Stable Transitivity of Euclidean Group Extensions

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Abstract

Topological transitivity of noncompact group extensions of topologically mixing subshifts of finite type has been studied recently by Niţică. We build on these methods, and give the first examples of *stably* transitive noncompact group extensions of hyperbolic dynamical systems. Our examples include extensions of hyperbolic basic sets by the Euclidean group $\mathbf{SE}(n)$ for n even, $n \geq 4$.

1 Introduction

In this paper, we consider topological transitivity of noncompact group extensions of hyperbolic basic sets. In particular, we give examples of such extensions that are stably transitive. As far as we know, these are the first examples of noncompact group

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extensions that are stably transitive. Our results are based upon work of Niţică [9] which in turn is based upon work of Brin [2].

Suppose that $F : M \to M$ is a smooth diffeomorphism of a smooth Riemannian manifold M and that $f : X \to X$ is the restriction of F to a hyperbolic basic set X. Suppose that Γ is a finite-dimensional Lie group. Let $r \geq 1$ and define $\mathcal{Z}_r$ to be the space of C^r maps or *cocycles* $\zeta : M \to \Gamma$. Each cocycle $\zeta \in \mathcal{Z}_r$ induces a Γ-*extension* $f_\zeta : X \times \Gamma \to X \times \Gamma$ given by $f_\zeta(x, \gamma) = (fx, \gamma\zeta(x))$.

Definition 1.1 Fix $r \geq 1$, and define

$$Z = \{\zeta \in \mathcal{Z}_r \mid f_\zeta : X \times \Gamma \to X \times \Gamma \text{ is topologically transitive}\}.$$

If $\zeta \in \operatorname{Int} Z$, then f_ζ is *stably (topologically) transitive.*
If $\operatorname{Int} Z$ is dense in $\mathcal{Z}_r$, then we say that *generically f_ζ is stably transitive.*
If Z is a residual subset of $\mathcal{Z}_r$, then we say that *generically f_ζ is transitive.*

Let $\mathbf{SE}(n) = \mathbf{SO}(n) \ltimes \mathbb{R}^n$ denote the (special) Euclidean group generated by rotations and translations in n-dimensional space.

Theorem 1.2 *Let X be a hyperbolic basic set. Suppose that n is even and $n \geq 4$. Then generically $f_\zeta : X \times \mathbf{SE}(n) \to X \times \mathbf{SE}(n)$ is stably transitive.*

More generally, let $\rho : G \to \mathbf{SO}(n)$ be an orthogonal representation on $\mathbb{R}^n$ of a compact connected Lie group G. Form the semidirect product $\Gamma = G \ltimes \mathbb{R}^n$ with multiplication $(g_1, v_1) \cdot (g_2, v_2) = (g_1 g_2, v_1 + \rho_{g_1} v_2)$. We call Γ a *Euclidean-type group.*

Our results depend on the following three conditions:

(1) There exists a $g \in G$ such that $I - \rho_g$ is a nonsingular operator on $\mathbb{R}^n$.

(2) $n \geq 2$ and G acts irreducibly on $\mathbb{R}^n$.

(3) G is semisimple.

Remark 1.3 When n is odd, Condition (1) fails.

Suppose that $\Gamma = \mathbf{SE}(n)$. Then Condition (1) holds if and only if n is even, Condition (2) holds for all $n \geq 2$, and Condition (3) holds if and only if $n \neq 2$.

Theorem 1.2 is an immediate consequence of the next result.

Theorem 1.4 *Let X be a hyperbolic basic set and Γ be a Euclidean-type group.*
(a) If Condition (1) is satisfied, then generically $f_\zeta : X \times \Gamma \to X \times \Gamma$ is transitive.
(b) If Conditions (1)–(3) hold, then generically f_ζ is stably transitive.

Remark 1.5 It is an immediate consequence of our work (with $n = 0$) that extensions of hyperbolic basic sets by compact semisimple Lie groups G are generically stably transitive. In fact, they are stably ergodic (and stably mixing) see [5]. The nonsemisimple case requires additional hypotheses on the hyperbolic basic set [5, 12].

Although Euclidean-type groups include $\mathbb{R}^n$, our results do not apply to $\mathbb{R}^n$-extensions due to the failure of Condition (1). Recently, examples of stably transitive $\mathbb{R}^n$-extensions (with X Anosov) have been obtained by Niţică and Pollicott [10].

