_m$, $m \geq 2$, $\mathbb{D}_2^d$ and $\mathbb{D}_1$ have one-dimensional fixed-point subspaces.*

(b) The subgroups $\mathbb{D}_1^z$ and $\mathbb{Z}_2^-$ have two-dimensional fixed-point subspaces.

(c) The remaining nontrivial finite subgroups of $\mathbf{O}(3)$ have zero-dimensional fixed-point subspaces.

Proposition 8.3 *The inclusions between the finite subgroups of $\mathbf{O}(3)$ are as follows*

$$
\begin{array}{rcl}
\mathbb{I} \oplus \mathbb{Z}_2^c & \supset & \mathbb{I},\ \mathbb{D}_5 \oplus \mathbb{Z}_2^c,\ \mathbb{D}_3 \oplus \mathbb{Z}_2^c,\ \mathbb{D}_2 \oplus \mathbb{Z}_2^c \\
\mathbb{O} \oplus \mathbb{Z}_2^c & \supset & \mathbb{O},\ \mathbb{O}^-,\ \mathbb{D}_4 \oplus \mathbb{Z}_2^c,\ \mathbb{D}_3 \oplus \mathbb{Z}_2^c \\
\mathbb{T} \oplus \mathbb{Z}_2^c & \supset & \mathbb{T},\ \mathbb{D}_2 \oplus \mathbb{Z}_2^c,\ \mathbb{Z}_3 \oplus \mathbb{Z}_2^c \\
\mathbb{O}^- & \supset & \mathbb{T},\ \mathbb{D}_4^d,\ \mathbb{D}_3^z \\
\mathbb{I} & \supset & \mathbb{T},\ \mathbb{D}_5,\ \mathbb{D}_3 \\
\mathbb{O} & \supset & \mathbb{T},\ \mathbb{D}_4,\ \mathbb{D}_3 \\
\mathbb{T} & \supset & \mathbb{D}_2,\ \mathbb{Z}_3 \\
\mathbb{D}_m \oplus \mathbb{Z}_2^c & \supset & \mathbb{D}_k \oplus \mathbb{Z}_2^c,\ k \text{ divides } m,\ \mathbb{D}_m^d,\ (m \text{ even}),\ \mathbb{D}_m^z,\ \mathbb{D}_m \\
\mathbb{Z}_m \oplus \mathbb{Z}_2^c & \supset & \mathbb{Z}_k \oplus \mathbb{Z}_2^c,\ k \text{ divides } m,\ \mathbb{Z}_m^-,\ (m \text{ even}) \\
\mathbb{D}_{2m}^d & \supset & \mathbb{D}_{2k}^d,\ m/k \text{ odd},\ \mathbb{Z}_{2m}^-,\ \mathbb{D}_m^z,\ \mathbb{D}_m \\
\mathbb{Z}_{2m}^- & \supset & \mathbb{Z}_{2k}^-,\ m/k \text{ odd},\ \mathbb{Z}_m \\
\mathbb{D}_m^z & \supset & \mathbb{D}_k^z,\ k \text{ divides } m,\ \mathbb{Z}_m \\
\mathbb{D}_m & \supset & \mathbb{D}_k,\ k \text{ divides } m,\ \mathbb{Z}_m \\
\mathbb{Z}_m & \supset & \mathbb{Z}_k,\ k \text{ divides } m
\end{array}
$$

Using Propositions 8.1 and 8.3, we compute for each finite subgroup $\Gamma \subset \mathbf{O}(3)$ the isotropy subgroups I of Γ and for each I the subgroup I_R of I that is generated by reflections. By Theorem 3.2, the strongly admissible subgroups are those that lie between I and I_R for some isotropy subgroup I. Then the nonstrongly admissible subgroups are given by the nontrivial extensions of the strongly admissible subgroups.