The remainder of this paper is organized as follows. In Section 2, we recall the work of Niţică [9] on topological transitivity of skew products, and give an elementary proof of Theorem 1.4(a). In Section 3, we prove Theorem 1.4(b).

2 Transitivity of Euclidean-type group extensions

This section is divided into three subsections. In Subsection (a), we recall work of Niţică [9] on topological transitivity of skew products. In Subsection (b), we collect some elementary results on the structure of Euclidean-type groups. In Subsection (c), we prove Theorem 1.4(a) which gives a sufficient condition for transitivity of Euclidean-type group extensions.

(a) Topological transitivity of skew products

Suppose that Y is a complete metric space and that $T : Y \to Y$ is a homeomorphism. A partition $W = \{W(y)\}$ of Y is *T-invariant* if $T(W(y)) \subset W(T(y))$ for all $y \in Y$. A T-invariant partition W is *contracting* if for all $y_1, y_2 \in Y$ with $W(y_1) = W(y_2)$, we have that $d(f^N(y_1), f^N(y_2)) \to 0$ as $N \to \infty$. The partition is *expanding* if it is contracting for T^{-1}. Finally, the partition is of *foliation type* if given $\epsilon > 0$ and any $p \in Y$ and $q \in W(p)$ there exists $\delta(p, q, \epsilon)$ such that if C is a non-empty open set contained in the ball $B(p, \delta)$ then the set $B(q, \epsilon) \cap (\cup_{z \in C} W(z))$ contains a non-empty open set.

Fix two partitions W_1 and W_2 of Y. We say that a subset $A \subset Y$ is *accessible* (with respect to W_1 and W_2) if for any pair of points $p, q \in A$ there exists N and a chain $\{p = y_1, y_2, \ldots, y_N = q\}$ such that $y_{i+1} \in W_1(y_i) \cup W_2(y_i)$ for $i = 1, \ldots, N - 1$. The subset A is *ϵ-accessible* if instead of $y_N = q$, we have $d(y_N, q) < \epsilon$.

Theorem 2.1 (Niţică [9, Theorem 2.1]) *Let Y be a complete metric space, $T : Y \to Y$ a homeomorphism, and W_1, W_2 a pair of T-invariant partitions of foliation type, one contracting and one expanding. Assume that*

(i) Y is ϵ-accessible for all $\epsilon > 0$,

(ii) *For any open ball* $B \subset Y$ *and any positive integer* N *there is a positive integer* $m > N$ *such that* $T^m B \cap B \neq \emptyset$, *and*

(iii) *The conclusion of (ii) holds for* T^{-1}.

Then $T : Y \to Y$ *is topologically transitive.*

Now suppose that M is a smooth Riemannian manifold, and let $f : X \to X$ be the restriction of a smooth diffeomorphism $F : M \to M$ to a hyperbolic basic set X. Let Γ be a Euclidean-type group, and let $f_\zeta : X \times \Gamma \to X \times \Gamma$ be the Γ-extension induced by the C^r cocycle $\zeta : M \to \Gamma$.

Since Γ admits a left-invariant metric, the Γ-extension is automatically *partially hyperbolic* [3], and the stable and unstable partitions W^s and W^u for X induce (strong) stable and (strong) unstable partitions of foliation type W^{ss} and W^{uu} of $X \times \Gamma$, see [9]. To summarize, we have the following result.

Proposition 2.2 *Let* X *be a hyperbolic basic set and* Γ *be a Euclidean-type group. Then* W^{ss}, W^{uu} *are a pair of* f_ζ*-invariant partitions of foliation type, one contracting and one expanding.*

(b) Structure of Euclidean-type groups

Let Γ be a finite-dimensional Lie group. Suppose that $\gamma_1, \dots, \gamma_k \in \Gamma$ and let $\langle \gamma_1, \dots, \gamma_k \rangle$ denote the closed subgroup of Γ generated by $\gamma_1, \dots, \gamma_k$. Let $\langle \gamma_1, \dots, \gamma_k \rangle^+$ denote the closed semisubgroup generated by $\gamma_1, \dots, \gamma_k$. Define

$$
\begin{aligned}
C &= \{\gamma \in \Gamma : \langle \gamma \rangle \text{ is compact}\} \\
F_k &= \{(\gamma_1, \dots, \gamma_k) \in \Gamma^k : \langle \gamma_1, \dots, \gamma_k \rangle = \Gamma\} \\
F_k^+ &= \{(\gamma_1, \dots, \gamma_k) \in \Gamma^k : \langle \gamma_1, \dots, \gamma_k \rangle^+ = \Gamma\}
\end{aligned}
$$