If Γ contains no reflections, every subgroup of Γ is strongly admissible. This accounts for the class I subgroups $\Gamma = \mathbb{I}, \mathbb{O}, \mathbb{T}, \mathbb{D}_m, \mathbb{Z}_m$ and also for the subgroups $\mathbb{Z}_m \oplus \mathbb{Z}_2^c$ when m is odd and $\mathbb{Z}_{2m}^-$ when m is even. It remains to consider the cases when Γ contains reflections. In Table 3 we list the remaining subgroups Γ together with the isotropy subgroups of Γ which we denote by I.

A useful fact to bear in mind in computing cyclic extensions is that if Σ is a nontrivial cyclic extension of a dihedral group Δ, then Δ must be of index two in Σ.

Γ	I
$\mathbb{I} \oplus \mathbb{Z}_2^c$	$\mathbb{I} \oplus \mathbb{Z}_2^c$, $\mathbb{D}_5^z$, $\mathbb{D}_3^z$, $\mathbb{D}_2^z$, $\mathbb{Z}_2^-$, $\mathbf{1}$
$\mathbb{O} \oplus \mathbb{Z}_2^c$	$\mathbb{O} \oplus \mathbb{Z}_2^c$, $\mathbb{D}_4^z$, $\mathbb{D}_3^z$, $\mathbb{D}_2^z(e)$, $\mathbb{Z}_2^-(e)$, $\mathbb{Z}_2^-(f)$, $\mathbf{1}$
$\mathbb{O}^-$	$\mathbb{O}^-$, $\mathbb{D}_3^z$, $\mathbb{D}_2^z$, $\mathbb{D}_1^z$, $\mathbf{1}$
$\mathbb{T} \oplus \mathbb{Z}_2^c$	$\mathbb{T} \oplus \mathbb{Z}_2^c$, $\mathbb{D}_2^z$, $\mathbb{Z}_2^-$, $\mathbb{Z}_3$, $\mathbf{1}$
$\mathbb{D}_{2m}^d, m$ odd	$\mathbb{D}_{2m}^d$, $\mathbb{D}_m^z$, $\mathbb{D}_2^d$, $\mathbb{D}_1^z$, $\mathbb{Z}_2^-$, $\mathbf{1}$
$\mathbb{D}_{2m}^d, m$ even	$\mathbb{D}_{2m}^d$, $\mathbb{D}_m^z$, $\mathbb{D}_1$, $\mathbb{D}_1^z$, $\mathbf{1}$
$\mathbb{D}_m^z$	$\mathbb{D}_m^z$, $\mathbb{Z}_2^-$, $\mathbf{1}$
$\mathbb{D}_m \oplus \mathbb{Z}_2^c, m$ odd	$\mathbb{D}_m \oplus \mathbb{Z}_2^c$, $\mathbb{D}_m^z$, $\mathbb{D}_1^z$, $\mathbb{D}_1$, $\mathbf{1}$
$\mathbb{D}_m \oplus \mathbb{Z}_2^c, m$ even	$\mathbb{D}_m \oplus \mathbb{Z}_2^c$, $\mathbb{D}_m^z$, $\mathbb{D}_1^z$, $\mathbb{D}_2^d$, $\mathbb{Z}_2^-$, $\mathbf{1}$
$\mathbb{Z}_m \oplus \mathbb{Z}_2^c, m$ even	$\mathbb{Z}_m \oplus \mathbb{Z}_2^c$, $\mathbb{Z}_2^-$, $\mathbb{Z}_m$, $\mathbf{1}$
$\mathbb{Z}_{2m}^-, m$ odd	$\mathbb{Z}_{2m}^-$, $\mathbb{Z}_2^-$, $\mathbb{Z}_m$, $\mathbf{1}$

Table 3: Isotropy subgroups of the finite subgroups of $\mathbf{O}(3)$ that contain reflections

We end by describing as promised the notation for the subgroups of $\mathbb{O}$ and $\mathbb{O} \oplus \mathbb{Z}_2^c$. There are order two rotations in $\mathbb{O}$ that are conjugate in $\mathbf{O}(3)$ but not in $\mathbb{O}$. These are the rotations about axes that connect midpoints of opposite edges and faces respectively. As a result, $\mathbb{O}$ contains two nonconjugate copies of $\mathbb{D}_2$, the subgroup $\mathbb{D}_2(f)$ which contains the three face rotations and is normal, and the subgroup $\mathbb{D}_2(e)$ which contains two edge rotations and a face rotation.