Note that

$$C^k \cap F_k \subset F_k^+ \subset F_k. \tag{2.1}$$

Now suppose that $\Gamma = G \ltimes \mathbb{R}^n$ is a Euclidean-type group. Define $G_C \subset G$ to be the set

$$G_C = \{g \in G : I - \rho_g \text{ is a nonsingular operator on } \mathbb{R}^n\}.$$

Proposition 2.3 *(a)* $g \in G_C$ *if and only if* $(g, v) \in C$ *for all* $v \in \mathbb{R}^n$.
(b) If Condition (1) holds, then G_C *is open and dense in* G, *and* C *contains the open and dense subset* $G_C \times \mathbb{R}^n \subset \Gamma$.

Proof Compute that $(g,v)^m = (g^m, v_m)$, where

$$v_m = (I + \rho_g + \cdots + \rho_g^{m-1})v.$$

If $g \in G_C$, then $v_m = (I - \rho_g)^{-1}(I - \rho_g^m)v$ is bounded and $(g,v) \in C$. Otherwise, choosing $v \in \ker(I - \rho_g)$ yields an element $(g,v) \notin C$. This proves part (a).

To prove part (b), note that G_C is a Zariski open subset of G. Condition (1) guarantees that G_C is nonempty, and hence open and dense in G. By (a), $G_C \times \mathbb{R}^n \subset C$. ∎

Proposition 2.4 *Suppose that Γ is a Euclidean-type group. Then F_{n+3} is a residual subset of Γ^{n+3}.*

Proof We use the well-known facts that

(i) If $w_1, \ldots, w_{n+1} \in \mathbb{R}^n$, then generically $\langle w_1, \ldots, w_{n+1} \rangle = \mathbb{R}^n$ (cf. [9, Lemma 2.6]), and

(ii) If $g_1, g_2 \in G$, then generically $\langle g_1, g_2 \rangle = G$ (see [1]).

A standard argument shows that F_{n+3} is a countable intersection of open subsets of Γ^{n+3} which is Baire space, so it suffices to show that F_{n+3} is dense in Γ^{n+3}. Let $\Sigma = \langle \gamma_1, \ldots, \gamma_{n+3} \rangle$ and let $\Sigma_2 = \langle \gamma_1, \gamma_2 \rangle$. We show that $\Sigma = \Gamma$ for a dense subset of $(n+3)$-tuples $(\gamma_1, \ldots, \gamma_{n+3}) \in \Gamma^{n+3}$.

Write $\gamma_j = (g_j, v_j)$ where $g_j \in G$, $v_j \in \mathbb{R}^n$. By (ii), we may suppose that $\langle g_1, g_2, \rangle = G$ and hence $\pi(\Sigma_2)$ is dense in G. (Without compactness, we cannot deduce directly that $\pi(\Sigma_2) = G$.) Hence, for $j \geq 3$, we can perturb g_j to lie in $\pi(\Sigma_2)$. In particular, for each $j \geq 3$, we have two corresponding elements $\gamma_j = (g_j, v_j)$, $\delta_j = (g_j, w_j)$ in Σ where w_j depends only on γ_1 and γ_2. Hence we obtain pure translations $(e, z_j) = \gamma_j \delta_j^{-1}$, $j \geq 3$, where $z_j = v_j - w_j$. The translations v_j, $j \geq 3$ are still free to be perturbed, so by fact (i), we may suppose that $z_3, \ldots, z_{n+3}$ generate $\mathbb{R}^n$, so that $\{e\} \times \mathbb{R}^n \in \Sigma$. It follows that $(g_1, 0)$, $(g_2, 0) \in \Sigma$. Hence $G \times \{0\} \subset \Sigma$ and so $\Sigma = \Gamma$. ∎

(c) Criteria for transitivity

Corollary 2.5 *Let X be a hyperbolic basic set and Γ be a Euclidean-type group satisfying Condition (1). Then generically $f_\zeta : X \times \Gamma \to X \times \Gamma$ is ϵ-accessible for all $\epsilon > 0$.*

Proof By Proposition 2.3, we have that C contains an open and dense subset of Γ. By Proposition 2.4, F_{n+3} is residual. It follows from (2.1) that F_{n+3}^+ is residual. Genericity of ϵ-accessibility is immediate from the proof of [9, Lemma 2.2]. ∎

Proposition 2.6 *Let X be a hyperbolic basic set and Γ be a Euclidean-type group satisfying Condition (1). Then generically, $f_\zeta : X \times \Gamma \to X \times \Gamma$ has the property that recurrent points are dense in $X \times \Gamma$.*

Proof For $N \geq 1$, $f_\zeta^N(x, \gamma) = (f^N x, \gamma\zeta_N(x))$, where

$$\zeta_N(x) = \zeta(x)\zeta(fx)\cdots\zeta(f^{N-1}x). \tag{2.2}$$

If $f^p x = x$, then $f_\zeta^{pj}(x, \gamma) = (x, \gamma\zeta_p(x)^j)$ for all $j \geq 1$. Hence (x, γ) is recurrent if and only if $\zeta_p(x) \in C$. It follows from Proposition 2.3 that this is the case for an open and dense set of Γ-extensions. Since X contains a countable dense set consisting of periodic points, generically recurrent points are dense in $X \times \mathbf{SE}(n)$. ∎

Proof of Theorem 1.4(a) It follows from Proposition 2.2, Corollary 2.5 and Proposition 2.6 that the hypotheses of Theorem 2.1 are generically valid. ∎

3 Stable transitivity of Euclidean-type group extensions

In this section, we prove Theorem 1.4(b) which gives a sufficient condition for stable transitivity of Euclidean-type group extensions. We require the following two results which strengthen the conclusions of Propositions 2.4 and 2.6 respectively.

Theorem 3.1 *Suppose that Γ is a Euclidean-type group satisfying Conditions (1)–(3). Then F_2 is (Zariski) open and dense in Γ^2.*

Theorem 3.2 *Let X be a hyperbolic basic set and Γ be a Euclidean-type group satisfying Condition (1). Then an open and dense set of Γ-extensions $f_\zeta : X \times \Gamma \to X \times \Gamma$ have recurrent points dense in $X \times \Gamma$.*

The proof of part (b) of Theorem 1.4 now follows the proof of part (a). It remains to prove Theorems 3.1 and 3.2 respectively. This is done in Subsections (a) and (b) respectively.

(a) Proof of Theorem 3.1

Theorem 3.3 ([7, 4]) *Suppose that G is a compact connected semisimple Lie group. Then F_2 is a nonempty Zariski open subset of G^2.* ∎

(Openness and density of F_2 is due to Kuranishi [7]. Zariski openness was proved recently by Field [4].)

Lemma 3.4 *Let $\Gamma = G \ltimes \mathbb{R}^n$ be a Euclidean-type group satisfying Conditions (1) and (3) and with projection $\pi : \Gamma \to G$. Suppose that Σ is a finitely generated closed subgroup of Γ such that $\overline{\pi(\Sigma)} = G$. Then $\pi(\Sigma) = G$.*

Proof Let $\pi_* = (d\pi)_e : L\Sigma \to LG$ be the corresponding homomorphism of Lie algebras. Then it suffices to show that π_* is onto.

By Condition (3), LG can be written uniquely as a direct sum of simple Lie algebras $LG = L_1 \oplus \cdots \oplus L_r$. Moreover, G is a finite cover of a finite product of simple Lie groups $G_1, \ldots, G_r$ where $L_j = LG_j$. Let $p_j : G \to G_j$ denote the obvious projections and $\pi_j = p_j \circ \pi : \Sigma \to G_j$. Observe that $\pi_j(\Sigma)$ is dense in G_j.

Let $V = \pi_*(L\Sigma)$. Since $\pi(\Sigma)$ is dense in G it follows that V is invariant under the adjoint action of G on LG and so V is an ideal in LG. By the semisimplicity of LG again, V is a direct sum of some of the factors $L_1, \ldots, L_r$. Relabelling, we can write $V = L_1 \oplus \cdots \oplus L_s$ where $s \leq r$.