Corresponding to these subgroups of $\mathbb{O}$, $\mathbb{O} \oplus \mathbb{Z}_2^c$ contains three nonconjugate class III subgroups isomorphic to $\mathbb{D}_2$. These are obtained by multiplying any two nontrivial elements of $\mathbb{D}_2(e)$ and $\mathbb{D}_2(f)$ by $-I$. Of course $\mathbb{D}_2^z(f)$ arises from $\mathbb{D}_2(f)$. In addition we can choose an edge and a face from $\mathbb{D}_2(e)$ to obtain $\mathbb{D}_2^z(e)$ or two edges to obtain $\mathbb{D}_2^d$. This distinction is important for admissibility of the subgroups of the finite reflection group $\mathbb{O} \oplus \mathbb{Z}_2^c$. Of these three class III subgroups only $\mathbb{D}_2^z(e)$ is an isotropy subgroup, the others lying in $\mathbb{D}_4^z$. Hence only $\mathbb{D}_2^z(e)$ is strongly admissible. The other two subgroups are cyclic extensions of $\mathbb{Z}_2^-(f)$ and hence are nonstrongly admissible.

The class II subgroups $\mathbb{D}_2^z(e) \oplus \mathbb{Z}_2^c$ and $\mathbb{D}_2^z(f) \oplus \mathbb{Z}_2^c$ are not isotropy subgroups and hence are not strongly admissible. However, $\mathbb{D}_2^z(e) \oplus \mathbb{Z}_2^c$ is a cyclic extension of $\mathbb{D}_2^z(e)$ and hence is admissible.

A Dynamics on graphs

We consider continuous mappings $f : G \to G$ where G is a finite graph with edges $I_1, \ldots, I_m$ and f satisfies the following properties:

(i) $f(I_j)$ is the union of some I_i's for each j.

(ii) $f|_{I_{ij}}$ is C^2 and invertible, where for each i, j, $I_{ij} = I_i \cap f^{-1}(I_j)$.

(iii) There is an iterate f^q such that wherever defined, $|(f^q)'| \geq \theta > 1$.

(iv) For all j, $\bigcup_{p \geq 1} f^p(I_j) = G$.

The subsets I_{ij} form a Markov partition for the mapping f. It is a standard procedure using symbolic dynamics to show that f is semiconjugate to a subshift of finite type. Moreover the assumptions of piecewise invertibility in (ii) and of expansitivity in (iii) imply that the semiconjugacy is finite-to-one and is actually a conjugacy except on a subset of Lebesgue measure zero. It follows in the usual way that G is transitive under f, that periodic orbits are dense, and that there is sensitive dependence on initial conditions. Moreover, if we replace (iv) by the stronger aperiodicity assumption

(iv)$'$ There is an integer p such that $f^p(I_j) = G$ for all j,

then G is topologically mixing. Finally, the full strength of (ii) requiring twice-differentiability implies the existence of a Lebesgue-equivalent ergodic measure on G, see the Folklore Theorem of Adler and Flatto [1]. To summarize, we have the following result.

Proposition A.1 *Suppose that $f : G \to G$ is a continuous mapping satisfying properties (i)–(iv). Then*

(a) G is transitive.

(b) Periodic points are dense in G, and G has sensitive dependence on initial conditions.

(c) There is a finite Lebesgue-equivalent f-invariant ergodic measure μ on G. This measure is unique up to a scalar multiple.

If, in addition, f satisfies property (iv)′ then G is topologically mixing.

Remark A.2 It follows from the uniqueness in part (c) of the proposition that if G is a Σ-graph and f is Σ-equivariant, then μ is Σ-invariant. To see this, let $\sigma \in \Sigma$. Then the pull-back of μ by σ is also a finite Lebesgue-equivalent f-invariant ergodic measure and hence is the same as μ.

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