It remains to show that $s = r$. It suffices to show that for each j, there is a closed subgroup $T \subset G_j$ with $\dim T \geq 1$ such that $T \in \pi_j(\Sigma)$. Since $\pi_j(\Sigma)$ is a finitely generated group of matrices, Selberg's Lemma [13], [11, p.18] guarantees that there exists a torsion-free subgroup $N \subset \pi_j(\Sigma)$ of finite index. By Condition (1), $p_j(G_C)$ is open and dense in G_j so $N \cap p_j(G_C) \neq \{e\}$. In particular, there exists an element $g \in G_j$ of infinite order such that the corresponding element $\gamma \in \Sigma$ generates a compact subgroup $\langle\gamma\rangle \subset \Sigma$. Let $T = \pi_j\langle\gamma\rangle$. Then $T \subset \pi_j(\Sigma)$ is an infinite compact subgroup of G_j and so $\dim T \geq 1$ as required. ∎

Lemma 3.5 *Let $\Gamma = G \ltimes \mathbb{R}^n$ be a Euclidean-type group satisfying Condition (2) and with projection $\pi : \Gamma \to G$. Suppose that Σ is a closed subgroup of Γ such that $\pi(\Sigma) = G$. Then either $\Sigma = \Gamma$ or $\Sigma \cong G$.*

Proof Identify $\{e\} \times \mathbb{R}^n$ with $\mathbb{R}^n$ and let $\mathcal{L} = \Sigma \cap \mathbb{R}^n$. Then $\mathcal{L}$ is a closed subgroup of $\mathbb{R}^n$ and is given by $\mathcal{L} = \mathbb{R}^p \times \mathbb{Z}^q$ where $p, q \geq 0$. We claim that G preserves $\mathcal{L}$, so for any $(v, w) \in \mathbb{R}^p \times \mathbb{Z}^q$ and $g \in G$, we can write $\rho_g(v, w) = (v_g, w_g) \in \mathbb{R}^p \times \mathbb{Z}^q$. Since G is connected, $w_g \equiv w$ so that G acts trivially on $\mathbb{Z}^q$. By Condition (2), $\mathbb{Z}^q = 0$ so that $\mathcal{L} = \mathbb{R}^p$. Moreover, $\mathcal{L} = \mathbb{R}^n$ or $\mathcal{L} = 0$ as required.

To prove the claim, note that for each $g \in G$, there exists a $v_g \in \mathbb{R}^n$ such that $(g, v_g + \mathcal{L}) \subset \Sigma$, and v_g is unique modulo $\mathcal{L}$. We have $(g, v_g + \mathcal{L})^2 = (g^2, v_{g^2} + \mathcal{L})$, and so

$$v_{g^2} + \mathcal{L} = \rho_g(v_g + \mathcal{L}) + v_g + \mathcal{L}.$$

Let $w = v_{g^2} - \rho_g v_g - v_g$. Then $w + \rho_g\mathcal{L} = \mathcal{L}$. Since $0 \in \rho_g\mathcal{L}$, we deduce that $w \in \mathcal{L}$ and hence $\rho_g\mathcal{L} = \mathcal{L}$ as required. ∎

Lemma 3.6 *Suppose that* Γ *is a Euclidean-type group. Let*

$$A = \{(\gamma_1, \gamma_2) \in \Gamma^2 : \overline{\langle \gamma_1, \gamma_2 \rangle} \textit{ is not compact}\}$$

Then A *contains a nonempty Zariski open subset of* Γ^2.

Proof For $j = 1, 2$, write $\gamma_j = (g_j, v_j)$ where $g_j \in G$ and $v_j \in \mathbb{R}^n$. Define $D = G - G_C$ which is Zariski closed in G.

Observe that $\gamma_1^n = (g_1^n, nv_1)$ for all $v_1 \in \ker(I - \rho_{g_1})$. Hence γ_1 generates a noncompact group for all those v_1 that have a nonzero component in $\ker(I - \rho_{g_1})$. It follows that if $D = G$, then $\overline{\langle \gamma_1 \rangle}$ is noncompact for a nonempty Zariski open set of $\gamma_1 \in \Gamma$.

It remains to consider the case when $D \neq G$. Define

$$A' = \{(\gamma_1, \gamma_2) \in \Gamma^2 : g_1, g_2 \in G_C \text{ and } (I - \rho_{g_1})^{-1} v_1 \neq (I - \rho_{g_2})^{-1} v_2\}.$$

Clearly, A' is Zariski open and nonempty. We show that $A' \subset A$.

Suppose that $\Sigma = \overline{\langle \gamma_1, \gamma_2 \rangle}$ is compact. We must show that $(\gamma_1, \gamma_2) \notin A'$. The Euclidean-type action of Γ on $\mathbb{R}^n$ $(x \to \rho_g x + v)$ restricts to an action of Σ on $\mathbb{R}^n$. Choose $x_0 \in \mathbb{R}^n$ and define $y = \int_\Sigma \sigma x_0 \, d\mu$ where μ is normalized Haar measure on Σ. Let $\sigma_0 = (g_0, v_0) \in \Sigma$. Then

$$\sigma_0 y = \int_\Sigma \rho_{g_0} \sigma x_0 \, d\mu + v_0 = \int_\Sigma (\rho_{g_0} \sigma x_0 + v_0) d\mu = \int_\Sigma \sigma_0 \sigma x_0 \, d\mu = y.$$

Hence Σ fixes the point y. In particular, $\gamma_j y = y$ for $j = 1, 2$. This means that $(I - g_1) y = v_1$ and $(I - g_2) y = v_2$. If $g_1, g_2 \in G_C$, then $y = (I - g_1)^{-1} v_1 = (I - g_2)^{-1} v_2$. Hence $(\gamma_1, \gamma_2) \notin A'$ as required. ∎

Proof of Theorem 3.1 Let Σ denote the closed subgroup of Γ generated by γ_1 and γ_2. By Conditions (1) and (3), Theorem 3.3, and Lemma 3.4, there is a nonempty Zariski open subset $U \subset \Gamma^2$ consisting of pairs $(\gamma_1, \gamma_2) \in \Gamma^2$ for which $\pi(\Sigma) = G$. By Condition (2) and Lemma 3.5, either $\Sigma \cong G$ or $\Sigma = \Gamma$. But by Lemma 3.6, we can shrink U if necessary so that in addition Σ is noncompact and hence $\Sigma = \Gamma$. ∎

(b) Proof of Theorem 3.2

Suppose that $G \subset \mathbf{SO}(n)$ is a compact connected Lie group. Recall that the open set G_C consists of those elements of G such that $I - \rho_g$ is nonsingular.

Let $f : X \to X$ be a hyperbolic basic set and let $h : X \to G$. For $x \in X$ and $N \geq 1$, define

$$h_N(x) = h(x) h(fx) \cdots h(f^{N-1} x). \tag{3.1}$$

Suppose that $x \in X$ is a periodic point of period p. We say that x is *good* if $h_p(x) \in G_C$. Otherwise x is *bad*. Our main result in this section is the following.

Theorem 3.7 *Suppose that $h : X \to G$ is Lipschitz. Either good periodic points are dense in X, or all periodic points are bad.*

We require the following (strengthened) version of the Anosov closing lemma.

Lemma 3.8 (Anosov Closing Lemma [6, p. 269]) *Suppose that X is a hyperbolic basic set. There exist constants $C_1 \geq 1, \epsilon > 0$ such that if $d(f^n(v), v) < \epsilon$ for some $v \in X$, $n \geq 1$, then there exists $w \in X$ with $f^n w = w$ such that*

$$d(f^j v, f^j w) < C_1 \lambda^{\min(j,n-j)} d(f^n v, v),$$

for $j = 1, 2, \ldots, n$. ∎

Proof of Theorem 3.7 Suppose that good periodic points are not dense, and fix the periodic point $x_0 \in X$. We show that x_0 is bad. By passing to a power of f, we may suppose without loss that x_0 is a fixed point. Let $g_0 = h(x_0)$ and define $D = G - G_C$. We show that $g_0 \in D$.

Since good periodic points are not dense, there is a nonempty open subset $U_0 \subset X$ such that all periodic points in U_0 are bad. Let $U = \bigcup_{j \in \mathbb{Z}} f^j(U_0)$. Then U is an open dense subset of X and all periodic points in U are bad.

Let C_1, ϵ, μ be as in Lemma 3.8. Choose a periodic point z such that

(i) z and fz are within distance $\epsilon/2$ of x_0,

(ii) There is a $j \geq 1$ such that points within distance $C_1 d(z, fz)$ of $f^j z$ lie inside U.

(This can be achieved by first fixing $y \in U$, and then shadowing a pseudo-orbit that travels from x_0 to y and back again to x_0.) Let m denote the period of z and define $\sigma = h_m(z)$.

Fix $s \geq 1$ and note that $f^{ms+1} z = fz$ which is within distance ϵ of z. By Lemma 3.8, there exists a periodic point p of period $ms + 1$ such that

$$d(f^j p, f^j z) < C_1 \mu^{\min(j,ms+1-j)} d(z, fz) \leq C_1 d(z, fz),$$

for $j = 1, 2, \ldots, ms + 1$. It follows from property (ii) above that $f^j p \in U$ so that p is a bad periodic point. Hence $h_{ms+1}(p) \notin G_C$. By construction $h_{ms+1}(z) = \sigma^s h(fz)$.

We claim that there are universal constants K and α (depending only on $f : X \to X$ and $h : X \to G$) such that $\operatorname{dist}(\sigma^s h(fz), D) \leq K d(z, fz)^\alpha$. Since ϵ is arbitrarily small (so the three points x_0, z and fz are arbitrarily close to each other), $\sigma^s h(fz)$ is arbitrarily close to D and $h(fz)$ is arbitrarily close to g_0. At the same time, we can choose s so that σ^s is arbitrarily close to the identity. Altogether, we have that g_0 is arbitrarily close to the closed set D and hence $g_0 \in D$ as required.

It remains to verify the claim. The proof closely follows the proof of the Livšic periodic point theorem [6, p. 609]. Let M be the Lipschitz constant for h. It is easily verified by induction that

$$h_N(x) - h_N(y) = \sum_{j=0}^{N-1} h_j(x)\big[h(f^jx) - h(f^jy)\big]h_j(y)^{-1}h_{N-1}(y),$$

for all $x, y \in X$, $N \geq 1$. Hence

$$\begin{aligned}
\operatorname{dist}(\sigma^s h(fz), D) \leq \|h_{ms+1}(z) - h_{ms+1}(p)\| &\leq \sum_{j=0}^{ms} \|h(f^jz) - h(f^jp)\| \\
&\leq \sum_{j=0}^{ms} Md(f^jz, f^jp) \leq M\sum_{j=0}^{ms} C_1\mu^{\min(j,ms+1-j)}d(z,fz) \\
&= MC_1 d(z,fz)\sum_{j=0}^{ms}\mu^{\min(j,ms+1-j)} \\
&\leq 2MC_1(1-\mu)^{-1}d(z,fz).
\end{aligned}$$

The claim follows with $K = 2MC_1(1-\mu)^{-1}$. ∎

Proof of Theorem 3.2 Write elements of Γ as $\gamma = (g, v)$ where $g \in G$ and $v \in \mathbb{R}^n$. Similarly, write $\zeta(x) = (h(x), k(x))$ where $h : X \to G$ and $k : X \to \mathbb{R}^n$ are the restrictions (from M to X) of C^r maps. The heights $\zeta_N(x) \in \Gamma$ and $h_N(x) \in G$ (defined in (2.2) and (3.1) respectively) are related by the fact that $h_N(x)$ is the G-component of $\zeta_N(x)$ (cf. [8]).

Choose a periodic point $x_0 \in X$. By Proposition 2.3(b), G_C is open and dense in G, so x_0 is a good periodic point for an open and dense set of Γ-extensions f_ζ. But then by Theorem 3.7, good periodic points are dense in G. For such periodic points x, with period $p(x)$, we have $h_{p(x)}(x) \in G_C$ and hence, by Proposition 2.3(a), $\zeta_{p(x)}(x) \in C$. As shown in Proposition 2.6, it follows immediately that (x, γ) is recurrent in $X \times \Gamma$ for all $\gamma \in \Gamma$. Hence, recurrent points are dense as required. ∎

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